ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 4, 2012

A CORRESPONDENCE BETWEEN THE ISOBARIC RING AND MULTIPLICATIVE ARITHMETIC FUNCTIONS

TRUEMAN MACHENRY AND KIEH WONG

ABSTRACT. We give a representation of the classical theory of multiplicative arithmetic functions (MF) in the ring of symmetric polynomials written on the isobaric basis. The representing elements are recursive sequences of Schur-hook polynomials evaluated on subrings of the complex numbers. Multiplicative arithmetic functions are units in the Dirichlet ring of arithmetic functions, and their properties can be described locally, that is, at each prime number p. Our representation is, hence, a local representation. This representation enables us to clarify and generalize classical results, e.g., the Busche-Ramanujan identity, as well as to give a richer structural description of the convolution group of multiplicative functions. It is a consequence of the representation that the MFs can be defined in a natural way on the negative powers of prime p which, in turn, leads to a natural extension of Schur-hook polynomials to negatively indexed Schur-hook polynomials.

0. Introduction. In this paper we give a representation of the classical theory of multiplicative arithmetic functions (MF) in the ring of symmetric polynomials. The Dirichlet ring of arithmetic functions $\mathcal{A}^*$ is well known to be a unique factorization domain (see Cashwell and Everett [**4**]). Its ring theoretic properties have been investigated in, e.g., Rearick [**29, 30**], Shapiro [**33**], Carroll and Gioia [**3**], MacHenry [**22**], MacHenry and Tudose [**25**]. The multiplicative arithmetic functions are units in this ring, and their properties can be described locally, that is, at each prime number p, (see, e.g., McCarthy [**27**], Sivaramakrishnan [**35**] and Vaidyanathswamy, [**36**]). It is this local behavior which we take advantage of to construct a representation in terms of a certain class of symmetric polynomials called *weighted isobaric polynomials* [**25**]. It is advantageous to use the *isobaric* basis as a basis for the

2010 AMS *Mathematics subject classification.* Primary O5EO5, 11B37, 11B39, 11N99.
Keywords and phrases. Multiplicative arithmetic functions, symmetric polynomials, isobaric basis, linear recursions generalized Fibonacci sequences.
Received by the editors on March 13, 2009, and in revised form on January 7, 2010.
DOI:10.1216/RMJ-2012-42-4-1247

ring of symmetric polynomials; we describe this basis in Section 1. Henceforth, we refer to the symmetric polynomials in this basis as the ring of isobaric polynomials.

The link between the theory of symmetric polynomials and the theory of multiplicative, arithmetic functions is that of linear recurrence, especially the ideas contained in MacHenry and Wong [**25**] (also see Rutkowski [**32**], Lascoux [**19**] and Hou and Mu [**15**]). The isobaric ring contains a certain submodule, the submodule of *weighted isobaric polynomials* (WIP), which is generated as a $\mathbf{Z}$-module by the Schur-hook polynomials. This module has the property that it can be partitioned into sequences which are linear recursions (see [**25**]). It contains the sequence of Generalized Fibonacci Polynomials (GFP), and the sequence of Generalized Lucas Polynomials (GLP) (see MacHenry [**23**]). It turns out that each of these sequences, when the indeterminates are evaluated over a subring of the complex numbers, is the evaluation of a local sequence of multiplicative functions, i.e., a multiplicative function at a prime p. Moreover, every MF is represented locally by such sequences; in fact, the GFP-sequence is sufficient for this purpose. This fact brings the machinery of the isobaric ring to bear with respect to the convolution group of multiplicative functions. The importance of linear recursions in the theory of MF is recognized in Laohakosol and Pabhapote [**16**] and in Rutkowski [**32**]; however, the connection between multiplicative functions and symmetric functions and the power of the isobaric notation to simplify and reveal basic facts about the structure of MF is not made explicit in these papers.

The same machinery of linear recurrences in the isobaric setting was used in MacHenry and Wong [**26**] to study number fields. A consequence of the results in that paper implies a certain strong connection between the structure of number fields, the algebraic structure of multiplicative functions, and periodicity in the theory of recursion. The connection between $\mathcal{A}^*$ and the symmetric polynomials was exploited in [**22, 25**]. In the first of these papers it was used to prove that the group of multiplicative functions generated by the completely multiplicative functions is free abelian. In the second paper, a constructive procedure using isobaric polynomials was given for embedding this group into its divisible closure (also see [**3**]).

In Section 1, we define the *weighted isobaric polynomials* (WIP's) and give a formula for them independent of recursion.

In Section 2, the notion of the *core* polynomial is introduced and the *infinite companion matrix* and its properties are described, (see also [**25**], Lascoux [**20**]). This infinite matrix extends the WIP-sequences in the negative direction, that is, provides negatively-indexed functions as well as positively-indexed ones. Negatively indexed sequences of isobaric polynomials induce, using the linear recursion property, negatively indexed MFs. Also, in this section, generating functions are provided for the isobaric polynomials (and their MF counterparts).

In Section 3, we discuss the ring of arithmetic functions and introduce an important classification scheme for them.

In Section 4, the main theorem, the Correspondence Theorem (Theorem 1) asserting the relation between multiplicative arithmetic functions and the WIP-module is proved.

Given a prime p, let $\chi \in \mathcal{M}$ with local core $\mathrm{C}(X)$, k finite or infinite, and let $F_k(\mathbf{t})$ be the sequence of GFP's induced by this core, then

$$F_{k,n}(\mathbf{t}) = \chi(p^n).$$

In this section it is shown that, for each MF α, not only do we have its local representation in terms of GFP's, but in addition, each column of the infinite companion matrix also determines an MF. Each element in any column of A^∞ is a Schur-hook polynomial. (The negatively indexed one's provide an extension of the idea of Schur polynomials.) All of these Schur polynomials, the negatively-indexed ones as well as the positively-indexed ones, can be conveniently computed using Jacobi-Trudi formulae in their isobaric form.

In Section 5, we give a collection of examples showing the details of the application of the correspondence theorem.

In Section 6, we look at the theory of *specially multiplicative* arithmetic functions, the theorem of McCarthy and the Busche-Ramanujan identity from the point of view of isobaric representation, putting these ideas in a different and more transparent light. In particular, we show that the *specially multiplicative arithmetic* functions are not so special after all; the theorem of McCarthy is a trivially redundant assertion that specially multiplicative functions are quadratic ([**27**, Theorem 1.12, especially part (4)]). Thus, the recursion formula for multiplicative functions is a generalization of the McCarthy theorem. In the case of Busche-Ramanujan, we give a generalization and an interpretation. In the language of this paper, the (Busche-Ramanujan) theorem (see [**27, 34**]) becomes Proposition 3 which states that, when a multiplicative function is represented by the GFP F, then

(4.1)
$$F_{2,r+s} = F_{2,r}F_{2,s} + t_2F_{2,r-1}F_{2,s-1}$$
(4.2)
$$F_{2,r}F_{2,s} = F_{2,r+s} - t_2F_{2,r+s-2} + \cdots + (-t_2)^j F_{2,r+s-2j} + \cdots + (-t_2)^r F_{2,s-r}$$

where the degree of the core is 2, and $j = 1, \dots, r$.

In Section 7, the notions of *type* and *valence* are introduced. *Type* is a classification of the local convolution group of multiplicative arithmetic functions in terms of ranges and domains of its elements. There are four types

$$(fin, fin), (inf, fin), (fin, inf), and (inf, inf),$$

where, for example, (fin, inf) means that the domain contains only finitely many non-zero elements, and the range contains infinitely many non-zero elements. Here we show, e.g., that all of these types exist and are mutually exclusive, and that type 1 has only a single representative, the identity function.

Valence is also an ordered pair, namely, a pair (r, s), where r is the number of degree one factors in a multiplicative function χ and s is the number of inverses of degree 1 factors in χ, where χ is written in reduced form.

We also show how a theorem of Laohakosol and Pabhapote [**7**] extending Busche-Ramanujan identities to multiplicative functions of mixed type can be simplified and clarified.

In Section 8, we propose a classification system for MF which consists of the categories *degree*, *type* and *valence*, which enables us to take a refined look at the structure of MF's. The degree is the degree of the core polynomial; the type of an MF, as mentioned above, has to do with the sizes of its domain and range; and the valence, with its convolution structure. We discuss and extend some results of Laohakosol-Pabhapote, [**16**, Proposition 7, Theorem 9 and especially, Theorem 11] which gives a candidate for a generalization of the Busche-Ramanujan identity in terms of Schur-hook functions.

In Section 9, we discuss the Kesava Menon norm for MF in terms of the framework of this paper, showing that it is multiplicative and preserves degree. This is what the Kesava Menon norm looks like in isobaric terminology:

$$\mathbf{N}(F_n) = \sum_{j=0}^{2n} (-1)^j F_{2n-j} F_j$$

or, equivalently

$$\mathbf{N}(F_n) = \Big(2 \sum_{j=0}^{n-1} (-1)^j F_{2n-j} F_j\Big) + (-1)^n F_n^2.$$

In this section, we prove the multiplicative property of the Kesava Menon norm, that is, that the Kesava Menon norm preserves convolution products, in the framework of this paper.

1. Ring of isobaric polynomials.

Definition 1. For a fixed k, an *isobaric polynomial* is a polynomial of the form

$$P_{k,n}(t_1, \dots, t_k) = \sum_{\alpha \vdash n} C_\alpha t_1^{\alpha_1} \cdots t_k^{\alpha_k},$$
$$\alpha = (\alpha_1, \dots, \alpha_k), \quad \sum j\alpha_j = n, \quad \alpha_j \in \mathbf{N}.$$

The condition $\sum j\alpha = n$ is equivalent to: $(1^{\alpha_1}, \dots, k^{\alpha_k})$ is a partition of n, whose largest part is at most k, and we write this in the abbreviated, and somewhat unorthodox form, as $\boldsymbol{\alpha} \vdash n$. (Note that, given k the vector $\{\alpha_i\}$ is sufficient information for reconstructing the partition.) Thus, an isobaric polynomial of *isobaric degree* n is a polynomial whose monomials represent partitions of n with largest part not exceeding k. These polynomials form a graded commutative ring with identity under ordinary multiplication and addition of polynomials, graded by isobaric degree. This ring is naturally isomorphic to the ring of symmetric polynomials, where the isomorphism is given by the involution Ω:

$$t_j \rightleftarrows (-1)^{j+1} e_j$$

with e_j being the jth elementary symmetric polynomial in the kth gradation of the graded ring of symmetric polynomials on the monomial basis, (see [**21, 25**]). For example, if $k = 2$ and $\{\lambda_1, \lambda_2\}$ is the monomial basis, then for $j = 1, e_1 = \lambda_1 + \lambda_2$ and for $j = 2$ $e_2 = -\lambda_1\lambda_2$. This isomorphism associates the Complete Symmetric Polynomials (CSP) in the monomial ring with the Generalized Fibonacci Polynomials (GFP) in the isobaric ring. It also associates the Power Symmetric Polynomials (PSP) with the Generalized Lucas Polynomials (GLP) in the isobaric ring, (see [**25**] and Macdonald [**21**]). We denote these two sequences of polynomials, respectively, $\{F_{k,n}\}$ and $\{G_{k,n}\}$ where k is the number of variables, and n is the *isobaric degree*. The correspondence between (CSP) and (GFP) and the correspondence between (PSP) and (GLP) can be shown inductively using the fact that (GFP) and (GLP) are linearly recursions of degree k; that is,

$$F_{k,n} = t_1 F_{k,n-1} + t_2 F_{k,n-2} + \cdots + t_k F_{n-k}$$

and

$$G_{k,n} = t_1 G_{k,n-1} + t_2 G_{k,n-2} + \cdots + t_k G_{n-k}$$

with initial conditions given by

$$F_{k,0} = 1, F_{k,-1} = 0, \ldots , F_{k,-k+1} = 0$$

and

$$G_{k,0} = k, G_{k,-1} = 0, \ldots , G_{k,-k+1} = 0.$$

In Section 2 we show that this choice of initial conditions arises in a natural way.

The (GFP) and (GLP) can be explicitly represented as follows:

$$F_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1 \cdots \alpha_k} t_1^{\alpha_1} \cdots t_k^{\alpha_k},$$
$$G_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1 \cdots \alpha_k} \frac{n}{|\alpha|} t_1^{\alpha_1} \cdots t_k^{\alpha_k},$$

where $|\alpha| = \sum \alpha_j$, $j = 1 + 2 + \cdots + k$.

Note that, when $k = 2$ and $t_1 = 1$, $t_2 = 1$, $F_{2,n}(1,1)$ and $G_{2,n}(1,1)$ are, respectively, the sequence of Fibonacci numbers and the sequence of Lucas numbers. On the other hand, for a fixed k, both the GFP and the GLP are linearly recursive sequences indexed by n and. as we shall see below, the indexing can be extended to the negative integers with preservation of the linear recursion property (see [**25, 26**]). All other recursive sequences of isobaric polynomials are linear combinations of isobaric reflects of sequences of Schur-hook polynomials (to be explicitly defined in Section 2) and form a (free) **Z**-module, the module of Weighted Isobaric Polynomials (WIP) (see [**24, 25**]). It is a remarkable fact that the only isobaric polynomials that can be elements in linearly recursive sequences of isobaric polynomials are those that occur in one of the sequences in the WIP-module ([**25**, Theorem 3.4]). Every sequence in the WIP-module can be presented in a closed form whose structure explains the term *weighted.*

$$P_{\omega,k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \ldots, \alpha_k} \frac{\sum \alpha_j \omega_j}{\sum \alpha_j} t_1^{\alpha_1} \cdots t_k^{\alpha_k}$$
$$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ is the *weight* vector, usually taken to be an integer vector. k and ω are fixed and n varies. Both the GFP's and the GLP's are weighted sequences, the weighting being given by, respectively, $(1, 1, \ldots, 1, \ldots)$ for the GFPs and $(1, 2, \ldots, j, \ldots)$ for the GLPs. The Schur-hook polynomial sequences, i.e., the columns of the infinite companion matrix (see Section 2) have weightings of the form $(1, 1, \ldots, 1, \ldots)$ or $\pm(0, \ldots, 1, \ldots, 1, \ldots)$, that is, 0 up to $k-1$ zeros as their leading coordinates and 1's elsewhere. The GLPs are alternating sums of all of the Schur-hooks of the same isobaric degree (see [**21, 24, 25**]).

Moreover, it is important to note that a new variable t_k will appear in a WIP polynomial $P_{\omega,k,n}$ for the first time when $n = k$. Thus, for fixed ω and k, the $P_{\omega,k,j}$ are the same for all $|j| < k$. We call this the *conservation principle*. We illustrate this with a listing of the first five GFPs, taking the point of view that k varies as n varies.

- $F_{k,1} = t_1$
- $F_{k,2} = t_1^2 + t_2$
- $F_{k,3} = t_1^3 + 2t_1t_2 + t_3$
- $F_{k,4} = t_1^4 + 3t_1^2t_2 + t_2^2 + 2t_1t_3 + t_4$
- $F_{k,5} = t_1^5 + 4t_1^3t_2 + 3t_1t_2^2 + 3t_1^2t_3 + 2t_2t_3 + 2t_1t_4 + t_5$.

We need another concept which binds these various sequences together, namely, that of the *core polynomial.*

Definition 2. Given a set of variables $\mathbf{t} = (t_1, \dots, t_k)$, the *core* polynomial is:

$$[t_1, \dots, t_k] = X^k - t_1X^{k-1} - \cdots - t_k.$$

This polynomial is related to the various sequences of isobaric polynomials by the two fundamental theorems of symmetric functions, the first being that the ring of symmetric functions is generated by the elementary symmetric functions and the second, that the coefficients of a monic polynomial are (up to signs) elementary symmetric functions of the roots. A rather striking way of immediately achieving this connection is through the *companion matrix* (CM).

2. The companion matrix and core polynomials. Given the core polynomial, $[t_1, \dots, t_k]$, the companion matrix is:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & t_1 \end{pmatrix}.$$

A useful property of companion matrix A is that, when A operates on A^n by multiplication on the left, say, the result is that all of the rows of A^n are shifted up one row, the first row disappears and the new last row is the result of A operating on the last row of A^n. The upshot is, of course, A^{n+1}. We can make use of this fact by making the following construction. Starting with the $k \times k$-matrix A, we construct a $k \times (k+1)$-matrix whose last row is the result of A operating on the right on the last row vector of A. We repeat this process on the $k \times (k+1)$-matrix just constructed. And so on. The limit of this process is a $k \times \infty$-matrix whose $k \times k$-contiguous blocks constitute the orbit of A operating on itself repeatedly. These blocks are the positive powers of A. If A is non-singular, an analogous process, starting with A using A^{-1} as the operator, produces a new top row each time whose $k \times k$-contiguous blocks are the negative powers of A. This produces a doubly-infinite (top and bottom) matrix with k columns, which we denote A^∞ and call the *infinite companion matrix* (see [**26**] and Chen and Louck [**5**]). We write the orbit matrix A^∞ described above as follows:

$$
\begin{aligned}
A^\infty &= \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
(-1)^{k-1}S_{(-2,1^{k-1})} & \cdots & -S_{(-2,1)} & S_{(-2)} \\
(-1)^{k-1}S_{(-1,1^{k-1})} & \cdots & -S_{(-1,1)} & S_{(-1)} \\
(-1)^{k-1}S_{(0,1^{k-1})} & \cdots & -S_{(0,1)} & S_{(0)} \\
(-1)^{k-1}S_{(1,1^{k-1})} & \cdots & -S_{(1,1)} & S_{(1)} \\
(-1)^{k-1}S_{(2,1^{k-1})} & \cdots & -S_{(2,1)} & S_{(2)} \\
(-1)^{k-1}S_{(3,1^{k-1})} & \cdots & -S_{(3,1)} & S_{(3)} \\
(-1)^{k-1}S_{(4,1^{k-1})} & \cdots & -S_{(4,1)} & S_{(4)} \\
\cdots & \cdots & \cdots &
\end{pmatrix} \\
&= ((-1)^{k-j}S_{(n,1^{k-j})})_{k\times\infty}.
\end{aligned}
$$

Here is a typical $k \times k-$ block in A^∞ when $k = 3$:

$$
A^n = \begin{pmatrix}
S_{(n-2,1^2)} & -S_{(n-2,1)} & S_{(n-2)} \\
S_{(n-1,1^2)} & -S_{(n-1,1)} & S_{(n-1)} \\
S_{(n,1^2)} & -S_{(n,1)} & S_{(n)}
\end{pmatrix}.
$$

Note that

$$
S_{(n-2)} = F_{3,n-2}, S_{(n-1)} = F_{3,n-1}, S_{(n)} = F_{3,n}.
$$

This matrix can be regarded as recording all of the elements of the free-abelian group (an infinite cyclic group) generated by matrix A. A^n

is specifically the $k \times k$-block whose lower right hand element in the representation above is denoted by $S_{(n)} = F_{k,n}$, $n \in \mathbf{Z}$. It turns out that the entries in this matrix are the positive and negatively-indexed Schur-hook polynomials induced by Young diagrams of arm-length n and leg-length $k-j$ (see [**21**, page 2] for this terminology) but appear in this matrix in their isobaric form (what has been called in previous papers, *isobaric reflects* (see [**24, 25**])). However, the idea of having the indexing range over the negative integers appears to be new. So we shall describe how the Jacobi-Trudi formula (see [**21, 25**]) can be used to produce isobaric polynomials in both the well-known case of non-negative indices as well as in the newly introduced negatively indexed case.

Let $(\theta_1, \theta_2, \dots, \theta_r)$ be a partition of n ($\sum \theta_i = n$), listed in weakly descending order (the partition $(1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k})$ written in this fashion would be $(k, \dots, k, \dots, 2, \dots, 2, 1, \dots, 1)$, each j written α_j times; thus, $\sum j\alpha_j = n$. Then the Jacoby-Trudi formula for the Schur polynomial on the isobaric basis induced by this partition is given by the determinant of the $|\alpha| \times |\alpha|$-matrix, where $|\alpha| = \sum \alpha_i$.

$$S(\theta_1, \theta_2, \dots, \theta_k) = \det(F_{\theta_i - i + j}).$$

It is straightforward to check that this is consistent with first computing the Schur functions on the monomial basis and then going to the isobaric basis by way of the mapping $t_j \to (-1)^{j+1} e_j$ above. It is also straightforward to show that, when the Schur polynomials are hook-polynomials, this is consistent with extending the sequences to the negatively indexed polynomials by linear recursion. Perhaps an example would be useful. Consider the Schur-hook polynomial denoted by $S_{(2,1^2)}$, a polynomial of isobaric degree 4.

$$S_{(2,1^2)} = \det \begin{pmatrix} F_2 & F_3 & F_4 \\ F_0 & F_1 & F_2 \\ 0 & F_0 & F_1 \end{pmatrix} = t_1 t_3 + t_4.$$

Note that the monomials in this polynomial have the property that the sum of the product of the exponent and index of the variables of the monomial is equal to the isobaric degree. That is, $\sum j\alpha_j = n$. This is a necessary condition for a symmetric polynomial to be isobaric. It is this fact which associates isobaric polynomials to partitions of the natural numbers and, so, to arithmetical number theory. We refer to the checking of this condition as *isobaric bookkeeping.* The isobaric property is multiplicative, that is, the product of two isobaric polynomials is isobaric, the isobaric degree of the product being the sum of the isobaric degrees of the factors.

An alternate way of deriving this polynomial is to take the *conjugate* partition obtained by reading the partition off of the conjugate Young diagram, obtained by reversing rows and columns of the original diagram. The conjugate partition of $(2, 1^2)$ is the partition $(3, 1)$. Then the Jacoby-Trudi formula written directly in terms of t's gives the computation

$$S_{(3.1)} = \det \begin{pmatrix} t_3 & -t_4 \\ 1 & t_1 \end{pmatrix}.$$

The general expression for computing Schur polynomials directly in terms of the variables is

$$S_{(\theta_1,\theta_2,\dots,\theta_k)} = \det\left((-1)^{\theta_i-i+j+1} t_{\theta_i-i+j}\right),$$

where we define $t_0 = 1$.

It is also useful to remind the reader that the GFPs are Schur-hook polynomials, namely, $F_{k,n} = S_{(n)}$, where it is understood that the partition considered is a partition of n with greatest part k.

The infinite companion matrix carries an extraordinary amount of information. For example, the right-hand column is just the sequence $\{F_{k,n}\}$ of GFPs. Each entry of the matrix gives the isobaric reflect of the Schur-hook polynomial induced by the Young diagram $\pm(n, 1^{k-j})$, the diagram with an arm of length n and a leg of length $k-j$. The negatively indexed symbols represent new *Schur-hook* polynomials whose existence is defined by this matrix. Each column is a doubly-infinite k-degree linear recursion determined by the coefficients $\{t_1, t_2, \dots, t_k\}$ of the core polynomial (note that the columns of the identity matrix contained in A^∞ serve as initial conditions). The sums of the diagonal elements in A^∞, that is, the traces of the elements of the infinite cyclic group generated by A, is just the (k-degree linear recursive) sequence $(\{G_{k,n}\})$, that is, the GLPs (see [**21**, 1.2]). Each row provides coefficients for a vector representation of powers of the roots of the core polynomial in terms of a basis consisting of the first $k-1$ powers of a root λ, namely, $\lambda^0, \dots, \lambda^{k-1}$. We note that, as k increases, the identity matrix inside of A^∞ increases in size and the non-zero terms of the negatively indexed sequences recede. In the limit, there is no non-trivial negatively indexed part to each sequence.

We have mentioned that the columns of the ∞-companion matrix are linear recursions with recursion coefficients $\{t_1, t_2, \dots, t_k\}$. In connection with this fact, we mention the following lemma which will be useful later:

Lemma 1 [**26**, Theorem 2.1]. *A linear recursion is periodic if and only if every root of the core polynomial is a complex root of unity. In particular, if the core polynomial is the cyclotomic polynomial $CP(n)$ of degree $\varphi(n)$, where φ is the Euler totient function, then its associated linear recursion is periodic with period n* (see [**17**]).

On the other hand, every linear recursion is periodic modulo the prime (p) for every rational prime p. The p-period of a linear recursion induced by $[t_1, \dots, t_k]$ satisfies $c_p[t] \leqslant p^k - 1$ [**6**].

Finally, we remark that, while the positively indexed Schur-hook polynomials are induced by partitions of the form $(1^r, n)$, n positive, it is consistent with the notation to regard the negatively indexed Schur-hooks as induced by the "signed" partition $(1^r, n)$ in the sense of George E. Andrews, (see [**1**]).

There are other algebraic structures that can be imposed on the isobaric ring, (see [**25**]). For subsequent use, we want to consider one such, a new product on the elements of the WIP-module, namely, the *convolution product.*

Definition 3. For a fixed k, the *convolution product* of two elements U_n and V_n in the WIP-module is the convolution $U_n * V_n = \sum_{j=0}^{n} U_j V_{n-j}$, where U_n and V_n are n-th terms in the WIP-module.

Remark 1. Taking the convolution product of t_n and, say F_n, we are regarding t_n as the sequence $\{t_j\}_0^n$. For example, the linear recursion given by $F_n - t_1 F_{n-1} - \cdots - t_k F_k$ is just $F_n * -t_n$. $t_j = (-1)^{n-1} S_{(1,1^{n-1})}$ is an entry in A^∞.

It turns out that this gives a graded group structure to the WIP-module. In particular, $-t_n$ is the convolution inverse of $F_{k,n}$, where $t_j = 0$ when $j > k$ and $t_0 = 1$ (recall that $-t_j$, $j = 0, \dots, k$ are the coefficients of the core polynomial). The fact that this is the convolution inverse is a consequence of the statement that $F_{k,n} = \sum_{j=1}^{k} t_j F_{k-j}$, that is, that the GFP-sequence is a kth order linear recurrence.

We have regarded the t_j's as indeterminates so far, and the core polynomial as a generic kth-degree polynomial. That is, we have been operating with polynomials, not polynomial functions; but there are many applications in which it is convenient to evaluate these polynomials over a suitable ring. It is in this context that the names Generalized Fibonacci and Generalized Lucas were chosen. As was pointed out in Section 1, if $k = 2$ and $t_1 = 1 = t_2$, then a GFP is just the Fibonacci sequence, and a GLP is the Lucas sequence. Taking this point of view, it is easy to show that every linearly recursive numerical sequence is contained as a sequence in the WIP-module.

For some purposes, we shall want to choose the evaluation ring to be $\mathbf{Z}$, but other rings will also be useful.

We record here the generating functions for elements in the WIP-module. For example, a generating function for a GFP is given by

$$H(y) = \frac{1}{1 - p(y)}, \qquad p(y) = t_1 y + \cdots + t_k y^k,$$

where $p(y)$ is the generating function for the convolution inverse of the GFP.

Recall from Section 1 that sequences in the WIP-module are defined by weight vectors ω. For an arbitrary sequence $\{P_{\omega,k,n}\}$ in the WIP-module, we have the generating function $\Omega(y) = \sum_{n \geqslant 0} P_{\omega,n} y^n$ given in closed form by

Proposition 1.

$$\Omega(y) = 1 + \frac{\omega_1 t_1 y + \omega_2 t_2 y^2 + \cdots + \omega_k t_k y^k}{1 - p(y)},$$

where $P_{\omega,0} = 1$, *(see* [**25**]).

In [**26**] we studied the sequences in the WIP-module with respect to periodicity and periodicity modulo a prime (a linear recursion is periodic if and only if every root of the core polynomial is a root of unity; on the other hand, every linear recursion is periodic modulo p for every prime p [**26**, Theorems 2.1 and 2.2]). If $c_p[t_1, \dots, t_k]$ denotes the period of $F_{k,n}(t_1, \dots, t_k)$ modulo p, then every sequence in the WIP-module has a period $c_p[t_1, \dots, t_k]$, and so $c_p[t_1, \dots, t_k]$ can be regarded as an invariant of the core polynomial $\mathcal{C}(X)(t_1, \dots, t_k)$. (Letting $p = 1$ takes care of the case covered by Theorem 2.1). The fact that every sequence in the WIP-module has the same p-period has consequences for other structures derived from the same core polynomial. This leads to results concerning the number fields obtained as quotients by an irreducible core polynomial, discussed in [**26**]. Another such application occurs in the ring of arithmetic functions, which we turn to now.

3. The ring of arithmetic functions. While the isobaric ring is a not-so-classical version of the well-known ring of symmetric functions, the elements in the ring of arithmetic functions have long been objects of study, though not usually from a structural point of view (but see [**4, 29, 30, 33**]) and, recently, Laohakosol and Pabhapote [**16**] and Laohakosol, Pabhapote and Wechwiriyakul [**17, 18**], Haukkanen [**8–14**]. It is possible that the relation between the two structures is implicitly well understood, but it is rather surprising that the relationship, to our knowledge, has not been made explicit in the literature. The connection is that the GFP's with the convolution product is locally isomorphic to the group of multiplicative functions under the convolution product, and by consequence, every sequence in the WIP-module yields a group of multiplicative functions that can be associated locally with a given multiplicative function. So now we shall review the facts about arithmetic functions that we need in order to show this connection (see, e.g., [**27**] or [**35**]).

We recall that arithmetic functions ($\mathcal{A}$) are functions from $\mathbf{N}$ to $\mathbf{C}$ and form a ring under the usual sum and product rule for functions. It is also usual to add convolution as an additional operation: $(\alpha * \beta)(n) = \sum_{d|n} \alpha(d)\beta(n/d)$. Call this structure in which the convolution product substitutes for the standard product of functions, $\mathcal{A}^*$. Then not only is $\mathcal{A}^*$ a ring, but it is a unique factorization domain (see [**4**]). An arithmetic function α is invertible with respect to convolution if and only if $\alpha(1) \neq 0$. α is a *multiplicative function* (MF) if and only if $\alpha(mn) = \alpha(m) * \alpha(n)$ whenever $(m, n) = 1$. Thus the MF's are just those AF's that are uniquely determined at powers of primes. Every non-zero MF has value 1 at 1 and is therefore invertible. From now on, we shall exclude the zero function from MF so that MF belongs to the group of units of $\mathcal{A}^*$. Denote the group of units in $\mathcal{A}^*$ by $\mathcal{M}$. Since multiplicative functions are determined locally, $\mathcal{M}$ is the direct sum of its local subgroups, $\mathcal{M}_p$.

An MF α is *completely multiplicative* if $\alpha(mn) = \alpha(m)\alpha(n)$ for all $m, n \in \mathbf{N}$. Let $\mathcal{L}$ be the subgroup of $\mathcal{M}$ generated by the CM functions. $\mathcal{L}$ is known to be a free abelian group (see [**22**]). We also have that $\mathcal{L}$ is the direct sum of its local groups $\mathcal{L}_p$. We shall often drop the index p in what follows when the context is clear.

A multiplicative function is called *positive* if it is the convolution product of CM functions, *negative* if it is the product of the inverses of CM functions and *mixed* if it is the convolution product of at least one non-identity CM function and one negative of a non-identity CM function and is in $\mathcal{L}$. (In Carroll and Gioia (see [**3**]), these are called *rational functions*).

If α is a positive element in $\mathcal{L}$, then α and its inverses are determined at each prime, that is, *locally*, by a monic polynomial of degree k, where k depends on the prime p and the number of CM factors of α. Call this polynomial the *local core* of the MF, denoted by $\mathrm{C}_p(X)$. It is, in fact, a generating function for the negative elements in $\mathcal{L}$. We write this polynomial as

$$X^k - t_1 X^{k-1} - \cdots - t_k$$

in the non-mixed case. Since it will turn out that MFs are locally recursive, this polynomial will also determine positive elements in $\mathcal{L}$. We can also classify functions in MF as being one of the four following *types* depending upon the sizes of their ranges and domains: (fin, fin), (∞, fin), (fin, ∞), (∞, ∞), where the notation here means the pair $(range, domain)$. The domain of such a function is the set of coefficients of the core polynomial

The first type are those MF's which have both range and domain finite; the second type, those that have an infinite range and a finite domain; the third type, those with a finite range and an infinite domain; and the fourth type has both range and domain infinite–*finite range* and *finite domain* mean that eventually all values are zero. In Theorem 7 and Corollary 10 we show that the set of type (1) functions contains only the identity, type (2) functions are just the *positive* MF's, type (3) the negative MF's and type(4) are *mixed*. The mixed type lead to power series generating functions as core 'polynomials.'

4. Relation between multiplicative functions and the WIP-module. Since a given core polynomial determines both the WIP-module (in particular, the infinite companion matrix) and determines a particular arithmetic function locally in $\mathcal{M}$, it is clear that there is a strong connection between the MF's and the WIP sequences. In fact, the GFP-sequence evaluated at the vector $\mathbf{t}$ is a (non-trivial) MF. So is the positively indexed part of every column in the matrix A^{∞}. Thus, every sequence in the WIP-module is by consequence also in MF. There are instances of MF's for which the core polynomial remains the same for all choices of the prime p (the MF τ, of degree 2 which counts the number of divisors of n, has the core polynomial (X^2-2X+1) for all p). There are also instances where the core polynomial has the same degree over all primes and the coefficients are given by the same functions of p (the MF σ of degree 2 which records the sum of the divisors of n where the core polynomial is given by $X^2-(p+1)X+p$). The first case can be regarded as a special case of the second.

We note that convolution preserves isobaric degree in the WIP-module and core degree in both the isobaric and the $\mathcal{L}$ cases. The analogue of *isobaric degree* for the multiplicative functions is just *power of the prime* p. The analogue of a function in $\mathcal{L}$ requiring infinitely many powers of the prime in its definition is that k is unbounded in the WIP-module; that is, that (t_j) is different from 0 for infinitely many j's or, equivalently, that the rows of the companion matrix are unbounded on the left. If we call the MF's, which are locally of degree k for all primes p, *k-uniform*, and the set of all k-uniform for all k, *uniform*, then it is easily seen that the uniform MF's form a graded group under convolution. At each level of the grading and for each prime p, the core polynomial induces a (cyclic) direct summand (the values of the MF at the powers of the prime p) and, at the same time, on the induced GFP. That is, for each k the subgroup at that level is a direct sum of cyclic subgroups, one such subgroup for each p. We refer to these subgroups as the local subgroups of degree k. These subgroups all have the same generic core, i.e., the cores of the elements in the subgroup are evaluations of the same generic polynomial.

So far we have glossed over the point that the output of the functions $F_{k,n}$ depends upon the choice of domain ring. It is clear that the output of such evaluations will always be elements in the same ring as that of the input; for example, an input of integers yields an output of integers. Integer inputs will often be used in the examples simply because so many classical MF's are of that sort. In general, if the evaluation ring is $\mathbf{R}$ (a subring of the complex numbers), we denote the subgroup in $\mathcal{M}$ that they generate, $\mathcal{M}_{\mathbf{R}}$.

With these remarks, we state the main theorem, the Correspondence Theorem.

Theorem 1. *Given a prime p, let $\chi \in \mathcal{M}$ with local core $C(X)$, k finite or infinite, and let $F_k(\mathbf{t})$ be the sequence of* GFP*'s induced by this core, then $F_{k,n}(\mathbf{t}) = \chi(p^n)$.*

Proof. As pointed out above, for each integer k, every polynomial in the WIP-module determined by k (i.e., whose companion matrix is $\infty \times k$) is a member of a k-linear recursive sequence; in particular, the GFP-sequence is one such sequence. Thus, $F_{k,n} = \sum_{j=0}^{k} t_j F_{k,k-j}$, $F_{k,0} = 1$, where $t_j, j = 1, \dots, k$ is the set of parameters that determine the recursion. Thus, every linear recursion of degree k is determined by choosing a set of values for t_j (see [**26**]). (k can be finite or infinite). It is clear such a choice of parameters determines a multiplicative arithmetic function locally–the recursive relation determines the convolution product.

The converse is also true; that is, each multiplicative function χ has a locally faithful representation as an evaluation of $F(t_1, \dots, t_j, \dots)$ in the GFP-sequence. For, given a prime p and the set of values $\chi(p^n) = a_n$, we can determine the t_j and $F_{k,j}(t_1, \dots, t_j, \dots)$ inductively in such a way that $F_{k,j}(t_1, \dots, t_j) = \chi(p^j), j \leqslant k$. Let $\chi(p^j) = a_j$, $j = 1, 2, \dots$, and let $F_{k,0} = a_0 = 1$. Let $t_1 = a_1 = F_{1,1}$. Suppose that t_j for $j < n+1$ has been defined and that $a_j = F_{k,j} < n+1$. We define t_{n+1} by

$$t_{n+1} = a_n - \sum_{j=1}^{n} t_j a_{j-1}.$$

That is,

$$t_{n+1} = \chi(p^n) - \sum_{j=1}^{n-1} t_j \chi(p^{j-1})$$

(cf. Proposition 4)

$$= F_{n+1,n+1} - \sum_{j=1}^{n} t_j F_{n,n-j+1}.$$

The theorem now follows by the recursive property of the GFP sequence and induction. $\square$

We shall use the notation $\chi \leftrightarrow F$ to mean that χ is the MF that corresponds to F in the sense of Theorem 1. The last equation in the proof reflects the fact that $F_{k+1,k+1} - F_{k,k+1} = t_{k+1}$. This is an example of the *conservation principle* referred to in Section 1 applied to the GFP sequence. The very fact that the construction in the proof of Theorem 1 is possible guarantees, by the way, that every multiplicative function is recursive.

A useful way of formulating the content of Theorem 1 is that each core polynomial of degree k induces k columns of linear recursions which can be taken as the k generators of a $\mathbf{Z}$-module of linear recursions, each of which is locally a multiplicative arithmetic function induced by the coefficients of the core polynomial. The generators are produced by taking the successive powers of the companion matrix associated with the core polynomial (producing the $k \times \infty$ infinite companion matrix).

An immediate consequence of Lemma 1 and the Correspondence Theorem is the following fact about multiplicative functions:

Theorem 2. *A multiplicative function is periodic if and only if every root of the core polynomial is a complex root of unity. On the other hand, every multiplicative function is periodic modulo the prime* (p) *for every rational prime* p*. The* p*-period of the multiplicative function satisfies* $c_p[t] \leqslant p_k$.

On the other hand, it is a simple consequence of the pigeonhole principle that every multiplicative function is locally periodic at each prime p.

Proposition 2 [**7**]. *If the core polynomial is of degree k, then*

$$c_p[t] = c_p[t_1, \dots , t_k] \leqslant p^k - 1.$$

5. Examples.

Example 1. Consider the multiplicative functions τ and σ, where τ is the function on $\mathbf{N}$ which counts distinct divisors of $n \in \mathbf{N}$, while σ is the divisor sum function, that is, $\sigma(n) = \sum_{d|n} d$. Both of these functions are multiplicative of degree 2, $\tau(p^n) = n + 1$ and $\sigma(p^n) = 1 + p + \cdots + p^n$. We can find the local core polynomial for τ by noting that $\tau(p) = 2$, $\tau(p^2) = 3$ and $\tau(p^3) = 4$. Using the induced GFP, $F_1 = t_1$, we have that $t_1 = \tau(p^1) = 2$. Then we note that $F_2 = t_1^2 + t_2 = \tau(p^2) = 3$ and we deduce that $t_2 = -1$. A similar computation for t_3 yields $t_3 = 0$. An induction using the recursive properties of the GFP sequence shows that $t_j = 0$ for all $j > 2$. Thus, the local core is the quadratic polynomial $X^2 - 2X + 1$, which incidentally, shows the well-known result that τ is the convolution product of two copies of ζ, where $\zeta(n) = 1$ for all values of n. ζ is a completely multiplicative function.

Example 2. If we carry out the same procedure for σ, we find that $t_1 = 1+p$, that $t_2 = -p$ and that the degree of σ is 2, that is, that $t_j = 0$ for $j > 2$, hence, the local core is $X^2 - (p + 1)X + p$. Again, since the local core has linear factors $X - p, X - 1$, σ is the convolution product $\zeta_1 * \zeta$ of two CM arithmetic functions, i.e., two degree 1 functions, where $\zeta_k(p^n) = p^{nk}$. Degree 2 uniform MF's are also called *specially multiplicative*. These are both examples of uniform MF's, i.e., they are elements of $\mathcal{L}$.

Definition 4. A multiplicative function has *valence* $\langle r, s\rangle$ if it is a convolution product of r completely multiplicative functions and s inverses of completely multiplicative functions.

Example 3. This same procedure applied to the Euler totient function ϕ shows that $t_j \neq 0$ for all $j > 0$. Thus, ϕ is uniform and is an example of a MF whose core is a power series. Its values are given by $F_{k,n}(t_1, \dots, t_k, \dots)$, $t_j = p - 1$ for all $j > 0$ and all $k > 0$ and all primes p. It is well known that $\phi = \zeta_1 * \mu$, where μ is the convolution inverse of ζ, i.e., $\phi = \zeta_1 * \zeta^{-1}$, which is called in [**16**] a function of valence $\langle 1, 1\rangle$ (see Definition 4) and type (∞, ∞).

This example leads to an interesting theorem:

Theorem 3. *Let $\alpha = \beta * \gamma^{-1}$ where β and γ are* CM*, $\beta \neq \gamma$, that is, α is of valence $\langle 1, 1\rangle$, then α is of type (inf, inf). That is, α has an infinite range and infinite domain*; *thus, α has an infinite core.*

Proof. We combine the techniques of calculation used before with the generating functions from Proposition 1 and the remarks just above it. Represent α, β, γ by F, F', F'', respectively. Letting the parameters for the two CM functions β, γ be given as $t'_1 \neq 0$, $t_j = 0$ otherwise, and $t''_1 \neq 0$, $t_j = 0$ otherwise; then the local core polynomials are $X - t'_1$, $X - t''_1$. We have that the generating function for the convolution product in terms of the parameters of the factors is

$$\frac{1 - t''_1 y}{1 - t'_1 y} = 1 + \sum_{n=1}^{\infty} (t'^n_1 - t'^{n-1}_1 t''_1) y^n = \sum_{n=0}^{\infty} F_n y^n.$$

Thus,

$$F_n = (t'^n_1 - t'^{n-1}_1 t''_1).$$

Using these values for F_n, the calculating methods employed above, and induction, we can deduce that

$$t_n = (-t''^n_1 + t'_1 t''^{n-1}_1).$$

But, since $t'_1 \neq 0$ and $t''_1 \neq 0$, this shows that both the range and domain of F_n are infinite, that is, α is of type (inf, inf). $\square$

Thus, the case for ϕ generalizes. It seems reasonable to conjecture that $\langle r, r\rangle$-functions have infinite cores.

It is also instructive to look at the convolution product $\tau * \sigma = \alpha$, a positive function. A calculation of the nature of those above shows that the parameters for $\tau * \sigma = \alpha$ are $u_1 = p+3$, $u_2 = -3(p+1)$, $u_3 = 4p+1$, $u_4 = p$, all non-zero, and $u_j = 0, j > 4$. Hence, the product is of degree 4 and the local core is $X^4-(p+3)X^3-3(p+1)X^2-(4p+1)X-p$, which is just the product of the local cores of the two factors. This suggests that we make the following definition:

Definition 4. For any MF, we define its *degree* to be k if the parameters t_j of the function have the property that $t_k \neq 0$ and $t_j = 0$, $j > k$, where k is either finite or infinite.

Corollary 4. *The positive part of each column in the infinite companion matrix is also an* MF*, hence in* $\mathcal{M}$*.*

Since the GFPs with non-negative indexes are now understood as corresponding to the elements of MF, and since the properties of the GFPs naturally extend the range of the MFs to negative powers of the prime p, it seems reasonable to extend group $\mathcal{L}$ to a larger group $\mathcal{L}^*$ to reflect this fact. Moreover, the negative part of each column in the infinite companion matrix is also in MF. So we have the following situation. The core polynomial determines a *principal* MF, the one determined in Theorem 1 by the GFP, and at the same time (a module of) *induced* MF's, those determined by all of the sequences in the WIP-module.

The fact that each column in the infinite companion matrix is a k-linear recursion is reflected in the fact that the *induced* MF's are determined locally by the first k powers of the prime, while the rest of the sequence is determined by linear recursion, the recursion constants being given by the vector $\mathbf{t}$.

In [**25**] it was shown that, in the case that the local core is irreducible, there is a strong relation between the WIP-module and the number fields associated with the field extension determined by the local core. This fact gives a three-way relation among the three structures: WIP-module, multiplicative arithmetic functions and number fields. In particular, it associates with every such number field, a special set of multiplicative functions.

Another question arises from the fact that the UFD $\mathcal{A}^*$ has a rich ideal structure [**2, 33**]. Is there a representation of these ideals in terms of symmetric polynomials?

6. Specially multiplicative arithmetic functions.

Remark 2. The main point in this section is that there is nothing special about specially multiplicative functions. Expressions of the sort contained in the McCarthy theorem discussed below are true for every multiplicative function and merely represent the fact that these functions are degree k recursive. The $B(p)$-term in the degree 2 case (that is, $-t_2$) can be replaced by $-t_2, \dots, -t_k$ in the degree k-case. The representation of B in terms of the original function generalizes to the representation $t_2, \dots, t_k$ as a function of its associated F-polynomials as indicated in Proposition 5 above.

McCarthy's theorem (see P.J. McCarthy [**27**]) states that a multiplicative function χ is specially multiplicative, that is, it is of degree 2 if and only if, for each prime p,

$$\chi(p^{n+1}) = \chi(p)\chi(p^n) - \chi(p^{n-1})B(p),$$

where $B(p) = \chi(p)^2 - \chi(p^2)$ and $B(p) \in CM$. Furthermore, degree 2 multiplicative arithmetic functions are characterized by the property that they admit a Busche-Ramanujan identity (see [**28**]).

Using Theorem 1 to translate these results into isobaric form, we get as a characterization of degree 2 MF's the following:

$$F_{2,n+1}(t_1,t_2) = t_1F_{2,n}(t_1,t_2) + t_2F_{2,n-1}(t_1,t_2),$$

or more succinctly,

$$F_{n+1} = t_1F_n + t_2F_{n-1}.$$

In particular, $B_\chi = B(p) = \chi(p)^2 - \chi(p^2)$ translates into $-t_2 = F_1^2 - F_2$. Indeed, using the main theorem of this paper, this is just the redundant statement that degree 2 cores induce linear recursions of degree 2.

It is also asserted in the McCarthy theorem that $B(p)$ is a completely multiplicative arithmetic function, that is, it has degree 1. This is a rather peculiar claim. Is this intended to be a definition of function B or is this deemed to be a consequence of its role in the theorem? If the latter, then we should be able to show that representing B by $B(u_1, \dots, u_j)$ has the consequence that $u_j = 0$ for $j > 1$. Under some rather natural assumptions leading to a definition of this function, this is not the case. The linear recursion for χ does define $B(p) = B_1$. But $B(p^n)$ is not determined by the linear recursion when $n > 1$. The proof that B is CM in [**27**] and again in [**35**] appear to define B arbitrarily for higher values of p as CM, that is, that $B(p^n) = B^n(p)$ for all $n \in \mathbf{N}$.

The Busche-Ramanujan identities for the specially multiplicative functions σ_k are the two statements:

$$\sigma_k(mn) = \sum_{d|(m.n)} \sigma_k\left(\frac{m}{d}\right)\sigma_k\left(\frac{n}{d}\right)\mu(d)\, d^k \tag{1}$$

$$\sigma_k(m)\sigma_k(n) = \sum_{d|(m.n)} d^k\sigma_k\left(\frac{mn}{d^2}\right) \tag{2}$$

where $\sigma_k(n) = \sum_{d|n} d^k$, $\sigma_1 = \sigma$, $\sigma_k = \zeta_{k-1} * \zeta$.

We translate this into isobaric notation. For simplicity, we take the case where $k = 1$, that is, $\sigma_1 = \sigma$. So we are interested in the identities:

$$\sigma(mn) = \sum_{d|(m.n)} \sigma\left(\frac{m}{d}\right)\sigma\left(\frac{n}{d}\right)\mu(d)\, d^k \tag{1'}$$

$$\sigma(m)\sigma(n) = \sum_{d|(m.n)} d\sigma\left(\frac{mn}{d^2}\right) \tag{2'}$$

Letting $t_1 = 1 + p$, $t_2 = -p$, the core coefficients for σ, and using Theorem 1, that is, that $F_n(t_1, t_2) = \sigma(p^n)$ since σ is an MF of degree 2, and letting $m = p^r$, $n = p^s$, $r \leqslant s$, the two identities become

Proposition 3 (Busche-Ramanujan) (see [**27, 34**]).

$$F_{2,r+s} = F_{2,r}F_{2,s} + t_2F_{2,r-1}F_{2,s-1} \tag{4.1}$$

$$\begin{aligned} F_{2,r}F_{2,s} = F_{2,r+s} - t_2F_{2,r+s-2} + \cdots \\ + (-t_2)^jF_{2,r+s-2j} + \cdots + (-t_2)^rF_{2,s-r} \end{aligned} \tag{4.2}$$

where the degree of the core is 2, *and* $j = 1, \dots, r$.

It is perhaps instructive to give a proof of these well-known relations in terms of the isobaric representation of MF's being discussed in this paper.

Proof. We consider the first of these identities. Omitting the degree index $k = 2$ on the GFP-symbols, and noting that (4.1) is true when $r = 1$, we have the basis for an induction. But

$$\begin{aligned}
F_{r+s} &= t_1 F_{r+s-1} + t_2 F_{r+s-2} \\
&= t_1 F_{(r-1)+s} + t_2 F_{(r-1)+(s-1)} \\
&= t_1(F_{r-1}F_s + t_2 F_{r-2}F_{s-1}) + t_2 F_{r-1}F_{s-1} + t_2^2 F_{r-2}F_{s-2} \\
&= t_2(t_1 F_{r-2}F_{s-1} + t_2 F_{r-2}F_{s-2}) + t_1 F_{r-1}F_s + t_2 F_{r-1}F_{s-1} \\
&= t_2(F_{r-2}F_s) + t_1 F_{r-1}F_s + t_2 F_{r-1}F_{s-1} \\
&= F_s(t_1 F_{r-1} + t_2 F_{r-2}) + t_2 F_{r-1}F_{s-1} \\
&= F_s F_r + t_2 F_{r-1}F_{s-1},
\end{aligned}$$

using only linear recursion and the induction hypothesis. This proves the first of the two identities.

We prove the following lemma from which, along with (4.1), (4.2) will follow.

Lemma 4.2.

$$t_2 F_{2,r-1}F_{2,s-1} + \sum_{j=1}^{r}(-t_2)^j F_{2,r+s-2j} = 0, \quad r \leqslant s.$$

Proof. We observe that the identity of the lemma holds when $\langle r, s\rangle = \langle 1, 1\rangle$. Suppose that it also holds for $2 < r+s < n$. Then, using the linear recursion property of the $F-$sequence, we have, for $k = 2$,

$$\begin{aligned}
&t_2(t_1 F_{2,r-1}F_{2,s-2} + t_2 F_{2,r-1}F_{2,s-3}) \\
&\qquad\qquad + \sum_{j=1}^{r}(-t_2)^j(t_1 F_{2,r+s-2j-1} + (-t_2)^j F_{2,r+s-2j-2}) \\
&\qquad = \left(t_2(t_1 F_{2,r-1}F_{2,s-2}) + \sum_{j=1}^{r}(-t_2)^j t_1 F_{2,r+s-2j-1}\right)
\end{aligned}$$

$$+\left(t_2(t_2F_{2,r-1}F_{2,s-3})+\sum_{j=1}^{r}(-t_2)^jF_{2,r+s-2j-2}\right)$$
$$=0. \qquad \square$$

It is clear from the proofs of the proposition and Lemma 2 that the only assumption made was that we are dealing with degree 2 multiplicative functions, so the results do indeed hold for all degree 2 MF's; moreover, it is hardly surprising that they do not hold for higher degree MF's, since our very assumption is that our linear recursions and hence our core is of degree 2, that $t_j = 0$, $j > 2$ (also see [**35**, page 282]).

Can the McCarthy characterization of specially multiplicative functions, that is, degree 2 functions, be generalized to finite higher degree functions? The B-function in McCarthy's theorem is just $-t_2$ at p, and the relation itself is just a statement of the linear recursive property of the F-functions that represent the multiplicative functions (see (4.5)). If we think of t_2 as the isobaric degree 2 term in the specially multiplicative case, then it is natural to think of t_k as the isobaric degree k term in the general case where the core polynomial has degree k. The analogue to the fact that $B(p) = F_1^2 - F_2 = -t_2$ is the following proposition which expresses the indeterminates, t_j in terms of the generalized Fibonacci polynomials, $F_{k,n}$.

Proposition 4. *Consider the* GFP $F_n(t_1,\dots,t_k)$, *and make the substitution* $t_j = (-1)^{j+1}F_j$. *Then* $t_n = F_n(F_1,\dots,(-1)^{j+1}F_j,\dots,(-1)^{n+1}F_n)$, $j = 1,2,\dots,n$.

Proof. $t_n = (-1)^{n+1}S_{(1,1^{n-1})}$ where $S_{(1,1^{n-1})}$ is the Schur-hook polynomial determined by the Young diagram with arm length 1 and leg length $n-1$, i.e., a vertical strip of length n, written on the isobaric basis. Using the Jacobi-Trudi formula applied to the GFPs, we have

$$S_{(1,1^{n-1})} = \det\begin{pmatrix} F_1 & F_2 & \cdots & F_n \\ 1 & F_1 & \cdots & F_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_1 \end{pmatrix} = (-1)^{n-1}t_n. \qquad \square$$

Here are some examples:

1. $t_1 = F_1$,
2. $-t_2 = F_1^2 - F_2$,
3. $t_3 = F_1^3 - 2F_1F_2 + F_3$. $\square$

Proposition 3 expresses the Busche-Ramanujan identities in terms of the GFP-representation, which in turn suggests a way of generalizing such identities to MF's of higher degree. Thus, one way to think of the Busche-Ramanujan identities is as an expression of F_{r+s} in terms of F_r and F_s together with a *remainder* term. In Theorem 11 near the end of the next section, Section 7, we have just such a generalization, which has the pleasant property of involving Schur-hook functions as coefficients.

7. Structure of the convolution group $\mathcal{M}$ of multiplicative functions revisited. Group $\mathcal{L}$ generated by the completely multiplicative functions, sometimes called the group of rational functions (e.g., see [**16, 32**]) contains elements of four kinds: the identity, positive elements (the semi-group generated by CM functions), negative functions (the inverses of the positive functions) and mixed elements (those which are convolution products of both positive and negative elements) as discussed in Section 2. Each element has a degree which is either infinite or a non-negative integer. The identity has degree 0, a positive element has positive degree, the degree of its core polynomial, or equivalently, the number of CM functions of which it is a product. (In [**22**] it is shown that the CM functions freely generate a free abelian group). Both negative and mixed functions have infinite degrees. Negative functions have power series cores, and a mixed function has a rational function for a core whose numerator is the core of the positive part and whose denominator is the core of the negative part.

In Section 2, the classification of elements of types (fin, fin), (∞, fin), (fin, ∞), (∞, ∞) in the group $\mathcal{M}$ was introduced, where *finite* range or domain means that the sequence is eventually constantly zero. Infinite means that the sequence has infinitely many non-zero values.

Clearly, the types are mutually exclusive; moreover, each type is non-empty. For example, the identity function is type 1. Type 2 consists of the positive functions, and type 3, the negative functions. Completely multiplicative functions, e.g., ζ, or any specially multiplicative function is of type 2, e.g., σ, while μ, a negative function, is of type 3, and, as we shall see, the Euler totient function, ϕ, is of type 4.

We now regard the $\{t_j\}$ as a set of values so that F is a particular numerical sequence $F(\mathbf{t})$, and, hence, a particular MF.

Proposition 5. *Let $\alpha \in MF$, let $\alpha \leftrightarrow F$ and let $\{t_j\}$ be the set of parameters for F. Let $\{s_j\}$ be the set of parameters for F^{-1} which represents α^{-1}. Then, for all j,*

$$F_j = -s_j, \qquad F_j^{-1} = -t_j.$$

Proof. $0 = F_1 * F_1^{-1} = F_1 + F_1^{-1} = t_1 + s_1$, so $t_1 = -s_1$; therefore,

$$F_1 = -s_1, \quad \text{and} \quad F_1^{-1} = -t_1.$$

So assume inductively that $F_j = -s_j$, $j = 2, \dots, n-1$ and that $F_j^{-1} = -t_j$, $j = 2, \dots, n-1$. Then we have that

$$F_n = \sum_{j=1}^{n-1} t_j F_{n-j} + t_n = -\sum_{j=1}^{n-1} t_j s_{n-j} + t_n.$$

Similarly,

$$F_n^{-1} = -\sum_{j=1}^{n-1} t_j s_{n-j} + s_n;$$

therefore,

$$F_n - F_n^{-1} = t_n - s_n.$$

Also,

$$\begin{aligned} F_n * F_n^{-1} &= F_n + \sum_{j=1}^{n-1} F_{n-j} F_j^{-1} + F_n^{-1} \\ &= F_n - \sum_{j=1}^{n-1} s_{n-j} t_j^{-1} + F_n^{-1} = 0. \end{aligned}$$

Thus,

$$F_n + F_n^{-1} = -\sum_{j=1}^{n-1} t_j s_{n-j} = F_n - t_n = F_n^{-1} - s_n.$$

So

$$2F_n = F_n - s_n$$

implies

$$F_n = -s_n,$$

and similarly,

$$F_n^{-1} = -t_n. \qquad \square$$

Corollary 5. *Let $F = F' * F''$ with t'_j, t''_j being the parameters of F', F'' and s'_j, s''_j the values of F'', F'', then*

$$-s_n = -s'_n + \sum_{j=1}^{n-1} s'_{n-j} s''_j - s''_n, \tag{1}$$

$$-t_n = -t'_n + \sum_{j=1}^{n-1} t'_{n-j} t''_j - t''_n, \tag{2}$$

equivalently,

$$F_n = -s'_n + \sum_{j=1}^{n-1} s'_{n-j} s''_j - s''_n; \tag{1'}$$

and

$$t_n = t'_n - \sum_{j=1}^{n-1} t'_{n-j} t''_j + t''_n. \tag{2'}$$

Proof. Expand the convolution product $F_n = \sum_{j=0}^{n} F'_{n-j} F''_j$ and apply the theorem to the factors. $\square$

Theorem 6.

$$\operatorname{core}(\alpha_1 * \alpha_2) = \operatorname{core}(\alpha_1)\operatorname{core}(\alpha_2) \quad \square \tag{3.1}$$

$$\deg(\alpha_1 * \alpha_2) = \deg(\alpha_1) + \deg(\alpha_2). \quad \square \tag{3.2}$$

Proof. From Corollary 5 we have

$$-t_n = -t'_n + \sum_{j=1}^{n-1} t'_{n-j} t''_j - t''_n.$$

But the $\{-t_j\}$ are just coefficients of the core polynomial of the convolution product $\alpha_1 * \alpha_2$; while $\{-t'_j\}$ and $\{-t''_j\}$ are, respectively, coefficients of the core polynomials of α_1 and of α_2. Thus, Corollary 5 is, at the same time, the formula for the coefficients of the core polynomial of the convolution product and of the coefficients of the product of the core polynomials $\operatorname{core}(\alpha_1)$ and $\operatorname{core}(\alpha_2)$. This proves (3.1). Equation (3.2) follows directly from (3.1). $\square$

Definition 6. A convolution product of $r + s$ factors is said to be in *normal form* if there are r degree 1 (i.e., completely multiplicative) factors and s inverses of degree 1 factors, and if no two factors in the product are mutually inverse to one another. By commutativity we can always write the positive (degree 1) factors first.

Theorem 7. *Let* $\alpha \in \mathbf{MF}$

(1) *α is type* $1 = (fin, fin)$ *if and only if it is the identity.*

(2) *If α is positive, it is type* 2, (∞, fin); *if α is negative, it is type* 3.

Proof. (1) Suppose that α and α^{-1} have a finite range, i.e., both $\alpha(p^n)$ and $\alpha^{-1}(p^n)$ have value zero for all values of $n > s > 0$. Let F represent α and F' represent α^{-1}. Suppose that t_m and F_n are the largest non-zero values of F and $-F'$, i.e., suppose that $t_m \neq 0$ and $F_n \neq 0$, but that $t_j = 0$ and $t_i = 0$ whenever $j > m > 0$ and $i > n > 0$, then consider the following equation resulting from the linear recursion property of the GFPs.

$$F_{m+n} = t_1 F_{m+n-1} \cdots + t_m F_n + t_{m+1} F_{n-1} + \cdots + t_{m+n} F_0.$$

Observe that the only term that survives on the right-hand side of the equation is $t_m F_n$, while the left-hand side is 0. So either t_m or F_m is not equal 0, a contradiction to which the convolution identity function is exempt. Thus, we have four mutually exclusive types, type 1 containing only the identity. (2) claims that functions with valence $\langle r, 0\rangle$ have infinite ranges and finite domains; while functions with valence $\langle 0, s\rangle$ have finite ranges and infinite domains. But these two propositions follow from the fact that we have defined functions to be positive if they are a product of CM functions and negative if they are the inverses of a product of CM functions, from Theorem 6, and from the fact that the core polynomials determine the number of parameters of a function. $\square$

Remark 3. We shall show that ϕ is of type 4. To show that ϕ is of type 4, we must show that both ϕ and ϕ^{-1} have infinite non-zero range. But, for any prime p and all n, $\phi(p^n) = p^n - p^{n-1} \neq 0$. ϕ^{-1} is also infinite: its parameters are $t_j = -(p-1)$ for all n. So let us suppose that $t_j = p - 1$ for $0 < j < n$. Then, by the recursive property of GFP-sequence and induction, we have that $F_n = t_1F_{n-1} + t_2F_{n-2} + \cdots + t_{n-1}F_1 + t_n$, that is,

(1)
$$t_n$$
(2)
$$= F_n - (t_1F_{n-1} + t_2F_{n-2} + \cdots + t_{n-1}F_1 + t_n)$$
(3)
$$= p^n - p^{n-1} - (p-1)(p^{n-1} - p^{n-2} + p^{n-2} + \cdots + (p-1))$$
(4)
$$= p^n - p^{n-1} - (p-1)(p^{n-1} - 1)$$
(5)
$$= p - 1.$$

Thus, we have shown that $t_j = p - 1 \neq 0$ for all j, and so $\overline{F}_j = -t_j \neq 0$ for all j, and thus that $\phi \in$ type 4. $\square$

We can also prove this by using the fact that $\phi = \zeta_1 * \mu$. Represent ζ_1 by F' and μ by F''; then by Corollary 5, $t'_1 = p$, $t'_j = 0$, $j > 1$; $s'_n = -p^n$; $s''_n = -p$, when $n = 1$ and 0 otherwise, and $t''_n = 1$ for all values of n. We can symbolize the product by $(\infty, finite) * (finite, \infty)$.

This property of ϕ turns out to be true for all totient functions, i.e., functions of valence $\langle 1, 1\rangle$.

Proposition 6. *$\alpha = \beta * \gamma^{-1}$ where β, γ are degree 1 functions, α is not the identity function. Then α is type (∞, ∞) and degree ∞.*

Proof. We represent α, β and γ by, respectively, F, F' and F'', with t_j, t'_j, t''_j; s_j, s'_j, s''_j as in Corollary 5. We observe that $F'_n = t_1'^n$ and $F''_n = t_1''^n$, and that neither of these values is 0. We also have that $F_n = t_1'^n - t_1'^{n-1}t''_1$, since $t''_j = 0, j > 1$, and $F_n = 0$ implies $t'_1 = t''_1$, contradicting the hypothesis; hence, the range of α is infinite. If we apply similar reasoning to F_n^{-1}, we find that it also has infinite range; thus, α is of type 4 and has infinite degree. $\square$

In [**16**, Corollary 2.4] 2005, Laohakosol and Pabhapote discussed Busche-Ramanujan identities and the Kesava Menon norm. We shall discuss the Kesava Menon norm in the next section. Now we wish to look at their theorem extending Busche-Ramanujan identities to multiplicative functions of mixed type, which we reproduce here.

Corollary 8. *Let $\chi \in MF$. Then the following hold.*

(i) *χ has valence $\langle 1, 1\rangle \iff$ for each prime p and each $n \in \mathbf{N}$, and there exists a complex number $T(p)$ such that*

$$\chi(p^n) = T(p)^{n-1}\chi(p).$$

(ii) *χ has valence $\langle 2, 0\rangle \iff$ for each prime p and each $n(\geqslant 2) \in \mathbf{N}$,*

$$\chi(p^{n+1}) = \chi(p)\chi(p^n) + \chi(p^{n-1})[\chi(p^2) - \chi(p)^2].$$

(iii) *χ has valence $\langle 1, s\rangle \iff$ for each prime p and each $\chi \in \mathbf{N}$, and there exist complex numbers $B_1(p), \dots, B_s(p)$ such that for all $\chi \geqslant s$,*

$$\chi(p^n) = \sum_{j=0}^{s} \rho(p)^{n-j} H_j$$

where

$$H_j = (-1)^j \sum_{1 \leqslant i_1 < i_2 < \cdots i_j \leqslant s} B_{i_1}(p) \cdots B_{i_j}(p), \quad H_0 = 1, \ \rho \in CM.$$

First we note that (ii) is just McCarthy's theorem (see [**27**]) discussed in Section 5, where we showed that $B(p)$ is parameter $-t_2$ of the positive degree two multiplicative function in question. We look now at parts (i) and (iii) of the Laohakosol-Pabhapote corollary. We shall, as is consistent with our practice in this paper, drop reference to the prime p since our theory is intrinsically local. So we now look at part (i).

Let $\chi = \theta * \rho^{-1}$ where $\theta, \rho \in CM$. (Recall the discussion of the Euler totient function φ above.)

Let

$$F(\mathbf{t}) \longleftrightarrow \chi, \quad F''(t') \longleftrightarrow \theta, \quad F''(t'') \longleftrightarrow \rho.$$

Then

$$t_1' \neq 0, \quad t_j' = 0,\ j > 1; \qquad t_1'' \neq 0, \quad t_j'' = 0,\ j > 1$$

Making use of Proposition 5, we have that

$$\overline{s''} = \overline{F''}_n = -t_1'', \quad \text{if } n = 1,\ = 0 \text{ if } n > 1.$$

From this we easily deduce that:

$$F_1 = t_1' - t_1'' = t_1,$$
$$\cdots,$$
$$F_n = (t')_1^{(n-1)}(t_1' - t_1'') = t_1'^{(n-1)}(t_1' - t_1'') = t_1'^{(n-1)} t_1 = t'^{(n-1)} F_1.$$

That is,

Proposition 7. *Suppose $\chi = \theta * \rho^{-1}$, θ and ρ are degree 1 functions, $\theta \neq \rho$ and $\theta \neq \delta \neq \rho$, and suppose that χ, θ and ρ are represented by, respectively, F, F' and F'', with parameters t_j, t_j', t_j''; s_j, s_j', s_j''. Then,*

$$F_n = t_1'^{(n-1)} F_1, \quad \textit{for all } n \in \mathbf{N}$$

where $\overline{F}'' = F^{-1}$.

Thus, the mysterious $T(p)$ in the original corollary is just one of the parameters determining the representing GFP-sequence just as in the case of McCarthy's theorem, this time for θ, it is just t_1'.

Next, we look at part (iii) of the corollary. We first consider a product of degree 3, as in Remark 3, i.e., a product $\chi = \theta * \rho^{-2}$ where θ is a positive function of degree 1 and ρ is a positive function of degree 2, and use the formalism of the previous example for representing χ, θ and ρ by GFP sequences. Thus, we have

$$F(t_1, t_2, t_3) = F'(t'_1) * \overline{F}''(t''_1, t''_2).$$

(Here we are thinking of $\overline{F}''$ not as an inverse, but as a function in its own right). With the help of Proposition 5 and Corollary 5, we easily find that

$$F_1 = t'_1 + t''_1 = t_1$$

and

$$\begin{aligned} F_2 &= F'_2 + F'_1\overline{F}''_1 + \overline{F}''_2 \\ &= t'2_1 + t'_1t''_1 + t''_2 \\ &= t'_1(t'_1 + t''_1) \\ &= t'_1F_1 - s''_2. \end{aligned}$$

In the same way,

$$F_3 = t'^2_1F_1 - t'_1s''_2 - s''_3.$$

Induction gives us

Proposition 8. *If χ has valence $\langle 1, s\rangle$ and $\chi \leftrightarrow F$, then*

$$F_n = (t')^{n-1}F_1 - t'^{n-2}_1 s''_2 - t'^{n-3}_1 s''_3.$$

This proposition is nothing but a thinly disguised version of the definition of a convolution product together with particular assumptions about the parameters of the factors together with Proposition 5. But it says all that the Laohakosol-Pabhapote result says and at the same time explicitly identifies the mysterious functions. Compare the previous remarks concerning the McCarthy theorem. The general case is now clear.

Theorem 9. *Let* $\chi = \theta * \rho$, *where* θ *is a positive degree* 1 *function and* ρ *is the convolution inverse of a degree* $(k-1)$ *positive function, and suppose that* χ *is non-trivial and in normal form. Representing these functions by* GFP *functions as above,* $\chi \leftrightarrow F(t_1, \dots, t_j, \dots), \theta \leftrightarrow F'(t_1')$, $\rho \leftrightarrow F''(t_1'', \dots, t_j'', \dots)$,

$$F_{k,n} = -\sum_{j=0}^{n} t_1'^{(n-j)} s_j'' = \sum_{j=0}^{n} t_1'^{(n-j)} F_j''.$$

Proof. $\theta \leftrightarrow F'$ is of type (∞, fin), and $\rho \leftrightarrow F''$ is of type (fin, ∞); thus, as a result of the assumptions on the factors and the freeness of the product, F is infinitely generated, and so is its inverse. Hence, F is of type (∞, ∞). We can symbolize this by $(\infty, fin) * (fin, \infty)$. By Corollary 5, we have that

$$F_n = -s_n' + \sum_{j=1}^{n-1} s_{n-j}' s_j'' - s_n'',$$

and since β is of degree 1,

$$-s_j' = F_j' = t_1'^j.$$

Therefore,

$$F_n = -\sum_{j=0}^{n} t_1'^{(n-j)} s_j'',$$

that is,

$$F_n = \sum_{j=0}^{n} t_1'^{(n-j)} F_j''. \qquad \square$$

Since χ in Theorem 9 has valence $\langle 1, r\rangle$, Theorem 9 is a generalization of Proposition 7.

The following corollary is a direct consequence of the remarks in the proof of the previous theorem concerning generators and the types of terms in the product:

Corollary 10. *A non-trivial product in normal form of valence $\langle r, s\rangle$, that is, a mixed product, is type (∞, ∞).*

In [**16**] the notion of *s-excessive* was introduced in connection with a generalization of the Busche-Ramanujan identities. Since the theory that we are dealing with is local, we shall refer the reader to Definition 3.1 of the paper just cited for the general definition and discussion of that concept and give here the local definition, which is suitable for this paper.

Definition 7. Two prime powers of the same prime p, say p^r and p^s with $s \leqslant r$, are said to be *e-excessive* if $r \leqslant s$ and $s - r = e$.

In Section 5, we suggested that Busche-Ramanujan identities can be regarded locally as expressing $F_n = F_{r+s}$ in terms of F_r and F_s. The following theorem, Theorem 11, is a local generalization of B-R identities to functions of arbitrary degree and generalizes to Theorem 3.2 in [**16**] when the functions here are restricted to the functions in that theorem and the result of Theorem 11 is globalized.

Theorem 11. *Let α be a multiplicative arithmetic function of degree k, and let r and s be two integers with $r \leqslant s$. Abbreviating $F_{k,n}$ as F_n, then*

$$F_n = F_{r+s} = \sum_{j=0}^{e+1} (-1)^j S_{(r,1^j)} F_{s-j},$$

where $S_{(r,1^j)}$ is an isobaric reflect of the Schur-hook function whose Young diagram has an arm of length r and a leg length of j.

Observe that these Schur-hook functions for a given r consist exactly of the elements of the r-th row, in order from right to left, of the companion matrix, and that $S_{(r,1^0)} = F_r$; so that this formula satisfies our interpretation of a generalization of the Busche-Ramanujan identity and includes Theorem 3.2 in [**16**]. Also observe that such a row is a vector representing the r-th power of a root of the core polynomial (see Section 2).

Proof of Theorem 11. First note that the companion matrix is *stable* in the sense that adding a $k+1$-st column on the left of an $\infty \times k$-companion matrix changes nothing in the original matrix. Thus, we might as well assume that we have infinitely many parameters for F. For a finite k we need only let $t_j = 0$ from a certain point onward. We shall also need to recall the fact that the columns of the companion matrix are linear recursions with respect to the parameters t_j. When $r = 0$ the theorem is just a statement of the fact that F is a linear recursion with parameters t_j. We proceed by induction on r with $e = r - s$.

$$\begin{aligned}
F_n &= F_{r+s} = t_1 F_{(r-1)+s} + t_2 F_{(r-2)+s} + \cdots + t_k F_{(r-k)+s} \\
&= t_1 \sum_{j=0}^{e+2} S_{(r-1,1^j)} F_{s-j} + t_2 \sum_{j=0}^{e+3} S_{(r-2,1^j)} F_{s-j} + \cdots \\
&\quad + \sum_{j=0}^{e+k+1} S_{(r-k,1^j)} F_{s-j} \\
&= \Bigg(\sum_{j=0}^{e+2} t_1 S_{(r-1,1^j)} + \sum_{j=0}^{e+3} t_2 S_{(r-2,1^j)} + \cdots \\
&\qquad + \sum_{j=0}^{e+k+1} t_k S_{(r-k,1^j)} \Bigg) F_{s-j} \\
&= \sum_{j=0}^{e+1} S_{(r,1^j)} F_{s-j}. \qquad \square
\end{aligned}$$

Corollary 2.6 in [**16**] simply states that, for a positive function of degree r, $t_j = 0$, $j > r$, $t_r \neq 0$. And Corollary 2.7 in [**16**] is a statement of the basic fact that the representing GFP-sequence is a k-order linear recursion whenever α is a multiplicative function of degree k, (see [**26**] or, indeed, [**23, 24, 25**]).

To conclude this section, we point out that in the formalism of this paper, using symmetric function theory, some *classical* theorems in multiplicative number theory reduce to rather prosaic statements, if not trivialities. An example of this is the binomial identity, (see [**8, 31, 35**]). We would like to use this instance to show how this is the case.

The binomial identity may be stated as follows:

Proposition 9. *Suppose that $\alpha = \gamma_1 * \gamma_2$ where γ_1 and γ_2 are completely multiplicative functions. Then α satisfies the binomial identity*

$$\alpha(p^n) = \sum_{j=0}^{\lceil n/2 \rceil} (-1)^j \binom{n-j}{j} \alpha(p)^{n-2j} (\gamma_1(p)\gamma_2(p))^j,$$

which in turn is just a special case of the general formula for Weighted Isobaric Polynomials, (see [**25**]*). If we use our* GFP *representation, then this result is just a case of the general formula for* GFP*'s as in Section* 1 *of this paper*; *namely,*

$$F_{k,n} = \sum_{\alpha \vdash n} \binom{|\alpha|}{\alpha_1, \dots, \alpha_k} t_1^{\alpha_1} \cdots t_k^{\alpha_k}.$$

Here is the formulation in our terms:

Proposition 10. *Let α be a positive MF of degree 2 represented by F, and let F' and F'' represent the convolution factors. Then*

$$F_{2,n} = F_n = \sum_{j=0}^{\lceil n/2 \rceil} (-1)^j \binom{n-j}{j} F_1^{n-2j} (-t_1' t_1'')^j.$$

Thus, we have

$$F_n = \sum_{j=0}^{\lceil n/2 \rceil} (-1)^j \binom{n-j}{j} F_1^{n-2j} (-t_2)^j.$$

As an example, let $n = 5$, and note that the partitions of 5 with $k = 2$ are just $(1^5), (1^3, 2), (1, 2^2)$. Then, we have,

$$F_{2,5} = \sum_{j=0}^{3} (-1)^j \binom{5-j}{j} F_1^{5-2j} (-t_2)^j = t_1^5 + 4t_1^3 t_2 + 3t_1 t_2^2.$$

8. Kesava Menon norm. Since our aim is to show the utility of expressing multiplicative function theory in terms of isobaric polynomials, we include a discussion of the Kesava Menon norm. The Kesava Menon norm $\mathbf{N}$ is defined on multiplicative functions and is given by

$$\mathbf{N}(\alpha)(n) = \sum_{d|n^2} \alpha(n^2)\lambda(d)\alpha(d)$$

where λ is the Liouville function. $\mathbf{N}$ is a multiplicative function. The Liouville function is defined by $\lambda(m) = (-1)^{\Omega(m)}$, where $\Omega(m)$ is the number of prime factors of m counting the multiplicity of a prime in m; in particular, $\lambda(p^n) = 1$ if n is even and -1 if n is odd. If α has degree 1 or 2, then $\deg \mathbf{N}(\alpha) = 1, 2$, respectively. What does the norm look like in isobaric notation?

Let $F_n(\mathbf{t}) \leftrightarrow \alpha, \alpha \in MF$ for a suitable choice of vector $\mathbf{t}$, and let $\mathbf{N}(F_n) = \mathbf{N}(\alpha)(p^n)$. Then

$$\mathbf{N}(F_n) = \sum_{j=0}^{2n} (-1)^j F_{2n-j} F_j$$

or, equivalently

$$\mathbf{N}(F_n) = 2\sum_{j=0}^{n-1} (-1)^j F_{2n-j} F_j + (-1)^n F_n^2.$$

By Corollary 5, (1′) and (2′), this can be written in the following way.

Proposition 11.

$$\mathbf{N}(F_n) = -2s_{2n} + 2\sum_{j=1}^{n-1} (-1)^j s_{2n-j} s_j + (-1)^n s_n^2$$

$$\mathbf{N}(F_n) = 2t_{2n} + 2\sum_{j=1}^{n-1} (-1)^j t_{2n-j} t_j + (-1)^n t_n^2.$$

It is well known that this norm is multiplicative (see [**27**, page 50]), that is: If α and $\beta \in$ MF, then

$$\mathbf{N}(\alpha * \beta) = \mathbf{N}(\alpha) * \mathbf{N}(\beta).$$

We give a proof here using the concepts of this paper.

First, we observe that

$$\mathbf{N}(F_0' * F_0'') = F_0'F_0'' = F_0'' = \mathbf{N}(F_0) = 1 = \mathbf{N}_0.$$

We prove the following lemma:

Lemma 3. *If* $F_1 = F_1' * F_1''$, *then*

$$\mathbf{N}(F_1) = \mathbf{N}(F_1' * F_1'') = \mathbf{N}(F_1') * \mathbf{N}(F_1'').$$

Proof. Let $\chi = \chi' * \chi''$ where χ, χ', χ'' are multiplicative functions, and let $\chi \leftrightarrow F$, $\chi' \leftrightarrow F'$ and $\chi'' \leftrightarrow F''$. Using the definition we have

$$\mathbf{N}_1(F) = 2F_2 - F_1^2 = 2(t_1^2 + t_2) - t_1^2 = t_1^2 + 2t_2.$$

Then, using Corollary 5, we have

$$\begin{aligned}(t_1' + t_1'')^2 + 2(t_2' - t_1't_1'' + t_2'') &= {t_1'}^2 + 2t_1't_1'' + {t_1''}^2 + 2t_2' - 2t_1't_1'' + 2t_2'' \\ &= ({t_1'}^2 + 2t_2') + ({t_1''}^2 + 2t_2''),\end{aligned}$$

which is

$$\mathbf{N}_1(F') + \mathbf{N}_1(F''),$$

and

$$\mathbf{N}_1(F'') + \mathbf{N}_1(F'') = \mathbf{N}_1' * \mathbf{N}_1''. \qquad \square$$

Theorem 12. *Let* $F = F' * F''$. *Then*

$$\mathbf{N}(F_n) = \mathbf{N}(F_n') * \mathbf{N}(F_n'');$$

that is, if $\mathbf{N}_n = \mathbf{N}(F_n)$, *then*

$$\mathbf{N}_n = \mathbf{N}_n' * \mathbf{N}_n''.$$

Proof. It is well known that the KM-norm is a multiplicative function and hence by the correspondence theorem is linearly recursive. Denoting the indeterminates of the MFs $\mathbf{N}$, $\mathbf{N}'$ and $\mathbf{N}''$ by n_r, n'_r and n''_r, and using Corollary 5, we have

$$n_r = n'_r - \sum_{j=1}^{r-1} n'_j n''_{r-j} + n''_r,$$

where $1 \leqslant j \leqslant r$. Then, using the recursion property of the norm function, we have

$$\mathbf{N}_r = \sum_{j=1}^{r} n_j \mathbf{N}_{r-j} = \sum_{j=1}^{r}(n'_j - \sum_{s=1}^{r-j} n'_{r-j-s} n''_s + n''_j)\mathbf{N}_{r-j},$$

where $1 \leqslant s \leqslant r$.

We let $n_u = 0$ when $u < 0$ and $n_u = 1$ when $u = 0$. By the inductive hypothesis:

$$\mathbf{N}_{r-j} = \mathbf{N}'_{r-j} * \mathbf{N}''_{r-j},$$

we have

$$\mathbf{N}_r = \sum_{j=1}^{r}(n'_j - \sum_{s=1}^{r-j} n'_{r-j-s} n''_s + n''_j) \sum_{j=1}^{r} \mathbf{N}'_{r-j} * \mathbf{N}''_{e-j},$$

and so

$$= \sum_{j=1}^{r} \left[(n'_j - \sum_{s=1}^{j-1} n'_{j-s} n''_s + n''_j) \left(\sum_{i=0}^{r-j} \mathbf{N}'_{r-j-i} \mathbf{N}''_i \right) \right],$$

where $0 \leqslant s \leqslant r$. Thus, we have

$$N_r = \sum_{j=1}^{r} [(n'_j + n''_j) \sum_{j=1}^{r} \mathbf{N}'_{r-j-i} \mathbf{N}''_i - \left(\sum_{s=1}^{j-1} n'_{j-s} n''_s \sum_{i=0}^{r-j} \mathbf{N}'_{r-j-i} \mathbf{N}''_i \right) \Bigg],$$

which we can write as

$$\begin{aligned}
N_r = &\sum_{j=1}^{r}\left(n'_j \sum_{i=0}^{r-j}\mathbf{N}'_{r-j-i}\mathbf{N}''_i\right)\\
&+\sum_{j=1}^{r}\left(n''_j \sum_{i=0}^{r-j}\mathbf{N}'_{r-j-i}\mathbf{N}''_i\right)\\
&-\left(\sum_{j=1}^{r}\left(\sum_{s=1}^{j-1}n'_{j-s}n''_s\sum_{i=0}^{r-j}\mathbf{N}'_{r-j-i}\mathbf{N}''_i\right)\right).
\end{aligned}$$

We can now shift the order of summation in the first and second summands on the right hand side, and using the fact that $N'_0 = 1$ and $N''_0 = 1$,

$$\begin{aligned}
N_r = &\sum_{i=0}^{r}\left(n'_j \sum_{j=1}^{r-i}\mathbf{N}'_{r-j-i}\mathbf{N}''_i\right)\\
&+\sum_{i=0}^{r}\left(n''_j \sum_{j=1}^{r-i}\mathbf{N}''_{r-j-i}\mathbf{N}'_i\right)\\
&+\sum_{j=1}^{r-2}n'_j\left(N''_{r-j}-\sum_{i=1}^{r-j}n''_j N''_{r-j-i}\right)\\
&+\left(\sum_{j=1}^{r-2}n''_j(N'_{r-j}-\sum_{i=1}^{r-j}n'_j N'_{r-j-i}\right).
\end{aligned}$$

Using the linear recursion property of the KM-norm in each of the four summands, this becomes

$$\begin{aligned}
\mathbf{N}_r &= \sum_{j=1}^{r}\mathbf{N}'_j\mathbf{N}''_{r-j}+\sum_{j=1}^{r-2}n'_j(\mathbf{N}''_{r-j}-\mathbf{N}''_{r-j})\\
&\quad+\sum_{j=1}^{r-2}n''_j(\mathbf{N}'_{r-j}-\mathbf{N}'_{r-j})\\
&=\sum_{j=1}^{r}\mathbf{N}'_j\mathbf{N}''_{r-j}\\
&=\mathbf{N}'_r * \mathbf{N}''_r.
\end{aligned}$$

Thus, together with Lemma 3, we have

$$\mathbf{N}(F' * F'') = \mathbf{N}(F') * \mathbf{N}(F''). \qquad \square$$

In particular, the theorem together with the proof show that any norm function on multiplicative functions that agrees with the KM-norm at all primes, i.e., at $n = 1$, agrees at all prime powers, that is, is the KM-norm.

Note that $\mathbf{N}_1 = G_{k,2}$ for all $k > 1$. While this is suggestive, it suggests the wrong thing. There does not seem to be such a relation to the GLPs for $n > 2$.

Lemma 4. *If* $\deg(\alpha) = 1$*, then* $\deg \mathbf{N}(\alpha) = 1$*.* (*See* [**27**, page 50].)

Theorem 13.

$$\deg(\mathbf{N}(\alpha)) = \deg(\alpha).$$

Proof. If $\deg(\alpha) = k$, then α is the convolution product of k degree 1 multiplicative functions. The theorem then follows from Theorem 12 and Lemma 4. $\square$

9. Examples.

Example 4. Let α be the arithmetic function whose values for $\alpha(p^n) = f_n$, for all primes p, where f_n is the n-th Fibonacci number. $f_0 = f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$, and represent α at each prime by F. It is easy to calculate that F has degree two, in fact, that $F_n(t_1, t_2) = F_n(1, 1)$. According to Theorem 13, $\deg \mathbf{N}(\alpha) = 2$ also. We can see this directly. $\mathbf{N}(F_1) = \mathbf{N}_1 = f_3 = 3$ and $\mathbf{N}_2 = f_5 = 8$. From the first of these two facts, we have that $n_1 = f_3 = 3$ and, from the second, that $\mathbf{N}_2 = f_5 = 8$, and since $\mathbf{N}_2 = n_1^2 + n_2$, we have that $n_2 = -1$. The same technique shows that $\mathbf{N}_3 = 0$. An easy induction yields $n_j = 0, j > 2$. We can then use the recursion property to show that $\mathbf{N}_n = f_{2n+1}$.

Example 5. For the multiplicative function τ (number-of-divisors function), using Theorem 13 and the methods above, we find that

$\deg \mathbf{N}(\tau) = 2 = \deg(\tau)$, where $t_1 = 1, t_2 = -1$ and $\mathbf{N}_n = n+1 = \tau(p^n)$, $n_1 = 2, n_1 = -1$.

Acknowledgments. The authors would like to thank an anonymous referee for many useful suggestions, both stylistic and mathematical, which are responsible for important improvements in this paper.

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York University, Toronto, Canada
Email address: machenry@mathstat.yorku.ca

Centennial College, Toronto, Canada
Email address: kkwong@centennialcollege.ca