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LINEAR MAPS PRESERVING GENERALIZED INVERTIBILITY ON COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let A and B be unital complex Banach algebras such that B is commutative and semi-simple. We study linear maps from A into B that preserve generalized invertibility.

1. Introduction and preliminaries. Let A be an algebra. An element $a \in A$ is generalized invertible (or regular) if there exists a $b \in A$ such that $aba = a$. We denote by $\mathcal{G}(A)$ the subset of all generalized invertible elements of A. If A is unital, obviously, $A^{-1} \subseteq \mathcal{G}(A)$, where A^{-1} denotes the group of all invertible elements of A. The well known Gleason-Kahane-Zelazko theorem [**11, 15, 24**] states that if A and B are complex unital Banach algebras such that B is commutative and semi-simple, and if $\phi : A \to B$ is a linear map preserving invertibility (i.e., $\phi(a) \in B^{-1}$ whenever $a \in A^{-1}$), then $\phi(1)^{-1}\phi$ is multiplicative. It seems natural to devote some attention to the case where ϕ preserves generalized invertibility instead of invertibility. The research into this area was initiated in the noncommutative case by Mbekhta, Rodman and Šemrl [**20**]. Afterwards, it was developed in several directions (see [**8, 9, 13, 18, 21**]). It should be pointed out that all these studies are closely connected with Kaplansky's conjecture [**17**]. For more details on this topic, the reader is referred to [**3, 10**].

Now let us define the basic concepts of this note. An algebra A is said to be semi-prime if the condition $aAa = \{0\}$ implies that $a = 0$, for all $a \in A$. Obviously, a semi-simple algebra is semi-prime. Let A be a semi-prime algebra. If A contains minimal left ideals, then the sum of all minimal left ideals is called the socle of A and is denoted by $\operatorname{soc} A$. If A does not have minimal left-ideals we define $\operatorname{soc} A = \{0\}$. It

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is well known that $\operatorname{soc} A$ is a two-sided ideal of A and coincides with the sum of all minimal right ideals of A. A minimal idempotent of A is a non-zero idempotent $e \in A$ such that eAe is a division algebra. For any minimal left ideal J of A, there exists a minimal idempotent $e \in A$ such that $J = Ae$. For basic results on the socle of an algebra, the reader is referred to [**7, 14**]. Moreover, recall that $\operatorname{soc} A \subseteq \mathcal{G}(A)$ [**6**]. As usual, $\operatorname{kh}(\operatorname{soc} A)$ denotes the intersection of all primitive ideals of A containing $\operatorname{soc} A$. It is easy to show that $\operatorname{rad}(A/\operatorname{soc} A) = \operatorname{kh}(\operatorname{soc} A)/\operatorname{soc} A$, and it is well known that $\operatorname{kh}(\operatorname{soc} A)$ and $\operatorname{soc} A$ have the same set of idempotents [**4**, page 107]. In particular, notice that if $a \in \operatorname{kh}(\operatorname{soc} A)$ and a has a generalized inverse, then $a \in \operatorname{soc} A$. Suppose from now on that the algebra A is unital, and let $a \in A$. Then $a + \operatorname{soc} A$ is invertible in $A/\operatorname{soc} A$ if and only if $a + \operatorname{kh}(\operatorname{soc} A)$ is invertible in $A/\operatorname{kh}(\operatorname{soc} A)$ [**4**, page 107]. An element $x \in A$ is said to be Fredholm if there exists a $y \in A$ such that $xy - 1 \in \operatorname{soc} A$ and $yx - 1 \in \operatorname{soc} A$ [**6**]. We denote by $\mathcal{F}(A)$ the set of Fredholm elements of A. Recall that, if $a \in A$ and there exists a $b \in A$ such that $aba - a \in \mathcal{G}(A)$, then $a \in \mathcal{G}(A)$ [**12**, page 246]. In view of this, we have $\mathcal{F}(A) \subseteq \mathcal{G}(A)$. If moreover A is a Banach algebra, then $\mathcal{F}(A)$ is open. Finally, suppose that A is a complex commutative Banach algebra. The maximal ideal space of A is denoted by Δ_A. If A is semi-simple, it is well known that $\operatorname{soc} A = \{0\}$ if and only if Δ_A has no isolated points (see for instance [**1**, page 255]). Moreover, an element $a \in A$ lies in $\operatorname{soc} A$ if its Gelfand transform $\widehat{a}$ has finite support.

2. Main result. Let A and B be two Banach algebras, and let $\phi : A \to B$ be a linear map. We shall say that ϕ preserves generalized invertibility if $\phi(a) \in \mathcal{G}(B)$ whenever $a \in \mathcal{G}(A)$. Now suppose that B is unital and semi-prime. Obviously, if $\phi(A) \subseteq \operatorname{soc} B + \mathbf{C}1$ or, more generally, $\phi(A) \subseteq \mathcal{F}(A)$, then ϕ preserves generalized invertibility. Thus, we will assume that our maps are surjective modulo the socle in our main results.

An indispensable tool in this section will be the following simple lemma.

Lemma 2.1. *Let A be a commutative complex unital semi-prime Banach algebra, and let $x \in A$. Then the following are equivalent.*

(i) $x \in \mathcal{F}(A)$,

(ii) *for every* $y \in A$, *there exists a* $\delta > 0$ *such that* $x + \lambda y \in \mathcal{G}(A)$ *for every scalar* λ *with* $|\lambda| < \delta$.

Proof. The implication i) $\Rightarrow$ ii) is trivial since the set of Fredholm elements of A is open and $\mathcal{F}(A) \subseteq \mathcal{G}(A)$.

ii) $\Rightarrow$ i). Suppose that ii) holds true for x. Since $x \in \mathcal{G}(A)$, the ideal Ax is closed. Fix $a \in A$. Then there exists a $\lambda \neq 0$ such that $x + \lambda a \in \mathcal{G}(A)$. This implies that there is a $z \in A$ with $a - \lambda z a^2 \in Ax$. As a result, every element in the Banach algebra A/Ax has a generalized inverse. Applying [**16**], we infer that the algebra A/Ax is finite-dimensional. Now choose $y \in A$ such that $x^2 y = x$. Then $(1 - xy)Ax = 0$. From this it follows that $\dim (1 - xy)^2 A < \infty$. Thus $1 - xy \in \operatorname{soc} A$. We have thereby shown that $x \in \mathcal{F}(A)$. This completes the proof. $\square$

Having the above lemma in hand, we can deduce the following characterization of the socle in a semi-prime commutative Banach algebra.

Lemma 2.2. *Let* A *be a commutative complex unital semi-prime Banach algebra, and let* $a \in A$. *Then the following two conditions are equivalent*:

(i) $a \in \operatorname{soc} A$.

(ii) *For every* $g \in \mathcal{G}(A)$, $a + g$ *has a generalized inverse.*

Proof. Since every element of $\operatorname{soc} A$ has a generalized inverse, the implication (i) $\Rightarrow$ (ii) follows from [**12**, 7.3.2.6].

To prove the converse, suppose that (ii) is satisfied. Then it follows from Lemma 2.1 that, for every $u \in \mathcal{F}(A)$, $a + u \in \mathcal{F}(A)$, and therefore, $a + u + \operatorname{soc} A$ is invertible in the algebra $A/\operatorname{soc} A$. By [**22**], we infer that $a + \operatorname{soc} A \in \operatorname{rad}(A/\operatorname{soc} A)$. As a result, $a \in \operatorname{kh}(\operatorname{soc} A)$. But $a \in \mathcal{G}(A)$; thus, $a \in \operatorname{soc} A$. $\square$

Theorem 2.3. *Let* A *and* B *be complex unital Banach algebras such that* B *is commutative and semi-prime. Let* $\phi : A \to B$ *be a linear map preserving generalized invertibility. Suppose, moreover, that* ϕ *is surjective modulo the socle of* B. *Then there exists a Fredholm element* u *of* B *such that the map* $\pi \circ (u\phi)$ *is multiplicative, where* $\pi : B \to B/\operatorname{kh}(\operatorname{soc} B)$ *is the canonical quotient map.*

Proof. We first show that $\pi \circ \phi$ preserves invertibility. Fix $x \in A^{-1}$, and let $b \in B$. Then there exist $a \in A$ and $s \in \operatorname{soc}(B)$ such that $b = \phi(a) + s$. Since A^{-1} is open, there exists a $\delta > 0$ with the property that $x + \lambda a \in A^{-1}$ for every complex number λ with $|\lambda| < \delta$. Hence, $\phi(x + \lambda a)$ has a generalized inverse in B for every λ with $|\lambda| < \delta$. Using once again [**12**, 7.3.2.6], we infer that $\phi(x) + \lambda b$ has a generalized inverse for every λ with $|\lambda| < \delta$. Applying Lemma 2.1, we conclude that $\phi(x) \in \mathcal{F}(B)$. Thus, $\pi \circ \phi$ preserves invertibility, as desired. Now it follows from the Gleason-Kahane-Zelazko theorem that the map $(\pi \circ \phi(1))^{-1}\pi \circ \phi$ is multiplicative. This completes the proof. □

Remark 2.4. Let A and B be complex unital semi-prime Banach algebras such that B is commutative, and let $\phi : A \to B$ be a linear map preserving generalized invertibility. Suppose, moreover, that ϕ is surjective modulo the socle of B. Then, using Lemma 2.1, we infer that ϕ preserves Fredholm elements (i.e., $\phi(x) \in \mathcal{F}(B)$ whenever $x \in \mathcal{F}(A)$, for every $x \in A$).

Given a topological space X and $F \subseteq X$, we denote by $\overline{F}$ the closure of F.

Remark 2.5. According to [**23**, Corollary 7.4], if $\mathcal{A}$ is a function algebra, then $\operatorname{kh}(\operatorname{soc} A) = \overline{\operatorname{soc} A}$. Let A be a commutative semi-simple Banach algebra. We do not know if the equality $\operatorname{kh}(\operatorname{soc} A) = \overline{\operatorname{soc} A}$ holds. Recall that the compactum of A is the set of all elements $x \in A$ such that the operator T_x defined by $T_x(a) = x^2 a$ for all $a \in A$ is compact. Using [**5**, Theorem 2.1] we see easily that $\overline{\operatorname{soc} A} \subseteq C(A) \subseteq \operatorname{kh}(\operatorname{soc} A)$. In [**2**], the author proves that $C(A) = \overline{\operatorname{soc} A}$, but his argument is not clear.

We are now in a position to prove our main result.

Theorem 2.6. *Let A and B be complex semi-prime unital Banach algebras, and suppose that B is commutative. Let $\phi : A \to B$ be a linear map preserving generalized invertibility. Suppose, moreover, that ϕ is surjective modulo the socle of B. Then* $\phi(\operatorname{soc} A) \subset \operatorname{soc} B$, $\phi(\operatorname{kh}(\operatorname{soc} A)) \subset \operatorname{kh}(\operatorname{soc} B)$. *Moreover, there exists an invertible element u of $B/\operatorname{kh}(\operatorname{soc} B)$ such that $u\varphi$ is multiplicative, where $\varphi : A/\operatorname{kh}(\operatorname{soc} A) \to B/\operatorname{kh}(\operatorname{soc} B)$ is the induced map.*

Proof. According to Theorem 2.3, it suffices to prove that $\phi(\operatorname{soc} A) \subset \operatorname{soc} B$ and $\phi(\operatorname{kh}(\operatorname{soc} A)) \subset \operatorname{kh}(\operatorname{soc} B)$. First observe that by Theorem 2.3, $\phi(1) \in \mathcal{F}(B)$. Choose $v \in B$ such that $v\phi(1) - 1 \in \operatorname{soc} B$, and define $\psi : A \to B$ by $\psi(x) = v\phi(x)$ for every $x \in A$. Since $\mathcal{F}(B) \subseteq \mathcal{G}(B)$, $v \in \mathcal{G}(B)$. Consequently, ψ preserves generalized invertibility. On the other hand, observe that ψ is surjective modulo $\operatorname{soc} B$. It follows from Theorem 2.3 that the map $\pi \circ \psi$ is multiplicative, where $\pi : B \to B/\operatorname{kh}(\operatorname{soc} B)$ is the canonical quotient map. Next we show that $\phi(\operatorname{soc} A) \subset \operatorname{soc} B$. Let e be a minimal idempotent of A. Since $e-1 \in \mathcal{F}(A)$, then $\psi(e-1) \in \mathcal{F}(B)$. But $\psi(1) - 1 \in \operatorname{soc} B$. This entails that $\psi(e) - 1 \in \mathcal{F}(B)$. Moreover, using the fact that $\pi \circ \psi$ is multiplicative, we infer that $\psi(e)(\psi(e)-1) \in \operatorname{kh}(\operatorname{soc} B)$. It follows that $\psi(e) \in \operatorname{kh}(\operatorname{soc} B)$. But $\psi(e) \in \mathcal{G}(B)$. Therefore, $\psi(e) \in \operatorname{soc} B$, and we get $\phi(e) \in \operatorname{soc} B$. We have thereby shown that $\phi(\operatorname{soc} A) \subset \operatorname{soc} B$.

Now let us show that $\phi(\operatorname{kh}(\operatorname{soc} A)) \subseteq \operatorname{kh}(\operatorname{soc} B)$. Since the map $\pi \circ \psi$ is multiplicative, it follows that $\pi \circ \psi$ is continuous [**7**, page 83]. Hence $\psi(\overline{\operatorname{soc} A}) \subseteq \operatorname{kh}(\operatorname{soc} B)$. Define $\tau : A/\overline{\operatorname{soc} A} \to B/\operatorname{kh}(\operatorname{soc} B)$ by $\tau(a + \overline{\operatorname{soc} A}) = \psi(a) + \operatorname{kh}(\operatorname{soc} B)$ for every $a \in A$. Observe that τ is a surjective linear map preserving invertibility with $\tau 1 = 1$; hence, $\tau(\operatorname{rad}(A/\overline{\operatorname{soc} A})) = \{0\}$, since the algebra $B/\operatorname{kh}(\operatorname{soc} B)$ is semi-simple. Thus, $\psi(\operatorname{kh}(\operatorname{soc} A)) \subset \operatorname{kh}(\operatorname{soc} B)$, as desired. This completes the proof. $\square$

Remark 2.7. Let K be a non-empty, compact, Hausdorff topological space. The commutative algebra of complex-valued, continuous functions on K is denoted by $\mathcal{C}(K)$. Let F be a closed subset of K. Suppose that there exists an homeomorphism $\eta : K \to F$. Then the linear map $\phi : \mathcal{C}(K) \to \mathcal{C}(K)$ defined by

$$\phi(g) = g|_F \circ \eta,$$

is multiplicative and surjective. Obviously, if $F \neq K$, then ϕ is not injective. Suppose, moreover, that K has no isolated points. Then $\operatorname{soc}(\mathcal{C}(K)) = \{0\}$. This simple example shows that the map φ in the above theorem is not injective in general.

3. General case. To conclude this paper, we give some results and comments about non surjective maps preserving generalized invertibility.

Remark 3.1. Let A, B be unital Banach algebras, and let $\phi : A \to B$ be an additive map. Following [**19**], we say that ϕ preserves strongly generalized invertibility if $\phi(y)$ is a generalized inverse of $\phi(x)$ whenever y is a generalized inverse of x, for every $x, y \in A$. Recall that y is a generalized inverse of x if $xyx = x$ and $yxy = y$. According to [**9**, Proposition 3.10], if B is commutative and ϕ preserves strongly generalized invertibility, then $\phi(1)\phi$ is multiplicative.

Let K be a non-empty, compact, Hausdorff topological space. For $f \in \mathcal{C}(K)$, we denote the zero set of f by $\mathcal{Z}(f)$. Let $\phi : A \to \mathcal{C}(K)$ be a linear map. Let K' be a closed subset of K. Then we denote by $\phi_{K'}$ the linear map $\phi_{K'} : A \to \mathcal{C}(K')$ with $\phi_{K'}(a)(u) = \phi(a)(u)$ for every $u \in K'$.

Lemma 3.2. *Let K be a non-empty, compact, Hausdorff topological space, and let A be a unital Banach algebra. Suppose that $\phi : A \to \mathcal{C}(K)$ is a linear map preserving generalized invertibility. Then the set $K' = \{u \in K : \phi(a)(u) \neq 0$ for all $a \in A^{-1}\}$ is closed. Moreover, the map $(\phi_{K'}(1))^{-1}\phi_{K'}$ is multiplicative.*

Proof. Pick $v \in \overline{K'}$, and suppose for a moment that there exists an $a \in A^{-1}$ such that $\phi(a)(v) = 0$. Since $\phi(a)$ has a generalized inverse, then the map $\phi(a)$ must vanish in a neighborhood of v. This contradicts our assumption on v and K'. Thus the set K' is closed. Now observe that the map $\phi_{K'}$ preserves invertibility. Applying the Gleason-Kahane-Zelazko theorem, we infer that $(\phi_{K'}(1))^{-1}\phi_{K'}$ is multiplicative. $\square$

One is tempted to expect in the above lemma that the set K' is also open. The following example shows that this is not true in general.

Example. Set $K = [-1, 0] \cup \{1/n : n \in \mathbf{N}^*\}$. For each $n \in \mathbf{N}^*$, define $\varphi_n \in \mathcal{C}(K)$ by $\varphi_n(x) = 1$ if $x \neq 1/n$ and by $\varphi_n(1/n) = 2$. We check easily that the set $\{\varphi_n : n \in \mathbf{N}^*\} \cup \{\mathbf{1}\}$ is linearly independent. Choose $\{f_j\}_j \subseteq \mathcal{C}(K)$ such that the set $\mathcal{B} = \{\varphi_n : n \in \mathbf{N}^*\} \cup \{\mathbf{1}\} \cup \{f_j\}$ is a Hamel basis of $\mathcal{C}(K)$. Define a linear map $\phi : \mathcal{C}(K) \to \mathcal{C}(K)$ by $\phi(g) = g$ for every $g \in \mathcal{B}$ with $g \notin \{\varphi_n : n \in \mathbf{N}^*\}$, $(\phi(\varphi_n))(1/n) = 0$, and $(\phi(\varphi_n))(x) = 1$ if $x \in K$ with $x \neq 1/n$. Suppose that $f \in \mathcal{C}(K)$

has a generalized inverse, and write $f = \alpha\mathbf{1} + \sum_n \alpha_n\varphi_n + \sum_j \beta_j f_j$, where α, α_n, $\beta_j \in \mathbf{C}$. Then, it is clear that $\phi(f)(x) = f(x)$ for every accumulation point of K. This entails that $\phi(f)$ has a generalized inverse. Now observe that $K \setminus K' = \{1/n : n \in \mathbf{N}^*\}$. Hence, the set K' is not open, as required.

Proposition 3.3. *Let K be a non-empty connected compact Hausdorff topological space, and let A be a unital Banach algebra. Suppose that $\phi : A \to \mathcal{C}(K)$ is a linear map preserving generalized invertibility. Then either $\phi(1)$ is invertible and the map $(\phi(1))^{-1}\phi$ is multiplicative, or ϕ has rank 1.*

Proof. Assume first that ϕ preserves invertibility. Then $\phi(1)$ is invertible and the map $(\phi(1))^{-1}\phi$ is multiplicative by the Gleason-Kahane-Zelazko theorem. Suppose now that there exists an $a \in A^{-1}$ such that $\phi(a) \notin A^{-1}$. Since ϕ preserves generalized invertibility, the set $\mathcal{Z}(\phi(a))$ is open. But evidently, $\mathcal{Z}(\phi(a))$ is also closed; therefore, $\mathcal{Z}(\phi(a)) = K$ and $\phi(a) = 0$. Next let $b \in A$. Then, for $\lambda \in \mathbf{C}$ with $|\lambda|$ sufficiently small, we have $a + \lambda b \in A^{-1}$. It follows that $\phi(a + \lambda b) = \lambda\phi(b) \in \mathcal{G}(\mathcal{C}(K))$. Thus, $\phi(b) \in \mathcal{G}(\mathcal{C}(K))$ for every $b \in A$. Now choose $b \in A$ such that $\phi(b) \neq 0$. Pick $u \in K$ with the property that $\phi(b)(u) \neq 0$. Let $b' \in A$. Then there exists a $\lambda \in \mathbf{C}$ such that $\phi(b' - \lambda b)(u) = 0$. As before, we conclude that $\phi(b' - \lambda b) = 0$. This completes the proof. ◻

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