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THE TOPOLOGICAL CENTER OF WEIGHTED SEMIGROUP ALGEBRAS WITH A STRICT TOPOLOGY

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ABSTRACT. For a family of a locally compact semigroup $\mathfrak{S}$ with a weight function ω, we have recently introduced and studied some locally convex topologies τ on the weighted semigroup algebra $M_a(S,\omega)$ and shown that the strong dual of $(M_a(\mathfrak{S},\omega),\tau)$ can be identified with a Banach space of certain functions on $\mathfrak{S}$. In this paper, we shall be concerned with the second dual of $(M_a(\mathfrak{S},\omega),\tau)$; using this duality, we first introduce and study an Arens multiplication on the second dual of $(M_a(\mathfrak{S},\omega),\tau)$. We then investigate the topological center of $(M_a(\mathfrak{S},\omega),\tau)$ for an extensive class of locally compact semigroups $\mathfrak{S}$. As a consequence, we conclude some results on Arens regularity and strong Arens irregularity of $(M_a(\mathfrak{S},\omega),\tau)$.

1. Introduction and preliminaries. Throughout this paper, we denote by $\mathfrak{S}$ a locally compact semigroup; that is, a semigroup with a locally compact Hausdorff topology under which the binary operation of $\mathfrak{S}$ is jointly continuous. We also assume that ω is a *weight function* on $\mathfrak{S}$; that is, a real-valued continuous function ω with the properties that $\omega(x)\geq 1$ and $\omega(xy)\leq\omega(x)\,\omega(y)$ for all $x,y\in\mathfrak{S}$.

Let $M(\mathfrak{S},\omega)$ denote the Banach space of all complex-valued regular Borel measures μ on $\mathfrak{S}$ for which

$$\|\mu\|_\omega := \int_{\mathfrak{S}} \omega(x)\,d|\mu|(x) < \infty,$$

and as usual write $M(\mathfrak{S})$ and $\|\mu\|$ for the case where $\omega(x)=1$ for all $x\in\mathfrak{S}$, where $|\mu|$ denotes the total variation of μ. Then $M(\mathfrak{S},\omega)$ is the

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dual of $C_0(\mathfrak{S}, 1/\omega)$ for the pairing

$$\langle \mu, \xi \rangle := \int_{\mathfrak{S}} \xi(x)\, d\mu(x)$$

for all $\mu \in M(\mathfrak{S}, \omega)$ and $\xi \in C_0(\mathfrak{S}, 1/\omega)$, the space of all complex-valued continuous functions ξ on $\mathfrak{S}$ such that ξ/ω vanishes at infinity. Moreover, $M(\mathfrak{S}, \omega)$ is a Banach algebra with respect to the convolution multiplication $*$ defined by the formula

$$\langle \mu * \nu, \xi \rangle = \int_{\mathfrak{S}} \int_{\mathfrak{S}} \xi(xy)\, d\mu(x)\, d\nu(y)$$

for all $\mu, \nu \in M(\mathfrak{S}, \omega)$ and $\xi \in C_0(\mathfrak{S}, 1/\omega)$; let us remark that the latter equality also holds for all $\xi \in L^1(\mathfrak{S}, |\mu| * |\nu|)$; see Wong [**24**].

The space of all measures $\mu \in M(\mathfrak{S})$ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from $\mathfrak{S}$ into $M(\mathfrak{S})$ are weakly continuous is denoted by $M_a(\mathfrak{S})$ (the same as $\widetilde{L}(\mathfrak{S})$ in Baker and Baker [**1**]), where δ_x denotes the Dirac measure at x. We call $\mathfrak{S}$ a *foundation semigroup* if $\mathfrak{S}$ coincides with the closure of the set

$$\bigcup \{\operatorname{supp}(\mu) : \mu \in M_a(\mathfrak{S})\}.$$

Also, the space of all measures $\mu \in M(\mathfrak{S}, \omega)$ such that $\omega\mu \in M_a(\mathfrak{S})$ is denoted by $M_a(\mathfrak{S}, \omega)$. Then $M_a(\mathfrak{S}, \omega)$ is a closed L-ideal of $M(\mathfrak{S}, \omega)$ called the weighted semigroup algebra of $\mathfrak{S}$, see Bami [**9**].

Let $\ell^1(\mathfrak{S}, \omega)$ denote the closed subalgebra of $M(\mathfrak{S}, \omega)$ consisting of all discrete measures. Let us point out that $M_a(\mathfrak{S}, \omega)$ and $M(\mathfrak{S}, \omega)$ coincide with $\ell^1(\mathfrak{S}, \omega)$ in the case where $\mathfrak{S}$ is discrete.

Also let $L^\infty(\mathfrak{S}; M_a(\mathfrak{S}, \omega))$ denote the space of all functions ξ on $\mathfrak{S}$ such that ξ/ω is bounded and μ-measurable for all $\mu \in M_a(\mathfrak{S})$. We identify functions in $L^\infty(\mathfrak{S}; M_a(\mathfrak{S}, \omega))$ that agree μ-almost everywhere for all $\mu \in M_a(\mathfrak{S})$, and for every $\xi \in L^\infty(\mathfrak{S}; M_a(\mathfrak{S}, \omega))$, define

$$\|\xi\|_{\infty,\omega} = \sup\{\|\xi/\omega\|_{\infty,|\mu|} : \mu \in M_a(\mathfrak{S})\},$$

where $\|\cdot\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Also, define the multiplication $\cdot_\omega$ on $L^\infty(\mathfrak{S}; M_a(\mathfrak{S}, \omega))$ by

$$\xi \cdot_\omega \eta = \xi\eta/\omega \quad (\xi, \eta \in L^\infty(\mathfrak{S}; M_a(\mathfrak{S}, \omega))).$$

It is known from Bami [9] that $L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ with the complex conjugation as involution, the multiplication $\cdot_\omega$ and the norm $\|\cdot\|_{\infty,w}$ is a commutative C^*-algebra with the identity element ω; see also Dales and Lau [4] for the group case. The duality

$$\langle \varrho(\xi), \mu\rangle := \langle \mu, \xi\rangle = \int_{\mathfrak{S}} \xi \, d\mu$$

for $\xi \in L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ and $\mu \in M_a(\mathfrak{S},\omega)$, defines a linear mapping ϱ from $L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ into the dual space $(M_a(\mathfrak{S},\omega), \|\cdot\|_\omega)^*$. It is known from Bami [9] that, if $\mathfrak{S}$ is a foundation semigroup with identity, then $L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ can be identified with $(M_a(\mathfrak{S},\omega), \|\cdot\|_\omega)^*$, see also Sleijpen [20].

We say that a function $\xi \in L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ *vanishes at infinity* if, for each $\varepsilon > 0$, there is a compact subset C of $\mathfrak{S}$ for which

$$\|\xi \, \chi_{\mathfrak{S}\setminus C}\|_{\infty,\omega} < \varepsilon;$$

that is, $|\xi(x)| < \varepsilon \, \omega(x)$ for μ-almost all $x \in \mathfrak{S} \setminus C$ ($\mu \in M_a(\mathfrak{S},\omega)$). We denote by $L_0^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ the C^*-subalgebra of $L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ consisting of all functions that vanish at infinity. Then $L_0^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ is the $\|\cdot\|_{\infty,\omega}$-closure of the space of all functions in $L^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ with compact support; for more details, see [16] by the authors and Rejali. In the case where $\mathfrak{S}$ is a locally compact group and $\omega(x) = 1$ for all $x \in \mathfrak{S}$, $L_0^\infty(\mathfrak{S}) := L_0^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$ has been introduced and studied by Lau and Pym [12].

We denote by $\mathcal{A}$ the set of increasing sequences of compact subsets in $\mathfrak{S}$ and by $\mathcal{B}$ the set of increasing sequences (b_n) of real numbers in $(0,\infty)$ with $b_n \to \infty$. For any $(A_n) \in \mathcal{A}$ and $(b_n) \in \mathcal{B}$, set

$$U((A_n),(b_n)) = \left\{ \mu \in M_a(\mathfrak{S},\omega) : \int_{A_n} \omega \, d|\mu| \le b_n \text{ for all } n \ge 1 \right\};$$

recall from the authors [13] that $U((A_n),(b_n))$ is a convex balanced absorbing set in the space $M_a(\mathfrak{S},\omega)$, and that the family $\mathcal{U}$ of all sets $U((A_n),(b_n))$ for $(A_n) \in \mathcal{A}$ and $(b_n) \in \mathcal{B}$, is a base of neighborhoods of zero for a locally convex topology $\beta^1(\mathfrak{S},\omega)$ on $M_a(\mathfrak{S},\omega)$ called *strict topology*. In the case where $\mathfrak{S}$ is a locally compact group and $\omega(x) = 1$ for all $x \in \mathfrak{S}$, this topology has been introduced and studied by Singh

[**20**]. Moreover, another locally convex topology on group algebras has been introduced and investigated by Grosser et al. [**8**]; see also Grosser [**7**] for a similar study on Banach modules.

Denote by $\sigma_0(\mathfrak{S},\omega)$ the weak topology $\sigma(M_a(\mathfrak{S},\omega), \varrho(L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))))$ and by $n(\mathfrak{S},\omega)$ the norm topology of $M_a(\mathfrak{S},\omega)$. Note that

$$\sigma_0(\mathfrak{S},\omega) \leq \beta^1(\mathfrak{S},\omega) \leq n(\mathfrak{S},\omega),$$

and therefore,

$$\varrho(L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))) \subseteq (M_a(\mathfrak{S},\omega), \beta^1(\mathfrak{S},\omega))^* \subseteq (M_a(\mathfrak{S},\omega), n(\mathfrak{S},\omega))^*.$$

In the case where $\mathfrak{S}$ is a foundation semigroup with identity, we have shown in [**13**] that $\beta^1(\mathfrak{S},\omega) = n(\mathfrak{S},\omega)$ if and only if $\mathfrak{S}$ is compact, and $\sigma_0(\mathfrak{S},\omega) = \beta^1(\mathfrak{S},\omega)$ if and only if $\mathfrak{S}$ is finite. In particular, if $\mathfrak{S}$ is infinite, then infinitely many locally convex topologies τ on $M_a(\mathfrak{S},\omega)$ exist with $\sigma_0(\mathfrak{S},\omega) \leq \tau \leq \beta^1(\mathfrak{S},\omega)$.

We now state the main result of the authors [**13**] which we need in the next section; first, let us denote by ϱ_0 the restriction of ϱ to $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$.

Theorem 1.1. *Let $\mathfrak{S}$ be a foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \leq \tau \leq \beta^1(\mathfrak{S},\omega)$. Then ϱ_0 is an identification between the Banach space $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ and the strong dual of $(M_a(\mathfrak{S},\omega),\tau)$. In particular, the adjoint ϱ_0^* of ϱ_0 is an identification between $(M_a(\mathfrak{S},\omega),\tau)^{**}$ and $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))^*$.*

In the present paper, we shall be concerned with the second dual of $M_a(\mathfrak{S},\omega)$ equipped with a locally convex topology τ such that $\sigma_0(\mathfrak{S},\omega) \leq \tau \leq \beta^1(\mathfrak{S},\omega)$. We first introduce and study an Arens multiplication on the second dual of $(M_a(\mathfrak{S},\omega),\tau)$. We then investigate the topological center of $(M_a(\mathfrak{S},\omega),\tau)$ for an extensive class of locally compact semigroups $\mathfrak{S}$. In particular, we obtain several results on Arens regularity and strong Arens irregularity of $(M_a(\mathfrak{S},\omega),\tau)$. It should be noted that the topological center of the second dual $\ell^1(\mathfrak{S})^{**}$ of $\ell^1(\mathfrak{S}) := \ell^1(\mathfrak{S},1)$ for a discrete semigroup $\mathfrak{S}$ has been studied by Dales, Lau and Strauss [**5**] and Lau [**10**].

2. Second dual of $M_a(\mathfrak{S},\omega)$ with strict topology. Let $\mathfrak{S}$ be a locally compact semigroup and ω be a weight function on $\mathfrak{S}$. Recall that $\mathfrak{S}$ is said to be *compactly cancelative* if $C^{-1}D$ and CD^{-1} are compact subsets of $\mathfrak{S}$ for all compact subsets C and D of $\mathfrak{S}$, where

$$C^{-1}D = \{s \in \mathfrak{S} : cs \in D \text{ for some } c \in C\}$$

and

$$CD^{-1} = \{s \in \mathfrak{S} : sd \in C \text{ for some } d \in D\}.$$

Now, suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. For each $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$ and $\mu \in M_a(\mathfrak{S},\omega)$, the functional $\phi\mu$ on $M_a(\mathfrak{S},\omega)$ is defined on $M_a(\mathfrak{S},\omega)$ by

$$\langle \phi\mu, \nu\rangle = \langle \phi, \mu * \nu\rangle \quad (\nu \in M_a(\mathfrak{S},\omega)).$$

Furthermore, for each $\Phi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$, the functional $\Phi\phi$ on $M_a(\mathfrak{S},\omega)$ is defined by

$$\langle \Phi\phi, \mu\rangle = \langle \Phi, \phi\mu\rangle \quad (\mu \in M_a(\mathfrak{S},\omega)).$$

We begin with the following key lemma which shows that $\phi\mu$ and $\Phi\phi$ are well defined and belong to $(M_a(\mathfrak{S},\omega),\tau)^*$.

Lemma 2.1. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$ and $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$. Then*

(i) *$\phi\mu \in (M_a(\mathfrak{S},\omega),\tau)^*$ for all $\mu \in M_a(\mathfrak{S},\omega)$.*

(ii) *$\Phi\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$ for all $\Phi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$.*

Proof. (i) Let $\mu \in M_a(\mathfrak{S},\omega)$. First, note that $\phi\mu \in (M_a(\mathfrak{S},\omega), n(\mathfrak{S},\omega))^*$ and thus $\varrho^{-1}(\phi\mu) \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. Also, for each $\nu \in M(\mathfrak{S},\omega)$ we have

$$\begin{aligned}\int_{\mathfrak{S}} \varrho^{-1}(\phi\mu)(x)\,d\nu(x) &= \langle \phi\mu, \nu\rangle \\ &= \langle \phi, \mu * \nu\rangle \\ &= \int_{\mathfrak{S}}\int_{\mathfrak{S}} \varrho_0^{-1}(\phi)(yx)\,d\mu(y)\,d\nu(x).\end{aligned}$$

We therefore have

$$\varrho^{-1}(\phi\mu)(x) = \int_{\mathfrak{S}} \varrho_0^{-1}(\phi)(yx)\, d\mu(y)$$

for ν-almost all $x \in \mathfrak{S}$ ($\nu \in M_a(\mathfrak{S}, \omega)$), and hence

$$\varrho^{-1}(\phi\mu) \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega));$$

indeed, it is known from [**16**, Proposition 2.3] that the function

$$x \longmapsto \int_{\mathfrak{S}} \psi(yx)\, d\mu(y)$$

is in $C_0(\mathfrak{S}, 1/\omega)$ for all $\psi \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$. That is, $\phi\mu \in (M_a(\mathfrak{S}, \omega), \tau)^*$.

(ii) Without loss of generality, we may assume that $\varrho_0^{-1}(\phi)$ is a non-negative function in $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$. Let $\Phi \in (M_a(\mathfrak{S}, \omega), \tau)^{**}$; to prove that $\Phi\phi \in (M_a(\mathfrak{S}, \omega), \tau)^*$, we also may assume that $\varrho_0^*(\Phi)$ is a positive functional on $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$. In view of Theorem 1.1, $\phi \in (M_a(\mathfrak{S}, \omega), n(\mathfrak{S}, \omega))^*$ and

$$\|\phi\| = \|\varrho_0^{-1}(\phi)\|_{\infty,\omega}.$$

For any $\nu \in M_a(\mathfrak{S}, \omega)$,

$$\begin{aligned}
|\langle \Phi\phi, \nu\rangle| &= |\langle \Phi, \phi\nu\rangle| \\
&= |\langle \varrho_0^*(\Phi), \varrho_0^{-1}(\phi\nu)\rangle| \\
&\le \|\varrho_0^*(\Phi)\|\, \|\varrho_0^{-1}(\phi\nu)\|_{\infty,\omega} \\
&\le \|\varrho_0^*(\Phi)\|\, \|\phi\|\, \|\nu\|_\omega.
\end{aligned}$$

It follows that $\Phi\phi \in (M_a(\mathfrak{S}, \omega), \|\cdot\|_\omega)^*$. Since $\mathfrak{S}$ is a foundation semigroup with identity,

$$\varrho^{-1}(\Phi\phi) \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega)).$$

It remains to show that $\varrho^{-1}(\Phi\phi)/\omega$ vanishes at infinity.

To show this, note that $\varrho_0^{-1}(\phi) \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$, and so for each $0 < \varepsilon < 1$, there is a compact subset A of $\mathfrak{S}$ with $\varrho_0^{-1}(\phi)(t) < \varepsilon\, \omega(t)$

for μ-almost all $t \in \mathfrak{S} \setminus A$ ($\mu \in M_a(\mathfrak{S},\omega)$). Choose a functional $\Psi \in L_0^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))^*$ and a compact set B in $\mathfrak{S}$ such that

$$\|\varrho_0^*(\Phi) - \Psi\| < \varepsilon \quad \text{and} \quad \langle \Psi, \xi \rangle = \langle \Psi, \chi_B \, \xi \rangle$$

for all $\xi \in L_0^\infty(\mathfrak{S}; M_a(\mathfrak{S},\omega))$, see [**16**, Proposition 2.4]. Then, for each positive measure $\sigma \in M_a(\mathfrak{S},\omega)$ with $\|\sigma\|_\omega = 1$ and $\operatorname{supp}(\sigma) \subseteq \mathfrak{S} \setminus AB^{-1}$, there is a compact subset C of $\mathfrak{S}$ for which

$$C \subseteq \mathfrak{S} \setminus AB^{-1} \quad \text{and} \quad (\omega\sigma)(\mathfrak{S} \setminus C) < \varepsilon.$$

On the one hand, since $C^{-1}A \cap B = \varnothing$, it follows that

$$\|\varrho_0^{-1}(\phi\sigma)\,\chi_B\|_{\infty,\omega} < \varepsilon\,(\|\phi\| + 1);$$

indeed, for each $x \in \mathfrak{S} \setminus C^{-1}A$ we get $Cx \subseteq \mathfrak{S} \setminus A$, and hence,

$$\begin{aligned}
\int_{\mathfrak{S}} \varrho_0^{-1}(\phi)(yx)\,d\sigma(y) &\le \int_{\mathfrak{S}\setminus C} \varrho_0^{-1}(\phi)(yx)\,d\sigma(y) \\
&\quad + \int_C \varrho_0^{-1}(\phi)(yx)\,d\sigma(y) \\
&\le \omega(x) \int_{\mathfrak{S}\setminus C} \frac{\varrho_0^{-1}(\phi)(yx)}{\omega(yx)}\,d(\omega\sigma)(y) \\
&\quad + \omega(x) \int_C \frac{\varrho_0^{-1}(\phi)(yx)}{\omega(yx)}\,d(\omega\sigma)(y) \\
&\le \varepsilon\,\omega(x)\,(\|\varrho_0^{-1}(\phi)\|_{\infty,\omega} + \|\sigma\|_\omega) \\
&\le \varepsilon\,\omega(x)\,(\|\phi\| + 1);
\end{aligned}$$

recall from (i) that

$$\varrho^{-1}(\phi\sigma)(x) = \int_{\mathfrak{S}} \varrho_0^{-1}(\phi)(yx)\,d\sigma(y) \ge 0$$

for ν-almost all $x \in \mathfrak{S}$ ($\nu \in M_a(\mathfrak{S},\omega)$); thus,

$$\varrho^{-1}(\phi\sigma)(x) \le \varepsilon\,\omega(x)\,(\|\phi\| + 1)$$

for ν-almost all $x \in \mathfrak{S} \setminus C^{-1}A$ ($\nu \in M_a(\mathfrak{S},\omega)$).

On the other hand, $\varrho^{-1}(\Phi\phi)$ is a positive function in $L^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$; indeed, for each positive measure $\nu \in M_a(\mathfrak{S},\omega)$ we have $\varrho_0^{-1}(\phi\nu) \geq 0$, and so

$$\begin{aligned}\int_{\mathfrak{S}} \varrho^{-1}(\Phi\phi)(x)\, d\nu(x) &= \langle \Phi\phi, \nu\rangle \\ &= \langle \varrho_0^*(\Phi), \varrho_0^{-1}(\phi\nu)\rangle \\ &\geq 0.\end{aligned}$$

We therefore have

$$\begin{aligned}\int_{\mathfrak{S}\setminus AB^{-1}} \varrho^{-1}(\Phi\phi)(x)\, d\sigma(x) &= \langle \Phi\phi, \sigma\rangle \\ &= \langle \varrho_0^*(\Phi), \varrho_0^{-1}(\phi\sigma)\rangle \\ &\leq |\langle \varrho_0^*(\Phi) - \Psi, \varrho_0^{-1}(\phi\sigma)\rangle| \\ &\quad + |\langle \Psi, \varrho_0^{-1}(\phi\sigma)\, \chi_B\rangle| \\ &\leq \|\varrho_0^*(\Phi) - \Psi\|\, \|\varrho_0^{-1}(\phi\sigma)\|_{\infty,\omega} \\ &\quad + \|\Psi\|\, \|\varrho_0^{-1}(\phi\sigma)\, \chi_B\|_{\infty,\omega} \\ &\leq \varepsilon\, [\|\phi\| + (\|\varrho_0^*(\Phi)\| + 1)(\|\phi\| + 1)].\end{aligned}$$

This shows that, if $\nu \in M_a(\mathfrak{S},\omega)$, then

$$\varrho^{-1}(\Phi\phi)(x) \leq \varepsilon\, \omega(x)\, [\|\phi\| + (\|\varrho_0^*(\Phi)\| + 1)(\|\phi\| + 1)].$$

for ν-almost all $x \in \mathfrak{S}\setminus AB^{-1}$; otherwise, there exist a positive measure $\sigma \in M_a(\mathfrak{S},\omega)$ and a σ-measurable set $D \subseteq \mathfrak{S} \setminus AB^{-1}$ with $\sigma(D) > 0$, $\|\sigma\|_\omega = 1$ and $\operatorname{supp}(\sigma) \subseteq D$ such that

$$\varrho^{-1}(\Phi\phi)(x) > \varepsilon\, \omega(x)\, [\|\phi\| + (\|\varrho_0^*(\Phi)\| + 1)(\|\phi\| + 1)].$$

for σ-almost all $x \in D$. Therefore,

$$\begin{aligned}\int_{\mathfrak{S}\setminus AB^{-1}} \varrho^{-1}(\Phi\phi)(x)\, d\sigma(x) &\geq \int_D \varrho^{-1}(\Phi\phi)(x)\, d\sigma(x) \\ &> \varepsilon\, [\|\phi\| + (\|\varrho_0^*(\Phi)\| + 1)(\|\phi\| + 1)] \\ &\quad \times \int_D \omega(x)\, d\sigma(x) \\ &= \varepsilon\, [\|\phi\| + (\|\varrho_0^*(\Phi)\| + 1)(\|\phi\| + 1)],\end{aligned}$$

which is a contradiction. It follows that

$$\varrho^{-1}(\Phi\phi) \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega)),$$

whence $\Phi\phi \in (M_a(\mathfrak{S}, \omega), \tau)^*$. $\square$

Proposition 2.2. *Let $\mathfrak{S}$ be a foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Then the convolution product on $M_a(\mathfrak{S}, \omega)$ is separately continuous with respect to the weak topology $\sigma_0(\mathfrak{S}, \omega)$, and the Mackey topology $\mu_0(\mathfrak{S}, \omega)$.*

Proof. The separate continuity of the convolution on $M_a(\mathfrak{S}, \omega)$ in the Mackey topology is an easy consequence of the separate continuity in the weak topology; see, for example, [**23**, Corollary 26.15]. So, we only need to note that the convolution is separately continuous in the weak topology by Lemma 2.1. $\square$

The following example shows that the convolution is, in general, not $\beta^1(\mathfrak{S}, \omega)$-separately continuous in $M_a(\mathfrak{S}, \omega)$ for all foundation semigroups with identity.

Example 2.3. Let $\mathfrak{S} = [1, \infty)$ and $\omega(x) = x$ for all $x \in \mathfrak{S}$. Then $\mathfrak{S}$ with the discrete topology and the operation $xy = \max\{x, y\}$ is a foundation semigroup with identity, and ω is a weight function on $\mathfrak{S}$. It is easy to see that $\mu \mapsto \mu * \delta_1$ is not $\beta^1(\mathfrak{S}, \omega)$-continuous on $M_a(\mathfrak{S}, \omega)$.

Proposition 2.4. *Let $\mathfrak{S}$ be a compactly cancelative semigroup with identity and ω a weight function on $\mathfrak{S}$. Then the convolution product in $M_a(\mathfrak{S}, \omega)$ is $\beta^1(\mathfrak{S}, \omega)$-continuous on bounded sets.*

Proof. Let (μ_α) be a bounded net convergent to zero in $\beta^1(\mathfrak{S}, \omega)$-topology and $\nu \in M_a(\mathfrak{S}, \omega)$. Let also $U((A_n), (b_n))$ be an arbitrary $\beta^1(\mathfrak{S}, \omega)$-neighborhood of zero. Choose compact set C with

$$|\nu|(\mathfrak{S} \setminus C) < b_1/2M,$$

where M is a bound for (μ_α). Set

$$V := U((A_n C^{-1}), (b_n/2\|\nu\|_\omega)).$$

Let α_0 be such that $\mu_\alpha \in V$ for all $\alpha \geq \alpha_0$. Then, for each $\alpha \geq \alpha_0$, we have

$$\begin{aligned}
|\mu_\alpha * \nu|(A_n) &\leq (|\mu_\alpha| * |\nu|)(A_n) \\
&= \int_C |\mu_\alpha|(A_n y^{-1})\, d|\nu|(y) + \int_{\mathfrak{S}\setminus C} |\mu_\alpha|(A_n y^{-1})\, d|\nu|(y) \\
&\leq \int_C |\mu_\alpha|(A_n C^{-1})\, \omega(y)\, d|\nu|(y) \\
&\quad + \int_{\mathfrak{S}\setminus C} |\mu_\alpha|(A_n y^{-1})\, \omega(y)\, d|\nu|(y) \\
&\leq \|\nu\|_\omega \frac{b_n}{2\|\nu\|_\omega} + \|\mu_\alpha\|_\omega \frac{b_1}{2M} \\
&\leq b_n.
\end{aligned}$$

Hence, $\mu_\alpha * \nu$ converges to zero in $\beta^1(\mathfrak{S}, \omega)$-topology. $\square$

Theorem 2.5. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S}, \omega)$ with $\sigma_0(\mathfrak{S}, \omega) \leq \tau \leq \beta^1(\mathfrak{S}, \omega)$. Then $(M_a(\mathfrak{S}, \omega), \tau)^{**}$ with the first Arens product $\odot$ can be identified with a Banach algebra, where $\Phi \odot \Psi$ is defined by the equation $\langle \Phi \odot \Psi, \phi\rangle = \langle \Phi, \Psi\phi\rangle$ for all $\Phi, \Psi \in (M_a(\mathfrak{S}, \omega), \tau)^{**}$ and $\phi \in (M_a(\mathfrak{S}, \omega), \tau)^*$.*

Proof. We only need to show that $\Phi \odot \Psi \in (M_a(\mathfrak{S}, \omega), \tau)^{**}$. First, note that $\Phi \odot \Psi$ is well defined by Lemma 2.1. Now, for each $\psi \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$ we have

$$\begin{aligned}
\langle \Phi \odot \Psi, \varrho_0(\psi)\rangle &= \langle \Phi, \Psi \varrho_0(\psi)\rangle \\
&= \langle \varrho_0^*(\Phi), \varrho_0^{-1}(\Psi \varrho_0(\psi))\rangle;
\end{aligned}$$

moreover, it follows easily that

$$\|\varrho_0^{-1}(\Psi \varrho_0(\psi))\|_{\infty,\omega} \leq \|\varrho_0^*(\Phi)\|\, \|\psi\|_{\infty,\omega}.$$

So, the linear functional Υ on $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S}, \omega))$ defined by

$$\Upsilon(\psi) = \langle \Phi \odot \Psi, \varrho_0(\psi)\rangle$$

for $\psi \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ is bounded by $\|\varrho_0^*(\Phi)\|\,\|\varrho_0^*(\Psi)\|$. In particular, $\Upsilon \in L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))^*$, and therefore $\Phi \odot \Psi = \varrho_0^{*-1}(\Upsilon) \in (M_a(\mathfrak{S},\omega),\tau)^{**}$. $\square$

In the following, denote by $L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ the C^*-subalgebra of those functions in $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ with continuous representatives.

Lemma 2.6. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. Then*

$$\varrho_0^{-1}(\phi\mu), \varrho_0^{-1}(\mu\phi) \in L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$$

for all $\mu \in M_a(\mathfrak{S},\omega)$ and $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$.

Proof. First, note that $\varrho_0^{-1}(\phi\mu)(x) = \langle \phi, \mu * \delta_x\rangle$ for ν-almost all $x \in \mathfrak{S}$ ($\nu \in M_a(\mathfrak{S},\omega)$); indeed,

$$\begin{aligned}\int_{\mathfrak{S}} \varrho_0^{-1}(\phi\mu)(x)\,d\nu(x) &= \langle \upsilon, \varrho_0^{-1}(\phi\mu)\rangle \\ &= \langle \phi\mu, \nu\rangle \\ &= \langle \phi, \mu * \nu\rangle \\ &= \int_{\mathfrak{S}} \langle \phi, \mu * \delta_x\rangle\,d\nu(x).\end{aligned}$$

Lemma 2.1 together with the weak continuity of the mapping $x \mapsto \mu*\delta_x$ from $\mathfrak{S}$ into $M(\mathfrak{S},\omega)$ imply that the function $x \mapsto \langle\phi, \mu * \delta_x\rangle$ is continuous on $\mathfrak{S}$; see [**9**]. Thus,

$$\varrho_0^{-1}(\phi\mu) \in L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega)).$$

Similarly, $\varrho_0^{-1}(\mu\phi) \in L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. $\square$

Proposition 2.7. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. Then $(M_a(\mathfrak{S},\omega),\tau)$ is a closed ideal in its second dual equipped with strong topology.*

Proof. That $M_a(\mathfrak{S},\omega)$ is closed in $(M_a(\mathfrak{S},\omega),\tau)^{**}$ follows from Theorem 1.1 and the fact that $\varrho_0^*(M_a(\mathfrak{S},\omega))$ is closed in $L_0^\infty(\mathfrak{S},M_a(\mathfrak{S},\omega))^*$.

Now, suppose that $\mu\in M_a(\mathfrak{S},\omega)$ and $\Phi\in(M_a(\mathfrak{S},\omega),\tau)^{**}$. We show that
$$\mu\odot\Phi\in(M_a(\mathfrak{S},\omega),\tau)^{**};$$
that $\Phi\odot\mu\in(M_a(\mathfrak{S},\omega),\tau)^{**}$ is similar. Since $M_a(\mathfrak{S},\omega)$ is an ideal in $M(\mathfrak{S},\omega)$, we have $\mu*\sigma\in M_a(\mathfrak{S},\omega)$, where σ is the restriction of $\varrho_0^*(\Phi)$ to $C_0(\mathfrak{S},1/\omega)$. So it suffices to show that
$$\mu\odot\Phi=\mu*\sigma.$$
To that end, let $\phi\in(M_a(\mathfrak{S},\omega),\tau)^*$. By Lemma 2.6 and its proof we have
$$\begin{aligned}\langle\mu\odot\Phi,\phi\rangle&=\langle\Phi,\phi\mu\rangle\\&=\langle\varrho_0^*(\Phi),\varrho_0^{-1}(\phi\mu)\rangle\\&=\int_{\mathfrak{S}}\langle\phi,\mu*\delta_x\rangle\,d\sigma(x)\\&=\int_{\mathfrak{S}}\langle\varrho_0^{-1}(\phi),\mu*\delta_x\rangle\,d\sigma(x)\\&=\int_{\mathfrak{S}}\int_{\mathfrak{S}}\varrho_0^{-1}(\phi)(yx)\,d\mu(y)\,d\sigma(x)\\&=\int_{\mathfrak{S}}\varrho_0^{-1}(\phi)(t)\,d(\mu*\sigma)(t)\\&=\langle\varrho_0^{-1}(\phi),\mu*\nu\rangle\\&=\langle\mu*\sigma,\phi\rangle.\end{aligned}$$
That is, $\mu\odot\Phi=\mu*\sigma$ as required. $\square$

3. Topological center of $M_a(\mathfrak{S},\omega)$ with strict topology. Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega)\le\tau\le\beta^1(\mathfrak{S},\omega)$. For any Ψ in $(M_a(\mathfrak{S},\omega),\tau)^{**}$, the map $\Phi\mapsto\Phi\odot\Psi$ is weak*-weak* continuous on $(M_a(\mathfrak{S},\omega),\tau)^{**}$. For an element Φ in $(M_a(\mathfrak{S},\omega),\tau)^{**}$, the map $\Psi\mapsto\Phi\odot\Psi$ is in general not weak*-weak* continuous on $(M_a(\mathfrak{S},\omega),\tau)^{**}$ unless Φ is in $M_a(\mathfrak{S},\omega)$.

The topological center of $(M_a(\mathfrak{S},\omega),\tau)^{**}$ with respect to $\odot$ is denoted by
$$\mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$$

and is defined to be the set of all $\Phi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$ for which the map $\Psi \mapsto \Phi \odot \Psi$ is weak*-weak* continuous on $(M_a(\mathfrak{S},\omega),\tau)^{**}$.

We are now ready to give the main result of this section.

Theorem 3.1. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. If $\Phi \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$ and μ is the restriction of $\varrho_0^*(\Phi)$ to $C_0(\mathfrak{S},1/\omega)$, then $\varrho_0^{-1}(\phi\mu) \in L^\infty_{0,c}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ for all $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$.*

Proof. Let $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$ and $\nu \in M_a(\mathfrak{S},\omega)$. Since

$$\varrho_0^{-1}(\phi\nu) \in L^\infty_{0,c}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$$

by Lemma 2.6, it follows that

$$\begin{aligned}\langle \nu, \phi\mu\rangle &= \langle \phi, \mu * \nu\rangle \\ &= \langle \mu, \varrho_0^{-1}(\nu\phi)\rangle \\ &= \langle \varrho_0^*(\Phi), \varrho_0^{-1}(\nu\phi)\rangle \\ &= \langle \Phi \odot \nu, \phi\rangle.\end{aligned}$$

Now, let $\Psi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$ and choose a net (ν_γ) in $M_a(\mathfrak{S},\omega)$ such that $\nu_\gamma \to \Psi$ in the weak* topology of $(M_a(\mathfrak{S},\omega),\tau)^{**}$. Since $\Phi \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$, the map $\Upsilon \mapsto \Phi \odot \Upsilon$ is weak*-weak* continuous on $(M_a(\mathfrak{S},\omega),\tau)^{**}$ and thus

$$\begin{aligned}\langle \Psi, \phi\mu\rangle &= \lim_\gamma \langle \nu_\gamma, \phi\mu\rangle \\ &= \lim_\gamma \langle \Phi \odot \nu_\gamma, \phi\rangle \\ &= \langle \Phi \odot \Psi, \phi\rangle.\end{aligned}$$

So, if (μ_α) is a net in $M_a(\mathfrak{S},\omega)$ with $\mu_\alpha \to \Phi$ in the weak* topology of $(M_a(\mathfrak{S},\omega),\tau)^{**}$, then

$$\langle \Psi, \phi\mu\rangle = \lim_\alpha \langle \Psi, \phi\mu_\alpha\rangle;$$

that is,

$$\langle \varrho_0^*(\Psi), \varrho_0^{-1}(\phi\mu)\rangle = \lim_\alpha \langle \varrho_0^*(\Psi), \varrho_0^{-1}(\phi\mu_\alpha)\rangle.$$

Since elements of $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))^*$ are of the form $\varrho_0^*(\Psi)$ for some Ψ in the second dual of $(M_a(\mathfrak{S},\omega),\tau)$, it follows that

$$\varrho_0^{-1}(\phi\mu_\alpha) \longrightarrow \varrho_0^{-1}(\phi\mu)$$

in the weak topology of $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. According to Lemma 2.6,

$$\varrho_0^{-1}(\phi\mu_\alpha) \in L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$$

for all α. Since $L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ is weakly closed in $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$, we conclude that $\varrho_0^{-1}(\phi\mu) \in L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. $\square$

Corollary 3.2. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. If $\Phi \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$ and μ is the restriction of $\varrho_0^*(\Phi)$ to $C_0(\mathfrak{S},1/\omega)$, then the function $x \mapsto \mu(Cx^{-1})$ is in $L_{0,c}^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ for all compact subsets C of $\mathfrak{S}$.*

Proof. Since $\varrho_0^{-1}(\varrho_0(\chi_C)\,\mu)(x) = \mu(Cx^{-1})$ for ν-almost all $x \in \mathfrak{S}$ ($\nu \in M_a(\mathfrak{S},\omega)$), the result follows from Theorem 3.1. $\square$

Let $\mathfrak{S}$, ω and τ be as in Theorem 2.5. The algebra $(M_a(\mathfrak{S},\omega),\tau)$ is called *Arens regular* if the map $\Psi \mapsto \Phi \odot \Psi$ is weak*-weak* continuous on $(M_a(\mathfrak{S},\omega),\tau)^{**}$ for all $\Phi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$, i.e.,

$$\mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**}) = (M_a(\mathfrak{S},\omega),\tau)^{**}.$$

As a consequence of Theorem 3.1, we obtain a necessary condition for Arens regularity of $(M_a(\mathfrak{S},\omega),\beta^1(\mathfrak{S},\omega))$. Arens regularity of $(M_a(\mathfrak{S},\omega),n(\mathfrak{S},\omega))$ has recently been studied by the authors and Rejali [**16**]; see also Dzinotyiweyi [**6**] and Rejali [**18**] for locally compact semigroups and Baker and Rejali [**2**] and Craw and Young [**3**] for discrete semigroups.

Corollary 3.3. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. If $(M_a(\mathfrak{S},\omega),\tau)$ is Arens regular, then*

$$\varrho_0^{-1}(\phi\mu), \varrho_0^{-1}(\mu\phi) \in L_{0,c}^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$$

for all $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$ and $\mu \in M(\mathfrak{S},\omega)$. In particular, the functions $x \mapsto \mu(Cx^{-1})$ and $x \mapsto \mu(x^{-1}C)$ are in $L_{0,c}^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ for all $\mu \in M(\mathfrak{S},\omega)$ and compact subsets C of $\mathfrak{S}$.

Proof. Let $\phi \in (M_a(\mathfrak{S},\omega),\tau)^*$ and $\mu \in M(\mathfrak{S},\omega)$. Let $m \in L_0^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))^*$ be an extension of μ from $C_0(\mathfrak{S},1/\omega)$ to $L_0^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. Then, by assumption,

$$\Phi := \varrho_0^{*-1}(m) \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**}).$$

So, by Lemma 3.1,

$$\varrho_0^{-1}(\phi\mu) \in L_{0,c}^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega)).$$

Now, let (μ_α) be a net in $M_a(\mathfrak{S},\omega)$ with $\mu_\alpha \to \Phi$ in the weak* topology of $(M_a(\mathfrak{S},\omega),\tau)^{**}$. Then, for any $\Psi \in (M_a(\mathfrak{S},\omega),\tau)^{**}$, we have $\Psi \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$, and therefore

$$\begin{aligned}\langle \Psi, \Phi\phi\rangle &= \langle \Psi \odot \Phi, \phi\rangle \\ &= \lim_{\alpha} \langle \Psi \odot \mu_\alpha, \phi\rangle \\ &= \lim_{\alpha} \langle \Psi, \mu_\alpha\phi\rangle.\end{aligned}$$

It follows that

$$\langle \varrho_0^*(\Psi), \varrho_0^{-1}(\Phi\phi)\rangle = \lim_{\alpha} \langle \varrho_0^*(\Psi), \varrho_0^{-1}(\mu_\alpha\phi)\rangle.$$

Thus,

$$\varrho_0^{-1}(\mu_\alpha\phi) \longrightarrow \varrho_0^{-1}(\Phi\phi)$$

in the weak topology of $L_0^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$. In view of Lemma 2.1, we have $\varrho_0^{-1}(\mu_\alpha\phi) \in L_{0,c}^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega))$ for all α. Consequently,

$$\varrho_0^{-1}(\Phi\phi) \in L_{0,c}^{\infty}(\mathfrak{S}, M_a(\mathfrak{S},\omega)).$$

The proof will be complete if we note that $\varrho_0^{-1}(\Phi\phi)$ is identical to the function $\varrho_0^{-1}(\mu\phi)$. For the last part, we only need to note that, for all $\mu \in M(\mathfrak{S},\omega)$ and compact subsets C of $\mathfrak{S}$, we have

$$\varrho_0^{-1}(\varrho_0(\chi_C)\,\mu)(x) = \mu(Cx^{-1})$$

and

$$\varrho_0^{-1}(\mu\,\varrho_0(\chi_C))(x) = \mu(x^{-1}C)$$

for ν-almost all $x \in \mathfrak{S}$ $(\nu \in M_a(\mathfrak{S},\omega))$. □

Let us recall that, for a semigroup $\mathfrak{S}$ with an identity element e, the group of units of $\mathfrak{S}$ is the set

$$\mathfrak{H}(e) := \{x \in \mathfrak{S} : \text{there is a } y \in \mathfrak{S} \text{ such that } xy = yx = e\}.$$

Now, let $\mathfrak{S}$, ω and τ be as in Theorem 2.5. The algebra $(M_a(\mathfrak{S},\omega),\tau)$ is called *strongly Arens irregular* if

$$\mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**}) = (M_a(\mathfrak{S},\omega),\tau).$$

In the case where $\mathfrak{S}$ is a locally compact group, strongly Arens irregularity of $M_a(\mathfrak{S},\omega)$ endowed with the norm topology has been studied by Dales and Lau [**4**] and Neufang [**17**].

Theorem 3.4. *Let $\mathfrak{S}$ be a compactly cancelative foundation semigroup with identity e such that $\mathfrak{H}(e)$ is open and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. Then $(M_a(\mathfrak{S},\omega),\tau)$ is strongly Arens irregular.*

Proof. Let $\Phi \in \mathcal{Z}_1((M_a(\mathfrak{S},\omega),\tau)^{**})$ and μ be the restriction of $\varrho_0^*(F)$ to $C_0(\mathfrak{S},1/\omega)$. It is sufficient to show that $\mu \in M_a(\mathfrak{S},\omega)$. It follows from Corollary 3.2 that the function $x \mapsto \mu(Cx^{-1})$ is in $L_{0,c}^{\infty}(\mathfrak{S},M_a(\mathfrak{S},\omega))$ for all relatively compact subsets C of $\mathfrak{S}$. In particular, $x \mapsto \mu(Cx^{-1})$ is equal almost everywhere to a continuous function on $\mathfrak{S}$ for all relatively compact subsets of $\mathfrak{S}$. Now, Theorem 4.4 in [**21**] implies that

$$\mu * \delta_x \in M_a(\mathfrak{S})$$

for all $x \in \mathring{\mathfrak{S}}$, where $\mathring{\mathfrak{S}}$ consists of all $x \in \mathfrak{S}$ that for every neighborhood U of x, the set $U^{-1}x \cap xU^{-1}$ is a neighborhood of e. Since $\mathfrak{H}(e)$ is open, $e \in \mathring{\mathfrak{S}}$ by Theorem 9.18 of [**22**]. Therefore, $\mu \in M_a(\mathfrak{S})$. This, together with the fact that $M_a(\mathfrak{S})$ is solid, implies that $\mu \in M_a(\mathfrak{S}, \omega)$. □

As a consequence of Theorem 3.4, we have the following result.

Corollary 3.5. *Let $\mathfrak{S}$ be a compact foundation semigroup with identity such that $\mathfrak{H}(e)$ is open. Then $(M_a(\mathfrak{S}), \|\cdot\|)$ is strongly Arens irregular, i.e.,*

$$\mathcal{Z}_1((M_a(\mathfrak{S}), \|\cdot\|)^{**}) = (M_a(\mathfrak{S}), \|\cdot\|).$$

Example 3.6. Let $\mathfrak{T}$ be a discrete finite semigroup with identity and $\mathfrak{G}$ a compact Hausdorff topological group. Let $\mathfrak{S} = \mathfrak{G} \times \mathfrak{T}$ be the direct product semigroup of $\mathfrak{G}$ and $\mathfrak{T}$. Then $\mathfrak{S}$ is a compact foundation semigroup with identity e for which $\mathfrak{H}(e)$ is open. Corollary 3.5 shows that $(M_a(\mathfrak{S}), \|\cdot\|)$ is strongly Arens irregular.

As another special consequence of Theorem 3.4, we have the main result of [**15**].

Corollary 3.7. *Let $\mathfrak{S}$ be a locally compact group and ω a weight function on $\mathfrak{S}$. Then $(M_a(\mathfrak{S}, \omega), \tau)$ is strongly Arens irregular for all locally convex topologies τ on $M_a(\mathfrak{S}, \omega)$ such that $\sigma_0(\mathfrak{S}, \omega) \le \tau \le \beta^1(\mathfrak{S}, \omega)$.*

The Arens regularity of $\ell^1(\mathfrak{S}, \omega)$ with the norm topology has been studied by several authors; see for example, Craw and Young [**3**] and Baker and Rejali [**2**]. As a consequence of Theorem 1.1, we have the following result.

Proposition 3.8. *Let $\mathfrak{S}$ be a discrete semigroup with identity and ω a weight function on $\mathfrak{S}$. Suppose that τ is a locally convex topology* *on* $\ell^1(\mathfrak{S},\omega)$ *with* $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. *Then*

$$\ell^1(\mathfrak{S},\omega) = \mathscr{Z}_1((\ell^1(\mathfrak{S},\omega),\tau)^{**}) = (\ell^1(\mathfrak{S},\omega),\tau)^{**}.$$

In particular, $(\ell^1(\mathfrak{S},\omega),\tau)$ *is Arens regular.*

Proposition 3.9. *Let* $\mathfrak{S}$ *be a compactly cancelative foundation semigroup with identity* e *such that* $\mathfrak{H}(e)$ *is open and* ω *a weight function on* $\mathfrak{S}$*. Suppose that* τ *is a locally convex topology on* $M_a(\mathfrak{S},\omega)$ *with* $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$. *Then* $(M_a(\mathfrak{S},\omega),\tau)$ *is Arens regular if and only if* $\mathfrak{S}$ *is discrete.*

Proof. The "if" part follows from Proposition 3.8. For the converse, let u be an element of $L_0^\infty(\mathfrak{S}, M_a(\mathfrak{S},\omega))^*$ with $\langle u,\xi\rangle = \xi(e)$ for all $\xi \in C_c(\mathfrak{S})$, the space of continuous functions with compact support. Theorem 1.1 together with the assumption implies that $u = \varrho^*(\mu)$ for some $\mu \in M_a(\mathfrak{S},\omega)$. In particular,

$$\xi(e) = \langle u,\xi\rangle = \langle \mu, \varrho(\xi)\rangle = \langle \mu,\xi\rangle$$

for all $\xi \in C_c(\mathfrak{S})$. It follows that $\mu = \delta_e$, the Dirac measure at e on $\mathfrak{S}$. Thus, $\delta_e \in M_a(\mathfrak{S})$; that is, $\mathfrak{S}$ is discrete; see [**1**, Theorem 2.8]. □

In conclusion, let us mention two natural conjectures for a compactly cancelative foundation semigroup $\mathfrak{S}$ with identity, a weight function ω on $\mathfrak{S}$, and a locally convex topology τ on $M_a(\mathfrak{S},\omega)$ with $\sigma_0(\mathfrak{S},\omega) \le \tau \le \beta^1(\mathfrak{S},\omega)$.

Conjecture 1. $(M_a(\mathfrak{S},\omega),\tau)$ *is Arens regular if and only if* $\mathfrak{S}$ *is discrete.*

Conjecture 2. $(M_a(\mathfrak{S},\omega),\tau)$ *is always strongly Arens irregular.*

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