

WEAKLY PROJECTIVE C^* -ALGEBRAS

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ABSTRACT. The noncommutative analog of an approximative absolute retract (AAR) is introduced, a weakly projective C^* -algebra. This property sits between being residually finite dimensional and projectivity. Examples and closure properties are considered.

1. Introduction. The noncommutative analogs of absolute retracts and absolute neighborhood retracts in the category of C^* -algebras are the projective ([8]) and semiprojective ([2]) C^* -algebras. In applications, semiprojectivity is often not the most desirable property; many authors have looked instead at weak semiprojectivity ([9]). For example, see [7, 11, 13, 21].

Using what are called approximative retracts, Clapp, many years ago in [6], defined approximative absolute retracts (AAR) and approximative absolute neighborhood retracts (AANR). The relation between AANR spaces and weakly semiprojective C^* -algebras will be explored elsewhere. Here, we get started on a noncommutative analog of AAR, the weakly projective C^* -algebra.

The class of weakly projective C^* -algebras has some of the expected closure properties. In addition, weak projectivity for A is enough to imply that A is residually finite dimensional.

In [5] it has been determined which compacta X have $C_0(X \setminus \{x_0\})$ projective—the dendrites. It would be nice to know when $C_0(X \setminus \{x_0\})$ is semiprojective, weakly projective or weakly semiprojective.

The reader is warned that what is called weak projectivity in [18] is weak semiprojectivity.

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There are potentially more definitions and results related to absolute neighborhood retracts than will be interesting when adapted to C^* -algebras. Some places these might be found are [22] and the more classic [4, 12]. For C^* -algebras recently found to be projective, see [5, 16, 20].

2. Approximative absolute retracts (AARs). In defining approximative absolute retracts we follow [6]. Recall that a *compactum* is a compact, metrizable space.

Definition 2.1. A compactum X is an *approximative absolute retract* (AAR) if, whenever X is a closed subset of a compactum Y , there is a sequence r_n of continuous functions $r_n : Y \rightarrow X$ so that

$$\lim_{n \rightarrow \infty} r_n(x) = x$$

uniformly over x in X .

We next use a pushout to get an approximate extension property. This is a variation on an old trick. See [12, Proposition 3.2].

Theorem 2.2. *Let X be a compactum. Then X is an AAR if, and only if, whenever Z is a closed subset of a compactum Y and $f : Z \rightarrow X$ is continuous, there is a sequence g_n of continuous functions $g_n : Y \rightarrow X$ for which $g_n(z) \rightarrow f(z)$ uniformly over z in Z . To summarize in a diagram:*

$$\begin{array}{ccc} & & Y \\ & \nwarrow g_n & \uparrow \\ X & \xleftarrow{f} & Z \end{array}$$

Proof. Suppose X is an AAR and we are given Y , Z and f as indicated. Take the pushout, or adjunction space:

$$\begin{array}{ccc} X \cup_Z Y & \xleftarrow{\iota_2} & Y \\ \uparrow \iota_1 & & \uparrow \\ X & \xleftarrow{f} & Z \end{array}$$

Notice that $X \cup_Z Y$ is a compact metrizable space and that ι_1 is an inclusion. We can apply the definition of AAR and find

$$\bar{r}_n : X \cup_Z Y \longrightarrow X$$

with $\bar{r}_n \circ \iota_1(w) \rightarrow w$ uniformly. Therefore, when z is in Z ,

$$\lim_n \bar{r}_n \circ \iota_2(z) = \lim_n \bar{r}_n \circ \iota_1(f(z)) = f(z)$$

uniformly, so we may set $g_n = \bar{r}_n \circ \iota_2$.

To prove the converse, assume that the second condition holds and that X is a closed subset of a compactum Y . We can find g_n as in this diagram

$$\begin{array}{ccc} & & Y \\ & \nwarrow g_n & \uparrow \\ X & \xleftarrow{\text{id}_X} & X \end{array}$$

with $g_n(x) \rightarrow \text{id}_X(x)$ uniformly for x in X . We set $r_n = g_n$. $\square$

Corollary 2.3. *Suppose X is a compactum. Then X is an AAR if, and only if, for every unital surjection $\pi : B \rightarrow C$ between separable, unital, commutative C^* -algebras, and for every unital $*$ -homomorphism $\varphi : C(X) \rightarrow C$, there is a sequence $\varphi_n : C(X) \rightarrow B$ of unital $*$ -homomorphisms so that $\pi \circ \varphi_n \rightarrow \varphi$.*

Proof. This is straightforward, except perhaps the meaning of the convergence. We require

$$\lim_{n \rightarrow \infty} \|\pi \circ \varphi_n(h) - \varphi(h)\| = 0$$

for each h in $C(X)$. $\square$

Of course, every AR is an AAR. To see examples of AARs that are not AR, we can use the following, a rewording of [6, Theorem 2.3].

Theorem 2.4. *Suppose X is a compactum and that $\theta_n : X \rightarrow X$ is a sequence of continuous functions that converges uniformly to the identity. If each $\theta_n(X)$ is an AAR, then X is an AAR.*

Proof. Let d be a compatible metric on X . Passing to a subsequence we may assume

$$d(\theta_n(x), x) \leq \frac{1}{n}$$

for all n and all x . Suppose X is a closed subset of a compactum Y . We apply Theorem 2.2 to $\theta(X)$ to find continuous r_n as in this diagram,

$$\begin{array}{ccccc} & & & & Y \\ & & & \nearrow r_n & \uparrow \\ X & \longleftrightarrow & \theta_n(X) & \xleftarrow{\theta_n} & X \end{array}$$

with

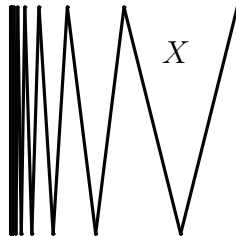
$$d(r_n(x), x) \leq \frac{1}{n}$$

for all x in X . Therefore,

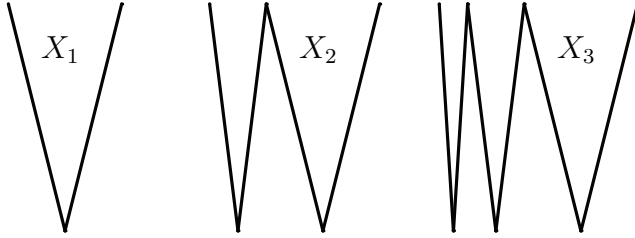
$$d(r_n(x), x) \leq d(r_n(x), \theta_n(x)) + d(\theta_n(x), x) \leq \frac{2}{n}$$

for all x in X . $\square$

Example 2.5 ([6, Example 2.2]). For an AAR that is not an AR, we have the topologist's sine curve



There is an increasing sequence of closed subsets X_n with dense union where each X_n is homeomorphic to a closed interval.



The map $r_n : X \rightarrow X_n$ that sends $X \setminus X_n$ horizontally to the left-most ascending segment in X_n , while fixing X_n , gives us

$$d(r_n(x), x) \leq 2^{-n+1},$$

and so X is an AAR. On the other hand, X is not path connected and so not an AR.

3. Pointed approximative absolute retracts. From the point of view of C^* -algebras, we need not only $C(X)$ for X a compactum, but most importantly also the ideals $C_0(U)$ for open subsets U . We could consider locally compact spaces (see [19]) but instead opt to look at pointed compacta. In terms of C^* -algebras, a pointed space translates to the surjection δ_∞ in the exact sequence

$$0 \longrightarrow C_0(X) \longrightarrow C(X^+) \xrightarrow{\delta_\infty} \mathbf{C} \longrightarrow 0.$$

In the noncommutative case we will of course look at λ in the exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \xrightarrow{\lambda} \mathbf{C} \longrightarrow 0.$$

We use $\tilde{A}$ to mean “add a unit, no matter what.” For a locally compact space X , we use X^+ to denote the one-point compactification. If X is compact, then X^+ has an extra, isolated point.

Definition 3.1. A pointed compactum (X, x_0) is a *pointed approximative absolute retract* if, whenever X is a closed subset of a compactum

Y , there is a sequence r_n of continuous functions $r_n : Y \rightarrow X$ so that

$$r_n(x_0) = x_0$$

for all n and

$$\lim_{n \rightarrow \infty} r_n(x) = x$$

uniformly over x in X .

Lemma 3.2. *Suppose x_0 is any point in a compactum X_0 . If (X, x_0) is a pointed approximative absolute retract, then X is an approximative absolute retract.*

Proof. Ignore x_0 . □

Example 3.3. If X is the topologist's sine curve, and if x_1 is the point on the bottom-left of X as drawn in Example 2.5, then (X, x_1) is not a pointed AAR.

Proof. By definition, X sits as a closed subset of the unit square S . For (X, x_1) to be an AAR, we would need $r_n : S \rightarrow X$ that fix x_1 and that come close to fixing elements of X . The points in X off the left edge are not path connected in X to x_1 , and the continuity of r_n forces $r_n(S)$ to be a subset of that left edge. This is a contradiction. □

Theorem 3.4. *Let X be a compactum and x_0 a point in X . Then (X, x_0) is a pointed AAR if and only if, whenever Z is a closed subset of a compactum Y , z_0 is a point in Z and $f : Z \rightarrow X$ is continuous with $f(z_0) = x_0$, there is a sequence g_n of continuous functions $g_n : Y \rightarrow X$ for which $g_n(z_0) = x_0$ for all n and $g_n(z) \rightarrow f(z)$ uniformly for z in Z .*

Proof. The proof of Theorem 2.2 can be modified as follows. In the adjunction space,

$$\iota_2(z_0) = \iota_1(f(z_0)) = \iota_1(x_0).$$

The $\bar{r}_n$ can now be found with the additional property $\bar{r}_n(\iota_1(x_0)) = x_0$, and so we find

$$g_n(z_0) = \bar{r}_n(\iota_2(z_0)) = \bar{r}_n(\iota_1(x_0)) = x_0. \quad \square$$

Corollary 3.5. *Suppose X is a compactum and x_0 is in X . Then (X, x_0) is a pointed AAR if, and only if, for every unital surjection $\pi : B \rightarrow C$ between separable, commutative C^* -algebras, and for every $*$ -homomorphism*

$$\varphi : C_0(X \setminus \{x_0\}) \longrightarrow C,$$

there is a sequence

$$\varphi_n : C_0(X \setminus \{x_0\}) \longrightarrow B$$

of $$ -homomorphisms so that $\pi \circ \varphi_n \rightarrow \varphi$.*

Proof. For locally compact spaces Λ and Ω , the pointed continuous maps from (Ω^+, ∞) to (Λ^+, ∞) are in one-to-one correspondence with the $*$ -homomorphisms from $C_0(\Lambda)$ to $C_0(\Omega)$. The $*$ -homomorphism $h \mapsto h \circ f$ will be a surjection if and only if $f : \Omega^+ \rightarrow \Lambda^+$ is injective. Convergence in $\text{hom}(C_0(\Lambda), C_0(\Omega))$ corresponds to uniform convergence of functions that preserve the points at infinity. The result follows. $\square$

Theorem 3.6. *Suppose X is a compactum, that $\theta_n : X \rightarrow X$ is a sequence of continuous functions that converges uniformly to the identity and that x_0 is a point in X that is fixed by all the θ_n . If each $(\theta_n(X), x_0)$ is a pointed AAR, then (X, x_0) is a pointed AAR.*

Proof. Just observe that, in the proof of Theorem 2.4, the r_n can now be found by fixing x_0 . $\square$

Example 3.7. If X is the topologist's sine curve, and if x_0 is the point on the top-right of X as drawn in Example 2.5, then (X, x_0) is a pointed AAR.

4. A noncommutative analog of AAR. From Corollary 3.5 we see how to define weak projectivity. In light of Examples 3.3 and 3.7, we will need to take care when dealing with unital C^* -algebras. We will, in fact, never define a notion of “weakly projective in the unital category,” but will define, for not-necessarily-unital C^* -algebras, the notion of “weakly projective relative to unital C^* -algebras.” This rather ruins the analogy with the topology but is more in keeping with how C^* -algebraists work. More than zero of us avoid the unital category for the simple reason that it does not allow for ideals.

Definition 4.1. Suppose A is a separable C^* -algebra. We say A is *weakly projective* if, for every $*$ -homomorphism $\varphi : A \rightarrow C$ and every surjection $\rho : B \rightarrow C$ of arbitrary C^* -algebras, there is a sequence $\varphi_n : A \rightarrow B$ of $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$.

By restricting what surjections ρ is allowed to be, we get weaker properties.

Definition 4.2. Suppose A is a separable C^* -algebra. We say A is *weakly projective with respect to unital C^* -algebras* if, for every $*$ -homomorphism $\varphi : A \rightarrow C$ and every unital surjection $\rho : B \rightarrow C$ between unital C^* -algebras, there is a sequence $\varphi_n : A \rightarrow B$ of $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$.

Obviously projective implies weakly projective and weakly projective implies weakly projective with respect to unital C^* -algebras.

Lemma 4.3. *If A is weakly projective with respect to unital C^* -algebras, then A does not have a unit.*

Proof. Suppose A is unital. Consider the interval over A ,

$$\mathbf{I}A = C([0, 1], A),$$

and the surjection found by evaluation at both endpoints,

$$\delta_0 \oplus \delta_1 : \mathbf{I}A \longrightarrow A \oplus A.$$

The $*$ -homomorphism $\iota_1 : A \rightarrow A \oplus A$ defined by $\iota_1(a) = (0, a)$ should lift approximately to $\psi_n : A \rightarrow \mathbf{I}A$. At 0, $\psi_n(1)$ will be a projection near 0, and so indeed $\psi_n(1)(0) = 0$ for large n . The only thing homotopic to 0 in the space of projections in A is 0 itself, so we conclude $\psi_n(1) = 0$ for large n . Therefore,

$$((\delta_0 \oplus \delta_1) \circ \psi_n)(1) = (0, 0)$$

will not converge to $\iota_1(1) = (0, 1)$. $\square$

Theorem 4.4. *If A is a separable C^* -algebra, then the following are equivalent:*

- (a) *A is weakly projective;*
- (b) *for all separable C^* -algebras B and C and for every $*$ -homomorphism $\varphi : A \rightarrow C$ and every surjection $\rho : B \rightarrow C$, there is a sequence $\varphi_n : A \rightarrow B$ of $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$.*

Proof. Certainly (a) implies (b). For the reverse, suppose (b) holds and $\varphi : A \rightarrow B/I$ is given. Let $a_1, a_2, \dots$ be dense in A . Pick any $b_1, b_2, \dots$ so that $\pi(b_k) = \varphi(a_k)$, and let B_0 denote the C^* -algebra generated by the b_k . This is separable. Let $I_0 = B \cap I$. If we let φ_0 denote φ but with codomain B_0/I_0 , we have the commutative diagram

$$\begin{array}{ccccc} B_0 & \xrightarrow{\iota} & B & & \\ \downarrow \pi_0 & & \downarrow \pi & & \\ A & \xrightarrow{\varphi_0} & B_0/I_0 & \xrightarrow{\iota} & B/I \\ & \searrow \varphi & & \nearrow & \end{array}$$

We know there are $\varphi_n : A \rightarrow B_0$ with $\pi_0 \circ \varphi_n(a) \rightarrow \varphi_0(a)$, and so $\iota \circ \varphi_n$ are the desired approximate lifts. $\square$

Corollary 4.5. *Suppose X is a locally compact, metrizable space. If $C_0(X)$ is weakly projective then (X^+, ∞) is a pointed AAR.*

Example 4.6. If X is the topologist's sine curve, and if x_1 is the point on the bottom-left of X as drawn in Example 2.5, then $C_0(X \setminus \{x_1\})$ is not weakly projective.

Theorem 4.7. *Suppose A is a separable C^* -algebra and that $\theta_n : A \rightarrow A$ is a sequence of $*$ -homomorphisms that converges to the identity map. If each $\theta_n(A)$ is weakly projective, then A is weakly projective. If each $\theta_n(A)$ is weakly projective with respect to unital C^* -algebras, then A is weakly projective with respect to unital C^* -algebras.*

Proof. Assume that the $\theta_n(A)$ are weakly projective. Suppose $\rho : B \rightarrow C$ is a surjection of C^* -algebras, and we are also given a $*$ -homomorphism $\varphi : A \rightarrow C$. If $a_1, a_2, \dots$ is a dense sequence in A , then we can pass to a subsequence of the θ_n so that

$$\|\theta_n(a_j) - a_j\| \leq \frac{1}{n} \quad (1 \leq j \leq n).$$

We are now looking at

$$\begin{array}{ccccc} & & & & B \\ & & & & \downarrow \rho \\ A & \xrightarrow{\theta_n} & \theta_n(A) & \hookrightarrow & A \xrightarrow{\varphi} C \end{array}$$

Since $\theta_n(A)$ is weakly projective, there are $*$ -homomorphisms φ_n as in this diagram

$$\begin{array}{ccccc} & & & & B \\ & & & \nearrow \varphi_n & \downarrow \rho \\ A & \xrightarrow{\theta_n} & \theta_n(A) & \hookrightarrow & A \xrightarrow{\varphi} C \end{array}$$

with

$$\|\rho \circ \varphi_n(\theta_n(a_j)) - \varphi(\theta_n(a_j))\| \leq \frac{1}{n} \quad (1 \leq j \leq n).$$

Then

$$\begin{aligned} \|\rho \circ \varphi_n \circ \theta_n(a_j) - \varphi(a_j)\| &\leq \|\rho \circ \varphi_n(\theta_n(a_j)) - \varphi(\theta_n(a_j))\| \\ &\quad + \|\varphi(\theta_n(a_j) - a_j)\| \\ &\leq \|\rho \circ \varphi_n(\theta_n(a_j)) - \varphi(\theta_n(a_j))\| \\ &\quad + \|\theta_n(a_j) - a_j\| \leq \frac{2}{n}, \end{aligned}$$

and so the $\varphi_n \circ \theta_n$ are the desired approximate lifts.

The proof of the second statement is nearly identical, starting with the extra assumptions that B , C and ρ are unital. $\square$

While Y^+ being an absolute retract does not generally lead to $C_0(Y)$ being projective, we do know that $C_0(0, 1]$ is projective. This is enough to get the following example. One could get more exotic examples by starting with more exotic projective C^* -algebras as seen, for example, in [17].

Example 4.8. If X is the topologist's sine curve, and if x_0 is the point on the top-right of X as drawn in Example 2.5, then $C_0(X \setminus \{x_0\})$ is weakly projective.

Example 4.6 and 4.8 show that it is possible to have $\tilde{A} \cong \tilde{B}$ with A weakly projective and B not weakly projective.

Theorem 4.9. *If A is a separable C^* -algebra, then the following are equivalent:*

- (a) *A is weakly projective with respect to unital C^* -algebras;*
- (b) *for all separable, unital C^* -algebras B and C and for every $*$ -homomorphism $\varphi : A \rightarrow C$ and every unital surjection $\rho : B \rightarrow C$, there is a sequence $\varphi_n : A \rightarrow B$ of $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$;*
- (c) *for all unital C^* -algebras B and C and for every unital $*$ -homomorphism $\varphi : \tilde{A} \rightarrow C$ and every unital surjection $\rho : B \rightarrow C$, there is a sequence $\varphi_n : \tilde{A} \rightarrow B$ of unital $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$;*
- (d) *for all separable, unital C^* -algebras B and C and for every unital $*$ -homomorphism $\varphi : \tilde{A} \rightarrow C$ and every unital surjection $\rho : B \rightarrow C$, there is a sequence $\varphi_n : \tilde{A} \rightarrow B$ of unital $*$ -homomorphisms so that $\rho \circ \varphi_n \rightarrow \varphi$.*

Proof. The proof of Theorem 4.4 works to show the equivalence of (a) and (b) so long as we set B_0 to be the C^* -subalgebra generated by the b_k and 1_B . Just as easily, we get the equivalence of (c) and (d).

Assume (a), and suppose that we are given B and C unital, a unital surjection $\rho : B \rightarrow C$, and $\varphi : \tilde{A} \rightarrow C$ unital. The assumption on A give us the ψ_n in this diagram,

$$\begin{array}{ccccc} & & & B & \\ & & \nearrow \psi_n & \downarrow \rho & \\ A & \xrightarrow{\quad} & \tilde{A} & \xrightarrow{\varphi} & C \end{array}$$

with $\rho \circ \psi_n(a) \rightarrow \varphi(a)$ for all a in A . We can extend ψ_n to a unital $*$ -homomorphism φ_n on $\tilde{A}$ by

$$\varphi_n(a + \alpha \mathbf{1}) = \psi_n(a) + \alpha \mathbf{1}_B.$$

Then

$$\begin{aligned} \rho \circ \varphi_n(a + \alpha \mathbf{1}) &= \rho(\psi_n(a) + \alpha \mathbf{1}_B) \\ &= \rho(\psi_n(a)) + \alpha \varphi(\mathbf{1}) \\ &\rightarrow \varphi(a) + \alpha \varphi(\mathbf{1}) \\ &= \varphi(a + \alpha \mathbf{1}), \end{aligned}$$

and we have verified (c).

Assume (c), and suppose B and C are unital and we are given a $*$ -homomorphism $\varphi : A \rightarrow C$ and a unital surjection $\rho : B \rightarrow C$. We can extend φ to a unital $\overline{\varphi}$ by

$$\overline{\varphi}(a + \alpha \mathbf{1}) = \varphi(a) + \alpha \mathbf{1}_C.$$

The assumption on A now gives us the unital $*$ -homomorphisms ψ_n in this diagram,

$$\begin{array}{ccccc} & & & B & \\ & & \nearrow \psi_n & \downarrow \rho & \\ A & \xrightarrow{\quad} & \tilde{A} & \xrightarrow{\overline{\varphi}} & C \\ & \searrow \varphi & & & \end{array}$$

with

$$\rho \circ \psi_n(a + \alpha \mathbf{1}) \longrightarrow \overline{\varphi}(a + \alpha \mathbf{1}).$$

We take for the needed approximate lifts the restriction of the ψ_n to A . We have verified (a). $\square$

Corollary 4.10. *Suppose X is a locally compact, metrizable space. If $C_0(X)$ is weakly projective with respect to unital C^* -algebras, then X^+ is an AAR.*

Corollary 4.11. *Suppose A and B are separable C^* -algebras. If A is weakly projective with respect to unital C^* -algebras and $A \cong \bar{B}$, then B is weakly projective with respect to unital C^* -algebras.*

We present the analogs of Theorems 2.2 and 3.4. We also include analogs of the fact that, if X is a compact subset of $[0, 1]^n$, that to prove X as an AAR, it suffices to show X is an approximate retract of $[0, 1]^n$. There is a similar statement involving the Hilbert cube.

The replacement for $[0, 1]^n$ is a projective C^* -algebra, such as the universal C^* -algebra generated by n -contractions. Such an object is an acquired taste, so we state our result to allow for a choice of projective C^* -algebra. The point is that, to test a given A , it suffices to work with a single map onto A from a single projective.

Theorem 4.12. *Suppose that A is a separable C^* -algebra. Each of the following two conditions is equivalent to A being weakly projective:*

- (a) *for every C^* -algebra B , and for every surjection $\rho : B \rightarrow A$, there is a sequence $\theta_n : A \rightarrow B$ of $*$ -homomorphisms so that $\rho \circ \theta_n(a) \rightarrow a$ for all a in A ;*
- (b) *there exists a projective C^* -algebra P and surjection $\rho : P \rightarrow A$ for which there is a sequence $\theta_n : A \rightarrow P$ of $*$ -homomorphisms so that $\rho \circ \theta_n(a) \rightarrow a$ for all a in A .*

Proof. Suppose A is weakly projective. Given $\rho : B \rightarrow A$ a surjection, we can approximately lift the identity map on A as in this diagram:

$$\begin{array}{ccc} & & B \\ & \nearrow \theta_n & \downarrow \rho \\ A & \xlongequal{\quad} & A \end{array}$$

We have proved (a), and it is obvious that (a) implies (b).

Suppose we are given $\varphi : A \rightarrow C$ and a surjection $\pi : B \rightarrow C$. Since P is projective, we can find ψ to make this diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & B \\ \downarrow \rho & & \downarrow \pi \\ A & \xrightarrow{\varphi} & C \end{array}$$

The maps $\varphi_n = \psi \circ \theta_n$ show that A is weakly projective. $\square$

Theorem 4.13. *Suppose that A is a separable C^* -algebra. Each of the following two conditions is equivalent to A being weakly projective with respect to unital C^* -algebras:*

- (a) *for every unital C^* -algebra B and for every unital surjection $\rho : B \rightarrow \tilde{A}$, there is a sequence $\theta_n : \tilde{A} \rightarrow B$ of unital $*$ -homomorphisms so that $\rho \circ \theta_n(a) \rightarrow a$ for all a in $\tilde{A}$;*
- (b) *there exists a projective C^* -algebra P and a unital surjection $\rho : P \rightarrow \tilde{A}$ for which there is a sequence $\theta_n : \tilde{A} \rightarrow P$ of unital $*$ -homomorphisms so that $\rho \circ \theta_n(a) \rightarrow a$ for all a in $\tilde{A}$.*

Proof. Suppose that A is weakly projective with respect to unital C^* -algebras. Given $\rho : B \rightarrow \tilde{A}$ a unital surjection, we can approximately lift the identity map on $\tilde{A}$, as in this diagram:

$$\begin{array}{ccc} & & B \\ & \nearrow \theta_n & \downarrow \rho \\ \tilde{A} & \xlongequal{\quad} & \tilde{A} \end{array}$$

We have proved (a). Again it is obvious that (a) implies (b).

Suppose that we are given $\varphi : A \rightarrow C$ and a unital surjection $\pi : B \rightarrow C$. Since C is unital, we can extend φ to a unital $*$ -homomorphism $\bar{\varphi} : \tilde{A} \rightarrow C$. Since P is projective and B is unital, we can find ψ a unital $*$ -homomorphism to make this diagram commute:

$$\begin{array}{ccccc} & & \tilde{P} & \xrightarrow{\psi} & B \\ & & \downarrow \rho & & \downarrow \pi \\ A & \hookrightarrow & \tilde{A} & \xrightarrow{\bar{\varphi}} & C \\ & \searrow \varphi & & \nearrow & \\ & & & & \end{array}$$

The maps $\varphi_n = \psi \circ \theta_n$ show that A is weakly projective with respect to unital C^* -algebras. $\square$

5. Properties of weakly projective C^* -algebras.

Definition 5.1. A quotient $B = A/I$ of a separable C^* -algebra A is an *approximate retract* of A if there is a sequence $\lambda_n : B \rightarrow A$ of $*$ -homomorphisms so that $\rho \circ \lambda_n(b) \rightarrow b$ for all b in B . Here ρ is the canonical surjection.

We use WP to stand for weakly projective.

Proposition 5.2. *An approximate retract of a separable WP C^* -algebra is WP.*

Proof. The proof is very similar to that of Theorem 4.7. $\square$

We use RFD to stand for residually finite dimensional. Recall that A is RFD if A has a separating family of finite dimensional representations. To read about other properties equivalent to this, see [1, 10].

Proposition 5.3. *An approximate retract of a separable RFD C^* -algebra is RFD.*

Proof. Given nonzero b in B , we may find m so that $\lambda_m(b) \geq \|b\|/2$. Now take a finite dimensional representation of A with $\pi(\lambda_m(b)) \neq 0$. Then $\pi \circ \lambda_m$ is a finite dimensional representation of B that does not send b to zero. $\square$

Theorem 5.4. *A C^* -algebra that is weakly projective with respect to unital C^* -algebras is RFD.*

Proof. If A is weakly projective with respect to unital C^* -algebras, it is an approximate retract of the unitization of a projective C^* -algebra.

Projective C^* -algebras are RFD ([17, Section 1]), and therefore so are their unitizations. $\square$

Lemma 5.5. *If A is weakly projective and D is semiprojective, then $[D, A]$ is trivial.*

Proof. Suppose $\varphi : D \rightarrow A$ is given. Let $\delta_1 : \mathbf{C}A \rightarrow A$ be the map defined on the cone over A by evaluation at 1. The weak projectivity of A provides us with $*$ -homomorphisms $\psi_n : A \rightarrow \mathbf{C}A$ with $\delta_1 \circ \psi_n \rightarrow \text{id}_A$. Let $\varphi_n = \delta_1 \circ \psi_n \circ \varphi$ so that $\varphi_n \sim 0$ and $\varphi_n \rightarrow \varphi$. By [2, Theorem 3.6] there is some n for which $\varphi_n \sim \varphi$. $\square$

Theorem 5.6. *If A is weakly projective, then $K_*(A) = 0$.*

6. Closure properties. The closure properties for projectivity found in [14] hold, and with practically the same proofs, for weak projectivity. The proofs involve hereditary subalgebras generated by positive elements, which are almost never unital, so we do not know about these closure properties for weak projectivity with respect to unital C^* -algebras.

Theorem 6.1. *If A is separable and weakly projective, then $\mathbf{M}_n(A)$ is weakly projective.*

Proof. The proof is very similar to that of [15, Theorem 10.2.3]. $\square$

Theorem 6.2. *Suppose that A_n is separable for all n (finite or countable list). Then $\oplus_n A_n$ is weakly projective if and only if each A_n is weakly projective.*

Proof. If the sum is WP, we use Proposition 5.2 and the fact that summand is a retract of a direct sum to conclude that each summand is WP.

For the converse we have, as in [15, Theorem 10.1.13], a way to lift orthogonal elements in the direct sum, each completely positive in A_n

and so can reduce to a lifting problem of the form

$$\begin{array}{ccc} & \bigoplus B_n & \\ & \downarrow \bigoplus \rho_n & \\ \bigoplus A_n & \xrightarrow{\bigoplus \varphi_n} & \bigoplus C_n \end{array}$$

Suppose that F is a finite subset of $\bigoplus A_n$ with $F = \{a_1, \dots, a_k\}$ and $a_j = \langle a_{j,n} \rangle$. There are $\psi_n : A_n \rightarrow C_n$ with

$$\|\rho_n \circ \psi_n(a_{j,n}) - \varphi_n(a_{j,n})\| \leq \varepsilon$$

for each j . Then

$$\begin{aligned} \left\| \left(\bigoplus \rho_n \right) \circ \left(\bigoplus \psi_n \right) (a_j) - \left(\bigoplus \varphi_n \right) (a) \right\| \\ = \sup \|\rho_n \circ \psi_n(a_{j,n}) - \varphi_n(a_{j,n})\| \end{aligned}$$

is also less than or equal to ε . $\square$

7. Questions. The Hilbert cube has nice properties, like local connectedness and the fixed-point property, and these get inherited by all ARs and, to a lesser extent, by all AARs. It would be nice to find similar properties of a “free” C^* -algebra (generated by a universal sequence of contractions).

Question 7.1. Does contractability plus weak projectivity imply projectivity?

This question is motivated by the commutative situation. See [6, Theorem 7.2]. An answer may be hard to find, as Lemma 5.5 shows that all the obvious invariants vanish on the weak projectives.

Question 7.2. Is the class of C^* -algebras that are weakly projective with respect to unital C^* -algebras closed under direct sums?

Question 7.3. Is the class of C^* -algebras that are weakly projective with respect to unital C^* -algebras closed under the formation of matrix algebras?

Question 7.4. For separable C^* -algebras, is it true that

$M_2(A)$ is weakly projective w.r.t. unital C^* -algebras
 $\implies A$ is weakly projective w.r.t. unital C^* -algebras?

Question 7.5. For separable C^* -algebras, is it true that

$M_2(A)$ is weakly projective $\implies A$ is weakly projective ?

See [3, Section 4] and [11, Section 3].

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