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ESSENTIAL NORM ESTIMATE OF A COMPOSITION OPERATOR BETWEEN BLOCH-TYPE SPACES IN THE UNIT BALL

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ABSTRACT. Let B_n be the unit ball of $\mathbf{C}^n$ and $\phi = (\phi_1, \dots, \phi_n)$ a holomorphic self-map of B_n. Let $p,\ q > 0$, be an estimate of the essential norm of a bounded composition operator C_ϕ induced by ϕ between the p-Bloch space $\beta^p(B_n)$ and q-Bloch space $\beta^q(B_n)$ given in this paper, as well as the corresponding results between the little p-Bloch space $\beta_0^p(B_n)$. As a consequence, a necessary and sufficient condition for the composition operator C_ϕ to be compact from $\beta^p(B_n)$ (or $\beta_0^p(B_n)$) into $\beta^q(B_n)$ (or $\beta_0^q(B_n)$) is obtained.

1. Introduction.

The class of all holomorphic functions with domain Ω will be denoted by $H(\Omega)$, where Ω is a bounded homogeneous domain in $\mathbf{C}^n$. Let ϕ be a holomorphic self-map of Ω, the composition operator C_ϕ induced by ϕ is defined by

$$(C_\phi f)(z) = f(\phi(z)),$$

for z in Ω and $f \in H(\Omega)$.

For $\Omega = B_n$ the unit ball of $\mathbf{C}^n$, Timoney [**10**] shows that $f \in H(B_n)$ is in the Bloch space $\beta(B_n)$ if and only if

$$\sup_{z \in B_n} (1 - |z|^2)|\nabla f(z)| < \infty.$$

This definition was the starting point for introducing the p-Bloch spaces [**21**].

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Let $0 < p < \infty$. We say that f belongs to the Bloch space $\beta^p(B_n)$, if $f \in H(B_n)$ and

$$\|f\|_{\beta^p} = |f(0)| + \sup_{z \in B_n} (1 - |z|^2)^p |\nabla f(z)| < \infty,$$

where

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \dots, \frac{\partial f(z)}{\partial z_n} \right).$$

It is an easy exercise to show that $\beta^p(B_n)$ is a Banach space with the norm $\|\cdot\|_p$ for $p \geq 1$; and for $0 < p < 1$, $\beta^p(B_n)$ is a non-locally convex topological vector space and $d(f,g) = \|f - g\|_p^p$ is a complete metric for it. Its proof idea is basic; we refer the reader to see the proof of Proposition 3.2 or the statement corresponding the Bloch-type space for the unit ball in [**21**]. It is also clear that any polynomial functions $P(z)$ are contained in $\beta^p(B_n)$. We define the closure in the Banach space $\beta^p(B_n)$ of the polynomial functions to be the little-Bloch space, denoted as $\beta_0^p(B_n)$.

We recall that the essential norm of a continuous linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}. \tag{1}$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to conditions for T to be compact.

In [**4**], Alfonso Montes-Rodriguez gave the exact essential norm of a composition operator on the Bloch space in the disc and obtained a different proof for the corresponding compactness results in [**5**]. About the essential norm of a composition operator between Hardy spaces, we refer the reader to see [**2**]. After that, Zhou and Shi generalized Alfonso's result to the polydisc in [**19**].

This paper gives some estimates of the essential norms of bounded composition operators C_ϕ between $\beta^p(B_n)$ ($\beta_0^p(B_n)$) and $\beta^q(B_n)$ (β_0^q $(B_n n)$), and generalizes the results on the Bloch space for the unit disc in [**4**]. Our main ideas of the proof come from [**19**].

In the following, we will use the symbol c to denote a finite positive number which does not depend on variables z, a, ω and may depend on some norms and parameters p, q, n, α, x, f, etc., not necessarily the same at each occurrence.

Our main results are the following:

Theorem 1. *Let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of B_n and $\|C_\phi\|_e$ the essential norm of a bounded composition operator $C_\phi : \beta^p(B^n) \to \beta^q(B^n)$; then there are $c_1,\ c_2 > 0$, independent of w, u such that*

$$c_1 \lim_{\delta\to 0} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\}\\ \mathrm{dist}\,(\phi(w),\partial B_n)<\delta}} X(w,u) \le \|C_\phi\|_e \le c_2 \lim_{\delta\to 0} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\}\\ \mathrm{dist}\,(\phi(w),\partial B_n)<\delta}} X(w,u). \tag{2}$$

where

$$X(w,u) :\overset{\triangle}{=} \frac{(1-|w|^2)^q}{(1-|\phi(w)|^2)^p}\left\{\frac{Q_{\phi(w),p}(J\phi(w)u, J\phi(w)u)}{Q_{w,q}(u,u)}\right\}^{1/2} \tag{3}$$

and $J\phi(w)$ denotes the Jacobian matrix of $\phi(w)$, $J\phi(w)u$ denotes a vector as following

$$J\phi(w) = \left(\frac{\partial \phi_j(w)}{\partial w_k}\right)_{1\le j,k\le n},$$
$$J\phi(w)u = \left(\sum_{k=1}^n \frac{\partial\phi_1(w)}{\partial w_k}u_k, \ldots, \sum_{k=1}^n \frac{\partial\phi_n(w)}{\partial w_k}u_k\right)^T;$$

and $Q_{w,q}(u,u)$ denotes as following:

when $q > 1/2$, $Q_{w,q}(u,u) = (1-|w|^2)|u|^2 + |\langle u,w\rangle|^2$;

when $q=1/2$, $Q_{w,q}(u,u)=(1-|w|^2)|u|^2\log^2(2/(1-|w|^2))+|\langle u,w\rangle|^2$;

when $0 < q < 1/2$, $Q_{w,q}(u,u) = (1-|w|^2)^{2p}|u|^2 + |\langle u,w\rangle|^2$.

The proof of Theorem 1 will be presented in Section 3.

By Theorem 1 and the fact that $C_\phi : \beta^p(B_n)$ (or $\beta_0^p(B_n)$) $\to \beta^q(B_n)$ (or $\beta_0^q(B_n)$) is compact if and only if $\|C_\phi\|_e = 0$, we obtain Corollary 1 at once.

Corollary 1. *Let $0 < p, q < \infty$, and let $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be a holomorphic self-map of B_n. Then the composition operator*

$C_\phi : \beta^p(B_n)$ (or $\beta^p_0(B_n)$) $\to \beta^q(B_n)$ (or $\beta^q_0(B_n)$) is compact if and only if C_ϕ is bounded and

$$\lim_{\delta\to 0} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} X(w,u) = 0. \tag{4}$$

The following two corollaries follow from Corollary 1.

Corollary 2. *$C_\phi : \beta^p(B_n) \to \beta^q(B_n)$ is compact if and only if* (2) *and* (4) *hold.*

Proof. If (2) holds, by Lemma 4, we know C_ϕ is bounded. Since (4) holds, it follows from Corollary 1 that C_ϕ is compact.

Conversely, if C_ϕ is compact, it is clear that C_ϕ is bounded, so by Lemma 4, (2) holds, and by Corollary 1, (4) holds. □

Similarly, by Lemma 6, Lemma 7 and Corollary 1, we have the following corollary.

Corollary 3. *$C_\phi : \beta^p_0(B_n) \to \beta^q_0(B_n)$ (or $\beta^q(B_n)$) is compact if and only if $\phi^r \in \beta^q_0(B_n)$ (or $\beta^q(B_n)$) for any multi-index r and* (2) *and* (4) *hold.*

2. Some lemmas. In order to prove Theorem 1, let us state a couple of lemmas.

Lemma 1 ([**15**, Lemma 2.2]). *Let $p > 0$. Then there is a constant $c > 0$ such that, for all $f \in \beta^p(B_n)$ (or $\beta^p_0(B_n)$) and for all $z \in B_n$, the estimate*

$$|f(z)| \le cG_p(z)\|f\|_{\beta^p} \tag{5}$$

holds, where the function $G_p(z)$ is defined as follows:

(i) *If $0 < p < 1$, then $G_p(z) = 1$;*

(ii) *If $p = 1$, then $G_p(z) = \ln(4/(1-|z|^2))$;*

(iii) *If $p > 1$, then $G_p(z) = 1/(1-|z|^2)^{p-1}$.*

Lemma 2 ([**10**, page 251], [**19**, Lemma 2.1]). *Let* $0 < p < \infty$, $f \in H(B_n)$. *Then* $f \in \beta^p(B_n)(\beta_0^p(B_n))$ *if and only if*

$$\sup_{\substack{u\in\mathbf{C}^n\setminus\{0\}\\ z\in B_n}} \frac{(1-|z|^2)^p|\nabla f(z)u|}{\sqrt{Q_{z,p}(u,u)}} < \infty;$$

furthermore,

$$\|f\|_{\beta^p} \approx |f(0)| + \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\}\\ z\in B_n}} \frac{(1-|z|^2)^p|\nabla f(z)u|}{\sqrt{Q_{z,p}(u,u)}},$$

where $M \approx N$ *means the two norms* M *and* N *are comparable, that is, there exist two positive constants* C_1 *and* C_2 *such that* $C_1 M \le N \le C_2 M$.

Lemma 3. *If* $w \in \{z \in C : |z| < 1\}$, $l \in \{1, \dots, n\}$, *denote*

$$f_w(z) = \frac{z_l - w}{(1-\overline{w}z_l)^p},$$
$$g_w(z) = \frac{a_2 z_2 + \cdots + a_n z_n}{(1-\overline{w}z_1)^p}.$$

Then $f_w, g_w \in \beta_0^p(B_n)$.

Proof. Since

$$\frac{\partial f_w(z)}{\partial z_l} = \frac{1}{(1-\overline{w}z_l)^p} + \frac{(z_l-w)p\overline{w}}{(1-\overline{w}z_l)^{p+1}}, \qquad \frac{\partial f_w(z)}{\partial z_k} = 0, \quad (k \ne l),$$

therefore,

$$(1-|z|^2)^p|\nabla f_w(z)| \le \frac{(1-|z|^2)^p}{(1-|z_l|)^p} + \frac{p(1-|z|^2)^p}{(1-|z_l|)^p}\left|\frac{z_l-w}{1-\overline{w}z_l}\right|;$$

thus, we get $\|f_w\|_{\beta^p} \le 1 + 2^p(p+1)$, that is, $f_w \in \beta^p(B_n)$. Since

$$(1-\overline{w}z_l)^{-p} = \sum_{k=0}^{\infty} \frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k z_l^k,$$

therefore

$$\begin{aligned}
f_w(z) &= (z_l - w)\sum_{k=0}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k z_l^k\\
&= \sum_{k=0}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k z_l^{k+1} - \sum_{k=0}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k w z_l^k\\
&= \sum_{k=0}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k z_l^{k+1} - \sum_{k=1}^{\infty}\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k w z_l^k - w\\
&= \sum_{k=0}^{\infty}\left(\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k - \frac{\Gamma(p+k+1)}{(k+1)!\Gamma(p)}(\overline{w})^{k+1}w\right) z_l^{k+1} - w\\
&= \sum_{k=0}^{\infty}\left(\frac{\Gamma(p+k)}{k!\Gamma(p)}(\overline{w})^k - \frac{\Gamma(p+k+1)}{(k+1)!\Gamma(p)}|w|^2\right)\overline{w}^k z_l^{k+1} - w.
\end{aligned}$$

Let

$$a_k = \left(\frac{\Gamma(p+k)}{k!\Gamma(p)} - \frac{\Gamma(p+k+1)}{(k+1)!\Gamma(p)}|w|^2\right)(\overline{w})^k,$$

and write

$$P_n(z) = \sum_{k=0}^{n} a_k z_l^{k+1} - w;$$

then

$$f_w(z) - P_n(z) = \sum_{k=n+1}^{\infty} a_k z_l^{k+1},$$

$$\frac{\partial(f_w - P_n)}{\partial z_l} = \sum_{k=n+1}^{\infty}(k+1)a_k z_l^k, \qquad \frac{\partial(f_w - P_n)}{\partial z_s} = 0, \quad (s \neq l)$$

and

$$\begin{aligned}
\|f_w - P_n\|_{\beta^p} &= |f_w(0) - P_n(0)|\\
&\quad + \sup_{z\in B_n}(1-|z|^2)^p|\nabla(f_w - P_n)(z)|\\
&= \sup_{z\in B_n}(1-|z|^2)^p\left|\frac{\partial(f_w - P_n)}{\partial z_l}(z)\right|\\
&\le \sup_{z\in B_n}(1-|z_l|^2)^p\sum_{k=n+1}^{\infty}(k+1)|a_k|\,|z_l|^k
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{k=n+1}^{\infty}(k+1)|a_k| \\
&\le \sum_{k=n+1}^{\infty}\frac{(k+1)\Gamma(p+k)}{k!\Gamma(p)}|w|^k \\
&\quad + \sum_{k=n+1}^{\infty}\frac{\Gamma(p+k+1)}{k!\Gamma(p)}|w|^{k+2}.
\end{aligned}$$

Since $|w|<1$ and the series $\sum_{n=1}^{\infty}[p(p+1)\cdots(p+k-1)]/k!|w|^k$ converges uniformly on any compact subset of B_n, therefore $\|f_w-P_n\|_{\beta^p}\to 0$. This shows that $f_w\in\beta_0^p(B_n)$.

Similarly, we can also prove that $g_w\in\beta_0^p(B_n)$; we omit the details here. □

Lemma 4 ([**15**, Corollary 1.4]; [**3**, Theorem 1]). *Let $0<p,q<\infty$ and $\phi=(\phi_1,\phi_2,\dots,\phi_n)$ be a holomorphic self-map of B_n. Then $C_\phi:\beta^p(B^n)\to\beta^q(B^n)$ is bounded if and only if*

$$\sup_{\substack{u\in\mathbf{C}^n\setminus\{0\}\\ w\in B_n}} X(w,u)<\infty. \tag{6}$$

where $X(w,u)$ has been given at (3).

Lemma 5. *$C_\phi:\beta_0^p(B_n)\to\beta_0^q(B_n)$ (or $\beta^q(B_n)$) is bounded if and only if $\phi^r\in\beta_0^q(B_n)$ (or $\beta^q(B_n)$) for any multi-index $r=(r_1,\dots,r_n)$ and* (6) *holds.*

Proof. First suppose (6) holds; by Lemma 4, we know that $C_\phi:\beta^p(B_n)\to\beta^q(B_n)$ is bounded. On the other hand, by the definition of $\beta_0^p(B_n)$, for any $f\in\beta_0^p(B_n)$, there exist polynomial functions $P_n(z)$ such that $\|f-P_n\|_{\beta^p}\to 0$ $(n\to\infty)$. So, when $n\to\infty$,

$$\|C_\phi f-C_\phi P_n\|_{\beta^q}=\|C_\phi(f-P_n)\|_{\beta^q}\le c\|f-P_n\|_{\beta^p}\longrightarrow 0.$$

But $C_\phi P_n\in\beta_0^q(B_n)$, by the assumption that $\phi^r\in\beta_0^p(B_n)$ for any multi-index $r=(r_1,\dots,r_n)$; thus, $C_\phi f\in\beta_0^q(B_n)$. This shows that $C_\phi:\beta_0^p(B_n)\to\beta_0^q(B_n)$ is bounded.

It is clear that $C_\phi : \beta_0^p(B_n) \to \beta^q(B_n)$ is bounded, because $\beta_0^q(B_n)$ is a subspace of $\beta^q(B_n)$.

Conversely, for any multi-index $r = (r_1, \dots, r_n)$, $z^r = z_1^{r_1} \cdots z_n^{r_n}$ is a polynomial; furthermore, $z^r \in \beta_0^p(B_n)$. If C_ϕ is bounded, then $C_\phi z^r = \phi^r \in \beta_0^q(B_n)$.

On the other hand, since C_ϕ is bounded, thus $\|C_\phi\|_{\beta^q} \le c\|f\|_{\beta^p}$ for all $f \in \beta_0^p(B_n)$. Notice that the test functions f_w and g_w used in Lemma 3 all belong to $\beta_0^p(B_n)$; using the same method as that of Lemma 3, we can show that (6) holds. $\square$

Lemma 6 ([**19**, Lemma 2.5]). *If $\{f_k\}$ is a bounded sequence in $\beta^p(B_n)$, then there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ which converges uniformly on compact subsets of B_n to a holomorphic function $f \in \beta^p(B_n)$.*

Lemma 7 ([**19**, Lemma 2.6]). *Let Ω be a domain in $\mathbf{C}^n$, $f \in H(\Omega)$. If a compact set K and its neighborhood G satisfy $K \subset G \subset \Omega$ and $\rho = \operatorname{dist}(K, \partial G) > 0$, then*

$$\sup_{z\in K} \left|\frac{\partial f}{\partial z_j}(z)\right| \le \frac{\sqrt{n}}{\rho} \sup_{z\in G} |f(z)| \quad (j = 1, \dots, n).$$

3. The proof of Theorem 1. Now we turn to the proof of Theorem 1. In the following, we are dealing with the case for $C_\phi : \beta^p(B_n) \to \beta^q(U^n)$, but if we note that the test functions f_m introduced below belong to $\beta_0^p(B_n) \subset \beta_{0*}^p(B_n) \subset \beta^p(B_n)$, the results in Theorem 1 also hold with minor modifications for the other cases.

To obtain the lower estimate in Theorem 1 we first prove the following Proposition 1.

Proposition 1. *For all $w \in B_n$ satisfying $|\phi(w)| > \sqrt{2/3}$, and for all $u \in \mathbf{C}^n \setminus \{0\}$, there is a function $g_{w,u}(z) \in \beta_0^p(B_n) \subset \beta^p(B_n)$ such that*

(i) *there exist $c_1,\ c_2 > 0$, independent of w and u, such that*

$$c_1 \le \|g_{w,u}\|_{\beta^p} \le c_2.$$

(ii) $\{g_{w,u}(z)\}$ *converges to zero uniformly for* $u \in \mathbf{C}^n \setminus \{0\}$ *and* z *on compact subsets of* B_n *when* $|\phi(w)| \to 1$;

(iii) *There is a constant* $c > 0$, *for all* $u \in \mathbf{C}^n \setminus \{0\}$ *and* $w \in B_n$,

$$(1 - |w|^2)^q |\nabla(g_{w,u} \circ \phi)(w)| \geq c \cdot X(w, u),$$

where $X(w, u)$ *is the same as Theorem* 1.

Proof. For all $w \in B_n$ satisfying $|\phi(w)| > \sqrt{2/3}$ and for all $u \in \mathbf{C}^n \setminus \{0\}$, there exists a unitary transformation U_w to make

$$\phi(w) = r_w e_1 U_w, \quad \text{where } r_w = |\phi(w)|, \quad e_1 = (1, 0, \dots, 0).$$

Next we break the proof into two cases.

(1) Assume $Q_{\phi(w),p}(J\phi(w)u, J\phi(w)u) \leq 2|\langle \phi(w), J\phi(w)u\rangle|^2$ $(\star)$. Taking

$$f_{w,u}(z) = \frac{1 - r_w^2}{(1 - r_w z_1)^p},$$

then

$$\frac{\partial f_{w,u}(z)}{\partial z_1} = \frac{(1 - r_w^2) p r_w}{(1 - r_w z_1)^{p+1}}, \qquad \frac{\partial f_{w,u}(z)}{\partial z_k} = 0, \quad k = 2, \dots, n.$$

It follows that

$$\begin{aligned}(1 - |z|^2)^p |\nabla f_{w,u}(z)| &= (1 - |z|^2)^p \frac{(1 - r_w^2) p r_w}{|1 - r_w z_1|^{p+1}} \\ &\leq p r_w (1 - r_w^2) \frac{(1 - |z_1|^2)^p}{(1 - r_w |z_1|)^{p+1}} \\ &\leq p r_w (1 - r_w^2) \frac{(1 - |z_1|^2)^p}{(1 - r_w)(1 - |z_1|)^p} \\ &= p r_w (1 + r_w)(1 + |z_1|)^p \leq 2p2^p.\end{aligned}$$

So $f_{w,u} \in \beta^p(B_n)$. As in the discussion of Lemma 3, we get $f_{w,u} \in \beta_0^p(B_n)$.

On the other hand, taking $z_0 = (z_1^0, 0, \dots, 0) = (r_w, 0, \dots, 0) \in B_n$. Then

$$\begin{aligned}(1-|z_0|^2)^p|\nabla f_{w,u}(z_0)| &= (1-|z_0|^2)^p \frac{(1-r_w^2)pr_w}{|1-r_w z_1^0|^{p+1}}\\ &= (1-r_w^2)^p \frac{(1-r_w^2)pr_w}{(1-r_w^2)^{p+1}}\\ &= pr_w > p\sqrt{2/3}.\end{aligned}$$

It follows that

$$p\sqrt{2/3} \le \sup_{z\in B_n}(1-|z|^2)^p|\nabla f_{w,u}(z)| \le 2p2^p.$$

Noticing that

$$\|f_{w,u}\|_{\beta^p} = 1-r_w^2 + \sup_{z\in B_n}(1-|z|^2)^p|\nabla f_{w,u}(z)|,$$

by the above discussion we get

$$p\sqrt{2/3} \le \|f_{w,u}\|_p \le 1+2p2^p.$$

At the same time, for fixed $z \in B_n$, it is clear that $\lim_{r_w\to 1}|f_{w,u}(z)| \to 0$ uniformly for z on compact subsets of B_n and $u \in \mathbf{C}^n - \{0\}$.

Let

$$g_{w,u}(z) = (f_{w,u}\circ U_w^{-1})(z) = f_{w,u}(zU_w^{-1});$$

then

$$\nabla g_{w,u}(z) = \nabla(f_{w,u}\circ U_w^{-1})(z) = \nabla(f_{w,u})(zU_w^{-1})(U_w^{-1})^T.$$

Since U_w is a unitary transformation (so is $(U_w^{-1})^T$), then we have

$$|\nabla g_{w,u}(z)| = |\nabla(f_{w,u})(zU_w^{-1})(U_w^{-1})^T| = |\nabla(f_{w,u})(zU_w^{-1})|.$$

Since $z \to zU_w^{-1}$ is a one-to-one mapping from B_n to B_n, we have

$$\begin{aligned}\sup_{z\in B_n}(1-|z|^2)^p|\nabla g_{w,u}(z)| &= \sup_{z\in B_n}(1-|zU_w^{-1}|^2)^p|\nabla f_{w,u}(zU_w^{-1})|\\ &= \sup_{v\in B_n}(1-|v|^2)^p|\nabla f_{w,u}(v)|.\end{aligned}$$

So

$$\|g_{w,u}\|_{\beta^p} = \|f_{w,u}\|_{\beta^p}.$$

This shows that (i) and (ii) hold.

For $u \in \mathbf{C}^n \setminus \{0\}$, by Lemma 2 we have

$$\begin{aligned}(7)\quad (1-|w|^2)^q|\nabla(g_{w,u}\circ\phi)(w)| &\geq A_1 \frac{(1-|w|^2)^q|\nabla(g_{w,u}\circ\phi)(w)u|}{\sqrt{Q_{w,q}(u,u)}}\\ &= A_1 \frac{(1-|w|^2)^q|\nabla g_{w,u}(\phi(w))J\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}}\\ &= A_1 \frac{(1-|w|^2)^q|\nabla f_{w,u}(\phi(w)U_w^{-1})(U_w^{-1})^TJ\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}}.\end{aligned}$$

Since $\phi(w) = r_w e_1 U_w$, we get $\phi(w)U_w^{-1} = r_w e_1$, then

$$\begin{aligned}\nabla f_{w,u}(\phi(w)U_w^{-1}) &= \nabla f_{w,u}(r_w e_1) = \nabla f_{w,u}(r_w, 0, \dots, 0)\\ (8)\qquad &= \left(\frac{(1-r_w^2)pr_w}{(1-r_w^2)^{p+1}}, 0, \dots, 0\right)\\ &= \left(\frac{pr_w}{(1-r_w^2)^p}, 0, \dots, 0\right).\end{aligned}$$

Combining (7) and (8), we have

$$\begin{aligned}(1-|w|^2)^q|\nabla(g_{w,u}\circ\phi)(w)|\\ &\geq A_1 \frac{(1-|w|^2)^q}{(1-r_w^2)^p}\frac{pr_w|e_1(U_w^{-1})^TJ\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}}.\end{aligned}$$

Since $\phi(w) = r_w e_1 U_w$ and $\overline{U_w} = (U_w^{-1})^T$, we get $\overline{\phi(w)} = r_w e_1 \overline{U_w} = r_w e_1 (U_w^{-1})^T$; thus,

$$\begin{aligned}(1-|w|^2)^q|\nabla(g_{w,u}\circ\phi)(w)| &\geq A_1 \frac{(1-|w|^2)^q}{(1-r_w^2)^p}\frac{p|\overline{\phi(w)}J\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}}\\ &= A_1 p\frac{(1-|w|^2)^q}{(1-r_w^2)^p}\frac{|\langle\phi(w), J\phi(w)u\rangle|}{\sqrt{Q_{w,q}(u,u)}}.\end{aligned}$$

Therefore, from our assumption in ($\star$), we get

$$\begin{aligned}(1-|w|^2)^q&|\nabla(g_{w,u}\circ\phi)(w)|\\ &\geq \frac{A_1p}{\sqrt{2}}\frac{(1-|w|^2)^q}{(1-r_w^2)^p}\left\{\frac{Q_{\phi(w),p}(J\phi(w)u,J\phi(w)u)}{Q_{w,q}(u,u)}\right\}^{1/2}\\ &=\frac{A_1p}{\sqrt{2}}X(w,u)\geq cX(w,u).\end{aligned}$$

So (iii) holds.

(2) Assume $Q_{\phi(w),p}(J\phi(w)u,J\phi(w)u) > 2|\langle\phi(w),J\phi(w)u\rangle|^2$ ($\star\star$). Write $J\phi(w)u=(\xi_1,\dots,\xi_n)^T$. For $j=2,\dots,n$, let $\theta_j=\arg\xi_j$ and $a_j=e^{-i\theta_j}$ if $\xi_j\neq 0$; or $a_j=0$ if $\xi_j=0$.

Case 1. When $q>1/2$, taking

$$f_{w,u}(z)=\frac{a_2z_2+\cdots+a_nz_n}{(1-r_wz_1)^p}\sqrt{1-r_w^2},$$

then

$$\begin{aligned}\frac{\partial f_{w,u}(z)}{\partial z_1}&=\frac{pr_w(a_2z_2+\cdots+a_nz_n)\sqrt{1-r_w^2}}{(1-r_wz_1)^{p+1}},\\ \frac{\partial f_{w,u}(z)}{\partial z_j}&=\frac{\sqrt{1-r_w^2}\,a_j}{(1-r_wz_1)^p},\quad (j=2,\dots,n).\end{aligned}$$

By simple calculation we can get

$$\begin{aligned}(9)\quad &|\nabla f_{w,u}(z)|\\ &=\left\{\left|\frac{pr_w(a_2z_2+\cdots+a_nz_n)\sqrt{1-r_w^2}}{(1-r_wz_1)^{p+1}}\right|^2+\sum_{j=2}^n\left|\frac{\sqrt{1-r_w^2}a_j}{(1-r_wz_1)^p}\right|^2\right\}^{1/2}\\ &\leq\frac{\sqrt{1-r_w^2}}{|1-r_wz_1|^p}\left\{\left|\frac{pr_w(a_2z_2+\cdots+a_nz_n)}{1-r_wz_1}\right|^2+(n-1)\right\}^{1/2}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&(1-|z|^2)^p|\nabla f_{w,u}(z)| \\
&\le (1-|z_1|^2)^p \frac{\sqrt{1-r_w^2}}{(1-r_w|z_1|)^p}\left\{\frac{p^2r_w^2|a_2z_2+\cdots+a_nz_n|^2}{(1-r_w|z_1|)^2}+(n-1)\right\}^{1/2} \\
&\le (1-|z_1|^2)^p \frac{\sqrt{1-r_w^2}}{(1-|z_1|)^p}\left\{\frac{p^2r_w^2(n-1)(|z_2|^2+\cdots+|z_n|^2)}{(1-r_w|z_1|)^2}+(n-1)\right\}^{1/2} \\
&\le 2^p\sqrt{1-r_w^2}\left\{\frac{p^2r_w^2(n-1)(1-|z_1|^2)}{(1-r_w|z_1|)^2}+(n-1)\right\}^{1/2} \\
&\le 2^p\sqrt{n-1}\sqrt{1-r_w^2}\left\{1+\frac{p^2r_w^2(1-|z_1|^2)}{(1-r_w|z_1|)^2}\right\}^{1/2} \\
&\le 2^p\sqrt{n-1}\left\{(1-r_w^2)+\frac{p^2r_w^2(1-|z_1|^2)(1-r_w^2)}{(1-r_w)(1-|z_1|)}\right\}^{1/2} \\
&\le 2^p\sqrt{(n-1)(1+4p^2)}.
\end{aligned}
$$

So $f_{w,u}\in\beta^p(B_n)$. As in the discussion of Lemma 3, we get $f_{w,u}\in\beta_0^p(B_n)$.

On the other hand, taking

$$z_0=(z_1^{(0)},\ldots,z_n^{(0)})=\left(r_w,\frac{1}{\sqrt{2}}\sqrt{1-r_w^2},0,\ldots,0\right).$$

Then

$$|z_0|^2=r_w^2+\frac{1}{2}(1-r_w^2)=\frac{1}{2}(1+r_w^2)<1,$$

thus $z_0\in B_n$. By (9) we have

$$
\begin{aligned}
&(1-|z_0|^2)^p|\nabla f_{w,u}(z_0)| \\
&\qquad\ge (1-|z_0|^2)^p\frac{\sqrt{1-r_w^2}}{|1-r_wz_1^{(0)}|^p}\left\{\frac{|pr_w(a_2z_2^{(0)}+\cdots+a_nz_n^{(0)})|^2}{|1-r_wz_1^{(0)}|^2}\right\}^{1/2} \\
&\qquad=\frac{(1-r_w^2)^p\sqrt{1-r_w^2}}{2^p(1-r_w^2)^p}\left|\frac{pr_wa_2z_2^{(0)}}{1-r_wz_1^{(0)}}\right| \\
&\qquad=\frac{(1-r_w^2)^p\sqrt{1-r_w^2}\,pr_w\sqrt{1-r_w^2}}{2^p\sqrt{2}(1-r_w^2)^p(1-r_w^2)} \\
&\qquad=\frac{pr_w}{2^p\sqrt{2}}\ge\sqrt{\frac{1}{3}}\frac{p}{2^p}.
\end{aligned}
$$

Since $f_{w,u}(0) = 0$, we can get

$$\sqrt{\frac{1}{3}}\frac{p}{2^p} \leq \|f_{w,u}\|_{\beta^p} \leq 2^p\sqrt{(n-1)(1+4p^2)}.$$

When $z \in B_n$ is fixed, it is clear that

$$\lim_{r_w \to 1} |f_{w,u}(z)| \longrightarrow 0.$$

As in the discussion of Case 1, we get $\|g_{w,u}\|_{\beta^p} = \|f_{w,u}\|_{\beta^p}$. So (i) and (ii) hold.

On the other hand, if $\rho_w = |\phi(w)| > \sqrt{2/3}$, notice that $J\phi(w)u = (\xi_1, \dots, \xi_n)^T$ and $\phi(w) = \rho_w e_1 U_w$, $\overline{U_w} = (U_w^{-1})^T$, then

$$|\xi_1| = \frac{1}{\rho_w}|\langle \phi(w), J\phi(w)u\rangle|.$$

It follows from our assumption in ($\star\star$) that

$$|\xi_1| \leq \frac{\sqrt{1-\rho_w^2}}{\rho_w}|J\phi(w)u|.$$

By simple calculation, it is easy to get

$$|\xi_1|^2 \leq \frac{1-\rho_w^2}{2\rho_w^2 - 1}(|\xi_2|^2 + \cdots + |\xi_n|^2).$$

Therefore,

$$\begin{aligned}
|\xi_1|^2 + \cdots + |\xi_n|^2 &\leq \frac{\rho_w^2}{2\rho_w^2 - 1}(|\xi_2|^2 + \cdots + |\xi_n|^2) \\
&\leq 2(|\xi_2|^2 + \cdots + |\xi_n|^2).
\end{aligned} \tag{10}$$

It follows from (10) and Lemma 2 that

$$
\begin{aligned}
&(1-|w|^2)^q|\nabla(g_{w,u}\circ\phi)(w)| \\
&\qquad \geq A_2\frac{(1-|w|^2)^q|(\nabla f_{w,u})(\phi(w)U_w^{-1})(U_w^{-1})^T J\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}} \\
&\qquad = A_2\frac{(1-|w|^2)^q}{(1-r_w^2)^p}\left\{\frac{(1-r_w^2)(|\xi_2|+\cdots+|\xi_n|)^2}{Q_{w,q}(u,u)}\right\}^{1/2} \\
&\qquad \geq A_2\frac{(1-|w|^2)^q}{(1-r_w^2)^p}\left\{\frac{(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)}{Q_{w,q}(u,u)}\right\}^{1/2} \\
&\qquad \geq \frac{A_2(1-|w|^2)^q}{\sqrt{2}(1-r_w^2)^p}\left\{\frac{(1-r_w^2)(|\xi_1|^2+\cdots+|\xi_n|^2)}{Q_{w,q}(u,u)}\right\}^{1/2} \\
&\qquad = \frac{A_2(1-|w|^2)^q}{\sqrt{2}(1-r_w^2)^p}\frac{\sqrt{1-r_w^2}|J\phi(w)u|}{\sqrt{Q_{w,q}(u,u)}} \\
&\qquad \geq \frac{A_2}{2}X(w,u)\geq cX(w,u).
\end{aligned}
$$

This is (iii).

Case 2. When $q=1/2$, taking

$$f_{w,u}(z)=(a_2z_2+\cdots+a_nz_n)\log^{-1}\frac{1}{1-r_w^2}\log^2\frac{1}{1-r_wz_1}.$$

Case 3. When $0<q<1/2$, taking

$$f_{w,u}(z)=(a_2z_2+\cdots+a_nz_n)\left\{\sqrt{1-r_w^2}-\frac{\sqrt{1-r_w^2}}{(1-r_wz_1)^p}\right\}.$$

By a similar discussion to Case 1, we can see that the functions above are just what we want, and (i), (ii) and (iii) all hold. So, the proof is completed. $\square$

Now, we are ready to prove Theorem 1. We begin by proving the *lower estimate.*

Let

$$F_{w,u}(z)=\frac{g_{w,u}(z)}{\|g_{w,u}\|_{\beta^p}},$$

where $g_{w,u}(z)$ is defined as in Proposition 1. It is clear that $\|F_{w,u}\|_{\beta^p} = 1$, and $F_{w,u}(z)$ converges to zero uniformly on compact subsets of B_n for all $u \in \mathbf{C}^n \setminus \{0\}$ when $|\phi(w)| \to 1$. Suppose $K : \beta^p(B_n) \to \beta^q(B_n)$ is compact; then $\|KF_{w,u}\|_{\beta^q} \to 0$ uniformly for $u \in \mathbf{C}^n \setminus \{0\}$ and z in compact subsets of B_n when $\delta \to 0$ (this is because $|\phi(w)| \to 1$ when $\delta \to 0$), so we have

$$\begin{aligned}\|C_\phi - K\| &= \sup_{\|f\|_{\beta^p}=1} \|(C_\phi - K)f\|_{\beta^q} \\ &\geq \sup_{\|f\|_{\beta^p}=1} (\|C_\phi f\|_{\beta^q} - \|Kf\|_{\beta^q}) \\ &\geq \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} (\|C_\phi F_{w,u}\|_{\beta^q} - \|KF_{w,u}\|_{\beta^q}) \\ &\geq \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} \|C_\phi F_{w,u}\|_{\beta^q} \\ &\quad - \sup_{u\in\mathbf{C}^n\setminus\{0\} \operatorname{dist}(\phi(w),\partial B_n)<\delta} \|KF_{w,u}\|_{\beta^q}.\end{aligned}$$

On the other hand, by Proposition 1, for $|\phi(w)| > \sqrt{2/3}$, we get

$$\begin{aligned}&\sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} \frac{\|g_{w,u}\circ\phi\|_{\beta^q}}{\|g_{w,u}\|_{\beta^p}} \\ &\qquad \geq \frac{1}{c_2} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} \|g_{w,u}\circ\phi\|_{\beta^q} \\ &\qquad \geq \frac{1}{c_2} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} \sup_{z\in B_n} (1-|z|^2)^q |\nabla(g_{w,u}\circ\phi)(z)| \\ &\qquad \geq \frac{1}{c_2} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} (1-|w|^2)^q |\nabla(g_{w,u}\circ\phi)(w)|.\end{aligned}$$

By Proposition 1, when $|\phi(w)| > \sqrt{2/3}$ and for all $u \in \mathbf{C}^n \setminus \{0\}$, we have

$$(1-|w|^2)^q |\nabla(g_{w,u}\circ\phi)(w)| \geq c \cdot X(w,u).$$

Therefore,

$$\|C_\phi - K\| \geq \frac{c}{c_2} \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} X(w,u) - \sup_{\substack{u\in\mathbf{C}^n\setminus\{0\} \\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} \|KF_{w,u}\|_{\beta^q}.$$

Letting $\delta \to 0$, we get

$$\|C_\phi - K\| \geq \frac{c}{c_2} \lim_{\delta \to 0} \sup_{\substack{u \in \mathbf{C}^n \setminus \{0\} \\ \text{dist}\,(\phi(w), \partial B_n) < \delta}} X(w,u).$$

It follows from the definition of $\|C_\phi\|_e$ that

$$\begin{aligned} \|C_\phi\|_e &= \inf\{\|C_\phi - K\| : K \text{ is compact}\} \\ &\geq \frac{c}{c_2} \lim_{\delta \to 0} \sup_{\substack{u \in \mathbf{C}^n \setminus \{0\} \\ \text{dist}\,(\phi(w), \partial B_n) < \delta}} X(w,u) \\ &= c_1 \lim_{\delta \to 0} \sup_{\substack{u \in \mathbf{C}^n \setminus \{0\} \\ \text{dist}\,(\phi(w), \partial B_n) < \delta}} X(w,u). \end{aligned}$$

This is the lower estimate.

To obtain the upper estimate in Theorem 1 we first prove the following Proposition 2. It derives from the paper [**19**], and the proof of (i), (ii) and (iv) is similar to that in Theorem 1.1 of [**19**], so we omit the details here.

Proposition 2. *Let ϕ be a holomorphic self-map of B_n. For $m = 2, 3, \ldots,$ we define the operators as follows*:

$$K_m f(w) = f\left(\frac{m-1}{m} w\right), \quad f \in H(B_n),\ w \in B_n.$$

Then the operators K_m have the following properties:

(i) *for all $f \in H(B_n)$, $K_m f \in \beta_0^p(B_n) \subset \beta^p(B_n)$*;

(ii) *For fixed m, K_m is compact on $\beta^p(B_n)$ or $\beta_0^p(B_n)$*;

(iii) *If $C_\phi : \beta^p(B_n)$ (or $\beta_0^p(B_n)$) $\to \beta^q(B_n)$ (or $\beta_0^q(B_n)$) is bounded, then $C_\phi K_m f \in \beta^q(B_n)$ (or $\beta_0^q(B_n)$) and $C_\phi K_m : \beta^p(B_n)$ (or $\beta_0^p(B_n)$) $\to \beta^q(B_n)$ (or $\beta_0^q(B_n)$) is compact*;

(iv) $\|I - K_m\| \leq 2$;

(v) $(I - K_m)f$ *tends to zero uniformly on compact subsets of B_n.*

Proof. (iii) By (i) and the fact that C_ϕ is bounded, the former is obvious. By (ii) and noting that C_ϕ is bounded, we get that $C_\phi K_m$ is compact.

(v) For any compact subset $E \in B_n$, there exists r $(0 < r < 1)$ such that $E \subset rB_n \subset B_n$. Write $r_m = (m-1)/m$; then $0 < r_m < 1$ and for all $z \in E$ we have

$$\begin{aligned}|(I-K_m)f(z)| &= |f(z) - f_m(z)| = |f(z) - f(r_m z)| \\ &= \left| \int_{r_m}^1 \frac{d}{dt}(f(tz))\,dt \right| = \left| \int_{r_m}^1 \sum_{k=1}^n \frac{\partial f}{\partial w_k}(tz) \cdot z_k\,dt \right| \\ &\le \sum_{k=1}^n \int_{r_m}^1 \left| \frac{\partial f}{\partial w_k}(tz) \right| dt.\end{aligned}$$

When $t \in [r_m, 1]$, $|tz| = t|z| < |z| \le r$ for all $z \in E$. But $(\partial f/\partial w_k)(w)$ is bounded uniformly on $r\overline{B_n}$; therefore, for all $z \in E$, $|(\partial f/\partial w_k)(tz)| \le M$. So when $m \to \infty$, we have

$$|(I-K_m)f(z)| \le nM(1-r_m) \longrightarrow 0.$$

Thus $(I-K_m)f$ tends to zero uniformly on compact subsets of B_n. □

Let us now return to the proof of the *upper estimate.*

First, for some $\delta > 0$, we denote that

$$\begin{aligned}G_1 &:= \{w \in B_n : \operatorname{dist}(\phi(w), \partial B_n) < \delta\} \\ G_2 &:= \{w \in B_n : \operatorname{dist}(\phi(w), \partial B_n) \ge \delta\} \\ G_2' &:= \{z \in B_n : \operatorname{dist}(z, \partial B_n) \ge \delta\}.\end{aligned}$$

Then $G_1 \cup G_2 = B_n$ and G_2' is a compact set of B_n, and $z = \phi(w) \in G_2'$ if and only if $w \in G_2$. For any $f \in \beta^p(B_n)$, write $\|f\| = \|f\|_{\beta^p}$; then by Lemma 2 and property (iv) of Proposition 2, we have

$$
\begin{aligned}
\|C_\phi\|_e &\le \|C_\phi - C_\phi K_m\| = \|C_\phi(I-K_m)\| = \sup_{\|f\|=1} \|C_\phi(I-K_m)f\|_{\beta^q} \\
&= \sup_{\|f\|=1}\Big[\sup_{w\in B_n}(1-|w|^2)^q|\nabla[(I-K_m)f\circ\phi](w)| \\
&\qquad\qquad + |[(I-K_m)f](\phi(0))|\Big] \\
&\le c_2 \sup_{\|f\|=1}\Big[\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in B_n}} \frac{(1-|w|^2)^q|\nabla[(I-K_m)f\circ\phi](w)|}{\sqrt{Q_{w,q}(u,u)}} \\
&\qquad \frac{(1-|\phi(w)|^2)^p}{(1-|\phi(w)|^2)^p}\frac{\sqrt{Q_{\phi(w),p}(J\phi(w)u,J\phi(w)u)}}{\sqrt{Q_{\phi(w),p}(J\phi(w)u,J\phi(w)u)}} \\
&\qquad\qquad + |[(I-K_m)f](\phi(0))|\Big] \\
&\le c_2 \sup_{\|f\|=1}\Big[\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in B_n}} X(w,u) \\
&\qquad \frac{(1-|\phi(w)|^2)^p|\nabla[(I-K_m)f](\phi(w))J\phi(w)u|}{\sqrt{Q_{\phi(w),p}(J\phi(w)u,J\phi(w)u)}} \\
&\qquad\qquad + |[(I-K_m)f](\phi(0))|\Big] \\
&\le c_2 \sup_{\|f\|=1}\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in B_n}} X(w,u)(1-|\phi(w)|^2)^p|\nabla[(I-K_m)f](\phi(w))| \\
&\quad + c_2\sup_{\|f\|=1}|[(I-K_m)f](\phi(0))| \\
&\le c_2\|I-K_m\|\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_1}} X(w,u) \\
&\quad + c_2 \sup_{\|f\|=1}\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|\phi(w)|^2)^p|\nabla[(I-K_m)f](\phi(w))| \\
&\quad + c_2\sup_{\|f\|=1}|[(I-K_m)f](\phi(0))| \\
&\le c_2 \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_1}} X(w,u) + I + II.
\end{aligned}
$$

By property (v) of Proposition 2 we know that $[(I-K_m)f](z)$ converges to zero uniformly on G_2', so $[(I-K_m)f](\phi(w))$ also converges to zero uniformly on G_2 for every fixed f. Next we prove that, for any $w \in G_2$ and $\|f\| = 1$, $I, II \to 0$ when $m \to \infty$. Since

$$\Big|[(I-K_m)f](\phi(0))\Big| = \left|f(\phi(0)) - f\left(\frac{m-1}{m}\phi(0)\right)\right|,$$

Let $F(t) = f(t\phi(0) + (1-t)((m-1)/m)\phi(0))$; thus,

$$\begin{aligned}\Big|[(I-K_m)f](\phi(0))\Big| &= \Big|\int_0^1 F'(t)dt\Big| \\ &\le \int_0^1 \left|\frac{\partial f}{\partial \zeta_k}\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\right. \\ &\qquad\qquad \left.\left(\phi_k(0) - \frac{m-1}{m}\phi_k(0)\right)\right| dt \\ &\le \int_0^1 \left|\nabla f\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\right| \\ &\qquad\qquad \cdot \frac{1}{m}|\phi(0)|\, dt \\ &\le \frac{1}{m}\int_0^1 \left|\nabla f\left(t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right)\right| dt.\end{aligned}$$

Since $(1-|z|^2)^p|\nabla f(z)| \le \|f\| = 1$, we get $|\nabla f(z)| \le (1-|z|^2)^{-p}$. On the other hand, when $0 < t < 1$, we have

$$\left(1 - \left|t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right|^2\right)^{-p} \le (1-|\phi(0)|^2)^{-p}.$$

So

$$\begin{aligned}\Big|[(I-K_m)f](\phi(0))\Big| &\le \frac{1}{m}\int_0^1 \left(1 - \left|t\phi(0) + (1-t)\frac{m-1}{m}\phi(0)\right|^2\right)^{-p} dt \\ &\le \frac{1}{m}(1-|\phi(0)|^2)^{-p} \longrightarrow 0 \quad (m \to \infty).\end{aligned}$$

Letting $m \to \infty$, we get $II \to 0$.

Letting $w \in G_2$ and $z = \phi(w)$, then

$$
\begin{aligned}
I &= c_2 \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|z|^2)^p |\nabla[(I-K_m)f](z)| \\
&= c_2 \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|z|^2)^p \left|\nabla f(z) - \frac{m-1}{m}\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\le c_2 \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|z|^2)^p \left|\nabla f(z) - \nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\quad + \frac{c_2}{m} \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|z|^2)^p \left|\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\le c_2 \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)(1-|z|^2)^p \left|\nabla f(z) - \nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\quad + \frac{c_2}{m} \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)\left(1-\left|\frac{m-1}{m}z\right|^2\right)^p \left|\nabla f\left(\frac{m-1}{m}z\right)\right| \\
&\le c_2 \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)\sum_{l=1}^n \left|\frac{\partial f}{\partial z_l}(z) - \frac{\partial f}{\partial z_l}\left(\frac{m-1}{m}z\right)\right| \\
&\quad + \frac{c_2}{m} \sup_{\|f\|=1} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u)\|f\| = I_1 + I_2.
\end{aligned}
$$

By Lemma 4 we get $\sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ w\in G_2}} X(w,u) < \infty$, and notice that $\|f\| = 1$, so it is easy to get that $I_2 \to 0$ when $m \to \infty$.

By a similar discussion in [**19**, page 185], we can get that $I_1 \to 0$ when $m \to \infty$. Now, letting $m \to \infty$ and $\delta \to 0$, we get the upper estimate:

$$
\|C_\phi\|_e \le c_2 \lim_{\delta\to 0} \sup_{\substack{u\in \mathbf{C}^n\setminus\{0\}\\ \operatorname{dist}(\phi(w),\partial B_n)<\delta}} X(w,u).
$$

So, the proof of Theorem 1 is finished. □

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