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SOME CLOSED-FORM EVALUATIONS OF INFINITE PRODUCTS INVOLVING NESTED RADICALS

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ABSTRACT. By applying double and triple angle identities for hyperbolic and trigonometric cosine functions, we obtain closed-form evaluations for two families of infinite products involving nested radicals. The first group of results represents a generalization of the classic Viéte infinite product expansion for $2/\pi$, while the second comprises variations on Viéte type infinite products and infinite products involving nested square roots of 2. In addition, specific examples of Viéte type infinite product expansions are presented for such numbers as $3\sqrt{3}/2\pi$ and $3/\pi$.

1. Introduction. In a series of papers (see [**1, 2**]) the author sought the closed-form evaluation of the functions $R_k(x) = \sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}$, consisting of k nested radicals, over the domain $[2,\infty)$, together with the closed-form evaluation of various definite integrals involving these functions. In particular, it was shown that the resulting algebraic expressions could be written in terms of a linear combination of the basis functions $((x+\sqrt{x^2-4})/2)^{1/2^k}$ and $((x+\sqrt{x^2-4})/2)^{-1/2^k}$. The approach taken to arrive at these closed-form expressions, which made use of a hyperbolic cosine identity, was motivated by the work of Servi (see [**3**]) who similarly employed a trigonometric cosine identity, to show that the class of functions consisting of $k-1$ nested radicals as follows

$$(1)\quad R(b_1,b_2,\cdots,b_k) = \frac{b_k}{2}\sqrt{2+b_{k-1}\sqrt{2+b_{k-2}\sqrt{2+\cdots+b_2\sqrt{2+z}}}},$$

where $b_i \in \{-1,0,1\}$, for $i \neq 1$, and $z = 2\sin(b_1\pi/4)$, with $b_1 \in [-2,2]$, could be evaluated in terms of $\cos(\theta)$, where θ was a rational multiple (dependent on b_i and z) of π. Later in the same paper, Servi also

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used this closed-form expression together with the identity $\sin(z)/z = \prod_{k=1}^{\infty} \cos(z/2^k)$ to re-derive the following infinite product expansion

$$\frac{2}{\pi} = \prod_{k=1}^{\infty} \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{k \text{ square roots}},$$

first discovered by Viéte in the sixteenth century. Following again from Servi's work, we intend in this paper to establish closed-form expressions for two different classes of infinite products involving nested radicals, one of which is given here as follows

$$\frac{\sqrt{x^2-1}}{\ln(x+\sqrt{x^2-1})} = \prod_{k=1}^{\infty} \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{x}{2}}}}}}_{k \text{ square roots}}.$$

This bears a marked resemblance to the Viéte infinite product above. As will be seen, the general approach taken here will be to exploit various double and triple angle identities for the hyperbolic and trigonometric cosine functions and to establish formulas similar to $\sin(z)/z = \prod_{k=1}^{\infty} \cos(z/2^k)$, from which the required infinite products will follow by insertion of an appropriate closed-form expressions for the functions $R_k(x)$ and $R(b_1, b_2, \dots, b_k)$.

2. Main results. We begin with the following technical lemma, which shows that the various nested radicals needed later can be evaluated in terms of a hyperbolic cosine function.

Lemma 1. *If $x \geq 1$ and k is a positive integer, then there exists a unique $\xi \geq 0$, such that*

$$r_k(x) = \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{x}{2}}}}}_{k \text{ square roots}} = \cosh\left(\frac{\xi}{2^k}\right),$$

while if $y = 2x \geq 2$, *then*

$$R_k(y) = \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+y}}}}_{k \text{ square roots}} = 2r_k(x) = 2\cosh\left(\frac{\xi}{2^k}\right).$$

Proof. As the mapping $\cosh(\cdot) : [0,\infty) \to [1,\infty)$ is a bijection, a unique ξ in $[0,\infty)$ must exist such that $x = \cosh(\xi)$. Using this hyperbolic function substitution, we show by induction that

$$r_k(x) = \cosh\left(\frac{\xi}{2^k}\right) \tag{2}$$

for $k = 1, 2, \ldots$. For $k = 1$, (2) follows directly from the double angle identity $\cosh(\phi/2) = \sqrt{1/2 + 1/2\cosh(\phi)}$ upon setting $\phi = \xi$. Suppose that $r_{\kappa-1}(x) = \cosh(\xi/2^{\kappa-1})$ for $\kappa \geq 2$. By definition, $r_\kappa(x) = \sqrt{1/2 + (1/2)r_{\kappa-1}(x)}$. From the inductive assumption together with the foregoing hyperbolic function identity we deduce, upon setting $\phi = \xi/2^{k-1}$, that (2) holds when $k = \kappa$. Hence, by induction (2) holds for all k. The second equality follows upon multiplying $r_k(x)$ by 2 and factoring this constant (written as $\sqrt{4}$) into each of the nested square roots. □

The first result we now prove deals with the closed-form evaluation of the following Viéte type infinite products and will be arrived at via an application of the standard formulas $\sinh(z)/z = \prod_{k=1}^{\infty} \cosh(z/2^k)$ and $\sin(z)/z = \prod_{k=1}^{\infty} \cos(z/2^k)$, valid for $z \neq 0$.

Theorem 1. *Consider a fixed real number* $x > 1$. *Then*

$$\frac{\sqrt{x^2-1}}{\ln(x+\sqrt{x^2-1})} = \prod_{k=1}^{\infty} \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{x}{2}}}}}}_{k \text{ square roots}}, \tag{3}$$

while if $-1 \le x < 1$*, then*

$$(4) \qquad \frac{\sqrt{1-x^2}}{\cos^{-1}(x)} = \prod_{k=1}^{\infty} \underbrace{\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{x}{2}}}}}}_{k \text{ square roots}}.$$

Proof. As $\cosh(\cdot) : (0, \infty) \to (1, \infty)$ is a bijection, there must exist a unique $\xi \in (0, \infty)$ such that $x = \cosh(\xi)$. After setting $z = \xi$ into the formula $\sinh(z)/z = \prod_{k=1}^{\infty} \cosh(z/2^k)$, we deduce from Lemma 1 that the right-hand side of (3) is equal to

$$(5) \qquad \prod_{k=1}^{\infty} \cosh\left(\frac{\xi}{2^k}\right) = \frac{\sinh(\xi)}{\xi}.$$

Now by definition $\xi = \cosh^{-1}(x)$, so via an application of the logarithmic form of the inverse hyperbolic cosine function we find that

$$\xi = \ln(x + \sqrt{x^2 - 1}).$$

Substituting this expression for ξ back into (5) and recalling $\sinh(\xi) = (e^{\xi} - e^{-\xi})/2$, one finds

$$\begin{aligned} \frac{\sinh(\xi)}{\xi} &= \frac{1}{2\ln(x + \sqrt{x^2-1})}\left(\frac{x + \sqrt{x^2-1}}{1} - \frac{1}{x + \sqrt{x^2-1}}\right) \\ &= \frac{1}{2\ln(x + \sqrt{x^2-1})}((x + \sqrt{x^2-1}) - (x - \sqrt{x^2-1})) \\ &= \frac{\sqrt{x^2-1}}{\ln(x + \sqrt{x^2-1})}, \end{aligned}$$

thus establishing (3). To demonstrate (4) first recall from (1) that the $k-1$ nested radical $R(b_1, b_2, \dots, b_k) = \cos(\theta)$, where θ is a rational multiple (dependent on b_i and z) of π. Via the main theorem of Servi (see [**3**, Theorem 1]), the explicit form of this θ is given by

$$(6) \quad \theta = \left(\frac{1}{2} - \frac{b_k}{4} - \frac{b_k b_{k-1}}{8} - \frac{b_k b_{k-1} b_{k-2}}{16} - \cdots - \frac{b_k b_{k-1} b_{k-2} \cdots b_1}{2^{k+1}}\right)\pi$$

where $z = 2\sin(b_1\pi/4)$ with $-2 \leq b_1 \leq 2$ and $b_i \in \{-1, 0, 1\}$ for all $i \neq 1$. Now for any $x \in [-1, 1)$ there exists a unique $b_1 \in [-2, 2]$ such that $x = z/2$. So by setting $b_2 = b_3 = \cdots = b_k = 1$ in (1) and (6) and factoring the multiplicative factor of $1/2$ (written as $1/\sqrt{4}$) in (1) into each of the $k-1$ nested square roots we find

$$\begin{aligned} R(b_1, 1, 1, \ldots, 1) &= \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{x}{2}}}}} \\ &= \cos\left(\frac{1}{2^{k-1}}\left(\frac{\pi}{2} - \frac{b_1\pi}{4}\right)\right). \end{aligned}$$

However since $\cos(\pi/2 - b_1\pi/4) = \sin(b_1\pi/4) = z/2 = x$ we deduce after setting $\theta = \pi/2 - b_1\pi/4 = \cos^{-1}(x)$ into the identity $\sin(\theta)/\theta = \prod_{k=1}^{\infty} \cos(\theta/2^k)$ that the right-hand side of (4) is equal to

$$\prod_{k=1}^{\infty} \cos\left(\frac{1}{2^k}\left(\frac{\pi}{2} - \frac{b_1\pi}{4}\right)\right) = \frac{\sin(\cos^{-1}(x))}{\cos^{-1}(x)} = \frac{\sqrt{1-x^2}}{\cos^{-1}(x)}.$$

as required. □

Clearly, substituting $x = 0$ into (4) results in the Viéte formula for $2/\pi$; however, (4) can also produce a myriad of other Viéte type infinite products, two of which are presented below and follow upon substituting $x = 1/2$ and $x = \sqrt{3}/2$ respectively into (4).

$$\begin{aligned} \frac{3\sqrt{3}}{2\pi} &= \prod_{k=1}^{\infty} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{3}{4}}}} \\ \frac{3}{\pi} &= \prod_{k=1}^{\infty} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \cdots + \frac{1}{2}\sqrt{\frac{2+\sqrt{3}}{4}}}}. \end{aligned}$$

To contrast equations (3) and (4), we now establish the second group of results, which unlike the previous, will follow from an application of two non-standard infinite product formulas derived using a triple angle identity for hyperbolic and trigonometric cosine functions.

Theorem 2. *Consider a fixed real number $x > 2$. Then*

$$\frac{x+1}{3} = \prod_{k=1}^{\infty}\Bigg(\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+x}}}}}_{k \text{ square roots}}-1\Bigg), \tag{7}$$

while if $-1 \leq x < 1$, then

$$\frac{2x+1}{3} = \prod_{k=1}^{\infty}\Bigg(2\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{x}{2}}}}}}_{k \text{ square roots}}-1\Bigg). \tag{8}$$

Proof. To prove (7) it will first be necessary to establish the finite product formula

$$\begin{aligned}\sinh(3z) = \sinh(z)(2\cosh(z)-1)&\left(2\cosh\left(\frac{z}{2^n}\right)+1\right)\\ &\prod_{r=1}^{n}\left(2\cosh\left(\frac{z}{2^r}\right)-1\right),\end{aligned} \tag{9}$$

valid for $n = 1, 2, 3, \ldots$, via the following inductive argument. Recalling the double angle identity $2\cosh(2\phi)+1 = 4\cosh^2(\phi)-1$, observe upon setting $\phi = z/2$ and $\phi = z$ that (9) must hold for $n = 1$, by virtue of the triple angle identity $\sinh(3z) = \sinh(z)(2\cosh(2z)+1)$. Suppose (9) holds for $n = \kappa$ where $\kappa \geq 1$. Then

$$\begin{aligned}\sinh(3z) = \sinh(z)(2\cosh(z)-1)&\left(2\cosh\left(\frac{z}{2^\kappa}\right)+1\right)\\ &\prod_{r=1}^{\kappa}\left(2\cosh\left(\frac{z}{2^r}\right)-1\right).\end{aligned} \tag{10}$$

Now substituting $\phi = z/2^{\kappa+1}$ again into the foregoing double angle identity, one finds that

$$\begin{aligned}2\cosh\left(\frac{z}{2^\kappa}\right)+1 &= 4\cosh^2\left(\frac{z}{2^{\kappa+1}}\right)-1\\ &= \left(2\cosh\left(\frac{z}{2^{\kappa+1}}\right)+1\right)\left(2\cosh\left(\frac{z}{2^{\kappa+1}}\right)-1\right),\end{aligned}$$

which upon replacing back into (10) implies that (9) holds for $n = \kappa + 1$. Hence, by induction (9) holds for all n. By dividing both sides of (9) by $\sinh(z)(2\cosh(z) - 1)$, for $z \neq 0$, and taking limits as $n \to \infty$, observe that

$$\frac{\sinh(3z)}{3(2\cosh(z) - 1)\sinh(z)} = \prod_{k=1}^{\infty} \left(2\cosh\left(\frac{z}{2^k}\right) - 1\right). \tag{11}$$

As the mapping $2\cosh(\cdot) : (0, \infty) \to (2, \infty)$ is a bijection and $x > 2$, a unique z in $(0, \infty)$ must exist such that $x = 2\cosh(z)$. Consequently, for this choice of z and by Lemma 1, the right-hand side of (11) is equal to $\prod_{k=1}^{\infty}(R_k(x) - 1)$, while the left-hand side of (11) can be evaluated in closed-form by using the logarithmic form of $z = \cosh^{-1}(x/2) = \ln((x + \sqrt{x^2 - 1})/2)$ as follows

$$\begin{aligned}
&\frac{\sinh(3z)}{3(2\cosh(z) - 1)\sinh(z)} \\
&\qquad = \frac{1}{3(x-1)} \frac{((x + \sqrt{x^2 - 4})/2)^3 - (2/(x + \sqrt{x^2 - 4}))^3}{((x + \sqrt{x^2 - 4})/2) - (2/(x + \sqrt{x^2 - 4}))} \\
&\qquad = \frac{1}{3(x-1)} \left\{ \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^2 + 1 + \left(\frac{2}{x + \sqrt{x^2 - 4}}\right)^2 \right\} \\
&\qquad = \frac{1}{3(x-1)} \left\{ \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^2 + 1 + \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^2 \right\} \\
&\qquad = \frac{x^2 - 1}{3(x-1)} = \frac{x+1}{3},
\end{aligned}$$

thus establishing (7). Finally, to prove (8) we shall need to first establish another finite product formula given as follows

$$\sin(3\theta) = \sin(\theta)(2\cos(\theta) - 1)\left(2\cos\left(\frac{\theta}{2^n}\right) + 1\right) \prod_{r=1}^{n} \left(2\cos\left(\frac{\theta}{2^r}\right) - 1\right), \tag{12}$$

valid for $n = 1, 2, 3, \ldots$. However this can be accomplished inductively in exactly the same manner as (9), using the double and triple angle cosine identities $2\cos(2\theta) + 1 = 4\cos^2(\theta) - 1$ and $\sin(3\theta) = \sin(\theta)(2\cos(2\theta) + 1)$, respectively. Now, by dividing both sides of (12) by $\sin(\theta)(2\cos(\theta)-1)$, for $\theta \neq 0$, and taking limits as $n \to \infty$, observe that

$$\frac{\sin(3\theta)}{3\sin(\theta)(2\cos(\theta)-1)} = \prod_{r=1}^{\infty}\left(2\cos\left(\frac{\theta}{2^r}\right)-1\right). \tag{13}$$

Since $-1 \leq x < 1$, there exists a unique θ in $(0,\pi]$ such that $x = \cos(\theta)$. Now by the argument used to establish equation (4) of Theorem 1, we have that

$$\cos\left(\frac{\theta}{2^k}\right) = \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{x}{2}}}}}}_{k \text{ square roots}};$$

moreover, the left-hand side of (13) can be simplified and evaluated in closed-form as follows

$$\frac{4\cos^2(\theta)-1}{3(2\cos(\theta)-1)} = \frac{4x^2-1}{3(2x-1)} = \frac{2x+1}{3},$$

as required. □

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