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GLOBALLY IMPULSIVE ASYMPTOTICAL SYNCHRONIZATION OF DELAYED CHAOTIC SYSTEMS WITH STOCHASTIC PERTURBATION

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ABSTRACT. In this paper, by applying the impulsive control approach, the globally asymptotical synchronization problem of delayed chaotic systems with stochastic perturbation is investigated. By establishing an L-operator differential inequality and using the properties of the M-cone, Hölder's inequality and stochastic analysis technique, we obtain some sufficient conditions ensuring the globally asymptotical p-stability of the error dynamical system. An example is also discussed to illustrate the efficiency of the results obtained.

1. Introduction. Since Pecora and Carrol presented the pioneering work of chaos synchronization in 1990, many researchers have done extensive work on this subject due to its potential applications in secure communication, chemical reactor, biological systems, information science, etc. In the meantime, many types of synchronization, such as anticipated, complete, generalized, phase, lag and exponential, have been presented in the past few years. A chaos synchronization problem means making two systems oscillate in a synchronized manner. In the real world, however, time delays are unavoidably encountered and their existence may lead to instability and oscillation in a real system. Therefore, chaos synchronization with time delays has received much attention [**3, 7, 8, 12, 13, 19**].

On the other hand, due to a real system is usually affected by external perturbations which in many cases are of great random uncertainties such as stochastic forces on the physical systems and noisy measure-

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ments caused by environmental uncertainties, a stochastic chaotic behavior should be produced instead of a deterministic one. Therefore, the stochastic perturbation should be taken into account in researching the synchronization of chaos systems. Recently, some stochastic synchronization results have been proposed [**5, 9, 11, 14, 16**].

As is well known, to achieve chaos synchronization a wide variety of approaches, such as the OGY method, coupling control, adaptive control method and time-delay feedback approach have been proposed. However, due to the difficulties in investigating stability of impulsive delayed differential equations (IDDE), there are few publications [**19**] dealing with the synchronization problem of coupled time-delayed stochastic chaotic systems via the impulsive control approach.

Motivated by the above discussions, this article is devoted to addressing the globally asymptotical synchronization problem of delayed chaotic systems with stochastic perturbation via the impulsive control approach. By establishing an L-operator differential inequality and using the properties of M-cone, Hölder's inequality and stochastic analysis technique, some sufficient conditions ensuring the globally asymptotical p-stability of the error dynamical system are obtained. Our method is simple and valid for analyzing chaotic systems with delays and stochastic perturbation, without using the Lyapunov functional and the differentiability of time-varying delays as needed in most other papers. An example is given to demonstrate the effectiveness of the theory results.

2. Preliminaries and model. Throughout this paper, unless otherwise specified, let E denote the n-dimensional unit matrix, $|\cdot|$ the Euclidean norm, $\mathcal{N} \triangleq \{1,2,\dots,n\}$, $R_+ = [0,\infty)$. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B$ ($A \leq B, A > B, A < B$) means that each pair of corresponding elements of A and B satisfies the inequality "$\geq$ ($\leq, >, <$)." Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

$C[X,Y]$ denotes the space of continuous mappings from topological space X to topological space Y. In particular, let $C \triangleq C[[-\tau,0],R^n]$.

$$PC[J,R^n] = \left\{ \psi : J \to R^n \;\middle|\; \begin{array}{c} \psi(s) \text{ is continuous for all but at most} \\ \text{countable points } s \in J \text{ and at these} \\ \text{points } s \in J,\ \psi(s^+) \text{ and } \psi(s^-) \text{ exist,} \\ \psi(s) = \psi(s^+) \end{array} \right\},$$

where $J \subset R$ is an interval, and $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$, respectively. In particular, let $PC \triangleq PC[[-\tau, 0], R^n]$.

For $x \in R^n$, $A \in R^{n\times n}$, we define

$$[x]^+ = (|x_1|, \ldots, |x_n|)^T \triangleq \mathrm{col}\,\{|x_i|\}_n, \ [A]^+ = (|a_{ij}|)_{n\times n}.$$

For $\varphi(t) \in C[J, R^n]$ or $\varphi(t) \in PC[J, R^n]$, we define

$$[\varphi(t)]_\tau = \mathrm{col}\,\{[\varphi_i(t)]_\tau\}_n, \qquad [\varphi_i(t)]_\tau = \sup_{-\tau\le s\le 0}\{\varphi_i(t+s)\}, \quad i \in \mathcal{N},$$

and $D^+\varphi(t)$ denotes the upper right derivative of $\varphi(t)$ at time t.

For any $\varphi(t) \in C$ or $\varphi(t) \in PC$, we always assume that φ is bounded and introduce the following norm:

$$\|\varphi\| = \sup_{-\tau\le s\le 0} |\varphi(s)|.$$

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let $PC^b_{\mathscr{F}_0}[[-\tau, 0], R^n]$ denote the family of all bounded $\mathscr{F}_0$-measurable, $PC[[-\tau, 0], R^n]$-valued random variables φ, satisfying $\|\varphi\|^p_{L^p} = \sup_{-\tau\le\theta\le 0} \mathbf{E}|\varphi(\theta)|^p < \infty$, where $\mathbf{E}$ denotes the expectation of a stochastic process.

It is assumed that the dynamics of the drive or master chaotic system is given by

$$(1) \qquad dx(t) = h(t)[A_0x(t) + Af(x(t)) + Bg(x(t - \tau(t))) + J]\, dt,$$

where $x \in R^n$ is the state vector, $h(t)$ is a scalar function, A_0, A, $B \in R^{n\times n}$ are constant matrices, f, $g : R^n \to R^n$ are continuous, $0 \le \tau(t) \le \tau$ (τ is a constant) is the time-delay, and $J \in R^n$ is a constant external input vector.

In order to deal with the synchronization problem of system (1) via impulsive control approach, we now integrate the impulse into it as follows:

(2)

$$\begin{cases} dx(t) = h(t)[A_0x(t) + Af(x(t)) + Bg(x(t - \tau(t))) + J]\, dt & t \ne t_k, \\ & t \ge t_0, \\ \Delta x = x(t_k^+) - x(t_k^-) = H_k x(t_k^-) \end{cases}$$

where t_k, $k = 1, 2, \ldots$ are constants and satisfy $t_0 \le t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

Inspired by [**7, 8, 15, 19**], we construct the response system as in the following form:

$$
(3)\qquad \begin{cases} dy(t) = h(t)[A_0 y(t) + Af(y(t)) + Bg(y(t-\tau(t))) \\ \qquad +J - W(y(t) - x(t))]\,dt + \sigma(t, e(t), e(t-\tau(t)))\,dw(t), \\ \qquad\qquad t \ne t_k,\ t \ge t_0, \\ \Delta y = y(t_k^+) - y(t_k^-) = -H_k e(t_k^-), \end{cases}
$$

where $W \in R^{n\times n}$ is a controller gain matrix to be designed later, $e(t) = y(t) - x(t)$ denotes the synchronization error, $\sigma : R_+ \times R^n \times R^n \to R^{n\times m}$ is continuous, and $w(t) = (w_1(t), \ldots, w_m(t))^T$ is an m-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$.

Therefore, the synchronization error between (1) and (3) can be expressed by the following dynamical system:

$$
(4)\qquad \begin{cases} de(t) = h(t)[(A_0 - W)e(t) + A(f(y(t)) - f(x(t))) \\ \qquad +B(g(y(t-\tau(t))) - g(x(t-\tau(t))))]\,dt \\ \qquad +\sigma(t, e(t), e(t-\tau(t)))\,dw(t), \qquad t \ne t_k,\ t \ge t_0, \\ e(t) = (E + H_k)e(t^-), \qquad t = t_k. \end{cases}
$$

Hence, the problem of synchronization between the drive system (1) and the response system (3) is shifted into the p-stability of the synchronization error system (4). In fact, from the analysis above, we can see that (1) and (3) are globally asymptotically synchronized if and only if the zero solution of (4) is globally asymptotically p-stable for any bounded initial condition. So, the globally impulsive asymptotical synchronization problem can be solved if the controller gain matrix W and controller impulse matrices H_k are suitably designed such that the zero solution of (4) is globally asymptotically p-stable.

Throughout this paper, we assume that, for any initial condition $\phi \in PC^b_{\mathscr{F}_0}[[-\tau, 0], R^n]$, there exists at least one solution of (4), which is denoted by $e(t, t_0, \phi)$, or $e(t)$, if no confusion occurs.

Definition 2.1. The zero solution of (4) is said to be globally asymptotically p-stable if it is p-stable and, moreover, for any given initial condition $\phi \in PC^b_{\mathscr{F}_0}[[-\tau, 0], R^n]$ such that $\lim_{t\to\infty} \mathbf{E}|e(t, t_0, \phi)|^p = 0$. In particular, it is said to be mean square asymptotically stable when $p = 2$.

Definition 2.2. The zero solution of (4) is said to be exponentially p-stable with exponential convergence rate λ if there is a pair of positive constants λ and γ such that for any solution $e(t, t_0, \phi)$ with the initial condition $\phi \in PC^b_{\mathscr{F}_0}[[-\tau, 0], R^n]$,

$$\mathbf{E}|e(t, t_0, \phi)|^p \leq \gamma \|\phi\|^p_{L^p} e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

In particular, it is said to be mean square exponentially stable when $p = 2$.

For an M-matrix D defined by [**4**], we define

$$\Omega_M(D) \triangleq \{z \in R^n \mid Dz > 0,\ z > 0\}.$$

Lemma 2.1 [**17**]. *For an M-matrix D, $\Omega_M(D)$ is nonempty, and for any $z_1,\ z_2 \in \Omega_M(D)$, we have*

$$k_1 z_1 + k_2 z_2 \in \Omega_M(D), \quad \textit{for all } k_1,\ k_2 > 0.$$

So $\Omega_M(D)$ is a cone without conical surface in R^n. We call it an"M-cone."

Lemma 2.2 ([**1**] Arithmetic-mean–geometric-mean inequality.) *For $x_i \geq 0$, $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$,*

$$\prod_{i=1}^n x_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i x_i, \tag{5}$$

the sign of equality holds if and only if $x_i = x_j$ for all $i, j \in \mathcal{N}$.

Lemma 2.3 ([**2**] Hölder's inequality). *If* $a_i \geq 0$*,* $b_i \geq 0$*,* $i \in \mathcal{N}$*,* $p > 0$*,* $q > 0$ *and* $(1/p) + (1/q) = 1$*, then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}. \tag{6}$$

3. ***L*-operator differential inequality.** In this section, we will first introduce a differential inequality with impulsive initial conditions and then establish an L-operator differential inequality.

Lemma 3.1 ([**6**, Lemma 5]). *Assume* $u(t) \in C[[t_0, \infty), \mathbf{R}^n]$ *satisfies that*

$$\begin{cases} D^+ u(t) \leq \varphi(t)\{Pu(t) + Q[u(t)]_\tau\} & t \geq t_0, \\ u(t_0 + \theta) = \phi(\theta) \in PC & \theta \in [-\tau, 0], \end{cases} \tag{7}$$

where $P = (p_{ij})_{n\times n}$ *and* $p_{ij} \geq 0$ *for* $i \neq j$*,* $Q = (q_{ij})_{n\times n} \geq 0$*,* $\varphi(t)$ *is a positive integral function and* $\sup_{t\geq t_0} \int_{t-\tau}^{t} \varphi(s)\,ds = K < \infty$ *and* $\lim_{t\to\infty} \int_{t_0}^{t} \varphi(s)\,ds = \infty$*. If* $D = -(P + Q)$ *is an* M*-matrix, then*

$$u(t) \leq z e^{-\lambda \int_{t_0}^{t} \varphi(s) ds}, \quad t \geq t_0, \tag{8}$$

provided that the initial conditions satisfy

$$u(t) \leq z e^{-\lambda \int_{t_0}^{t} \varphi(s) ds}, \quad t \in [t_0 - \tau, t_0], \tag{9}$$

where $z = \operatorname{col}\{z_i\} \in \Omega_M(D)$ *and the positive constant* λ *is determined by the following inequality*

$$[\lambda E + P + Q e^{\lambda K}] z < 0. \tag{10}$$

For the well known L-operator given by Itô's formula, we have the following theorem.

Theorem 3.1. *Let* $P = (p_{ij})_{n\times n}$ *and* $p_{ij} \geq 0$ *for* $i \neq j$*,* $Q(t) = (q_{ij}(t))_{n\times n} \geq 0$*,* $h(t)$ *a positive integral function and* $\sup_{t\geq t_0} \int_{t-\tau}^{t} h(s)$

$ds = \widehat{K} < \infty$ and $\lim_{t\to\infty}\int_{t_0}^t h(s)\,ds = \infty$, and let $D = -(P+Q)$ be a nonsingular M-matrix. Assume that there exist functions $V_i(x) \in C^2[R^n, R_+]$ such that, for the operator LV which is associated with system (4)*,*
(11)

$$LV_i(e) \le h(t)\sum_{j=1}^{n}\left[p_{ij}V_j(e) + q_{ij}V_j(e(t-\tau(t)))\right], \quad t \in [t_k, t_{k+1}),\ i \in \mathcal{N}.$$

Then

$$\mathbf{E}V_i(e(t)) \le z_i e^{-\lambda\int_{t_0}^t h(s)\,ds}, \quad t \in [t_k, t_{k+1}),\ i \in \mathcal{N}, \tag{12}$$

provided that the initial conditions satisfy

$$\mathbf{E}V_i(e(t)) \le z_i e^{-\lambda\int_{t_0}^t h(s)\,ds}, \quad t \in [t_k - \tau, t_k],\ i \in \mathcal{N}, \tag{13}$$

where $z = (z_1, z_2, \ldots, z_n)^T \in \Omega_M(D)$ and the positive constant λ satisfies the following inequality

$$[\lambda E + P + Qe^{\lambda\widehat{K}}]z < 0. \tag{14}$$

Proof. Similar to the proof of Theorem 3.1 in [**18**], this theorem can be proved by using Lemma 3.1. So the details are omitted here. □

4. Globally asymptotical p-stability. In this section, we will obtain several sufficient conditions ensuring the globally asymptotical p-stability of the zero solution of (4) by applying Theorem 3.1. We always suppose the following assumptions.

(A_1) For any x_j, $y_j \in R$, there exist nonnegative constants U_j, V_j such that

$$\begin{aligned}|f_j(x_j) - f_j(y_j)| &\le U_j|x_j - y_j|,\\ |g_j(x_j) - g_j(y_j)| &\le V_j|x_j - y_j|, \quad j \in \mathcal{N}.\end{aligned} \tag{15}$$

(A_2) $h(t)$ is a positive integral function, $\sup_{t\ge t_0}\int_{t-\tau}^t h(s)\,ds = \widehat{K} < \infty$ and $\lim_{t\to\infty}\int_{t_0}^t h(s)\,ds = \infty$.

(A_3) There exist nonnegative matrices $C = (c_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$ such that, for any x_i, $y_i \in R$, $i \in \mathcal{N}$,

$$|(\sigma_i(t, x_i, y_i))(\sigma_i(t, x_i, y_i))^T| \le \sum_{j=1}^{n} c_{ij} h(t)|x_i|^2 + \sum_{j=1}^{n} d_{ij} h(t)|y_i|^2. \tag{16}$$

(A_4) Let $A_0 - W = (\overline{a}_{ij})_{n\times n}$, $A = (a_{ij})_{n\times n}$. There exists a constant $p \ge 2$ such that $\widehat{D} = -(\widehat{P} + \widehat{Q})$ is an M-matrix, where

$$\begin{aligned} &\widehat{P} = (\widehat{p}_{ij})_{n\times n}, \qquad \widehat{p}_{ij} = (\overline{a}_{ij} + a_{ij}U_j) + (p-1)c_{ij}, \quad i \ne j, \\ &\widehat{p}_{ii} = p(\overline{a}_{ii} + a_{ii}U_i) + (p-1)\left(\sum_{j=1}^{n}(\overline{a}_{ij} + a_{ij}U_j + b_{ij}V_j)\right) \\ &\qquad + \frac{1}{2}(p-1)(p-2)\sum_{j=1}^{n}(c_{ij} + d_{ij}), \\ &\widehat{Q} = (\widehat{q}_{ij})_{n\times n}, \qquad \widehat{q}_{ij} = b_{ij}V_j + (p-1)d_{ij}. \end{aligned} \tag{17}$$

(A_5) Let $I_k = E + H_k$ and, for any $e(t) \in R^n$, there exist nonnegative matrices $M_k = (M_{k_{ij}})_{n\times n}$ such that

$$[I_k(e(t_k^-))]^+ \le M_k[e(t_k^-)]^+, \quad k = 1, 2, \ldots . \tag{18}$$

(A_6) The set $\Omega = \cap_{k=1}^{\infty}\Omega_\rho(\widehat{M}_k) \cap \Omega_M(\widehat{D})$ is nonempty, where $\widehat{M}_k = (\widehat{M}_{k_{ij}})_{n\times n}$ satisfies

$$\sum_{j=1}^{n} \hat{M}_{k_{ij}} \ge \left(\sum_{j=1}^{n} M_{k_{ij}}^{p/(p-1)}\right)^{p-1}. \tag{19}$$

(A_7) Let

$$\gamma_k = \max\{1, \rho(\widehat{M}_k)\} \tag{20}$$

and there exists a constant η such that

$$\frac{\ln \gamma_k}{\int_{t_{k-1}}^{t_k} h(s)\,ds} \le \eta < \lambda, \quad k = 1, 2, \ldots , \tag{21}$$

where the positive constant λ is determined by the following inequality:

$$\left[\lambda E + \widehat{P} + \widehat{Q}e^{\lambda \widehat{K}}\right] z < 0, \quad \text{for a given } z \in \Omega. \tag{22}$$

Remark 4.1. It is evident that system (4) has a zero solution $e(t) \equiv 0$ from (A_1), (A_3) and (A_5).

Theorem 4.1. *Assume that* (A_1)–(A_7) *hold. Then the zero solution of* (4) *is globally asymptotically* p*-stable, implying that the two systems* (1) *and* (3) *are globally asymptotically synchronized.*

Proof. Since $\widehat{D} = -(\widehat{P} + \widehat{Q})$ is a nonsingular M-matrix and the set Ω is nonempty, there must be a vector $z \in \Omega \subset \Omega_M(\widehat{D})$ such that

$$\widehat{D}z > 0, \quad \text{or} \quad \left[\widehat{P} + \widehat{Q}\right] z < 0.$$

By using continuity, we obtain that (22) has at least a positive solution λ.

Let $V_i(e(t)) = |e_i(t)|^p$, $p \geq 2$, $i \in \mathcal{N}$, where $e(t) = (e_1(t), \dots, e_n(t))^T$ is the solution of (4). Then

$$\begin{aligned} \frac{\partial V_i(e)}{\partial e_i} &= p|e_i|^{p-1}\mathrm{sgn}\,(e_i) = p|e_i|^{p-2}e_i, \\ \frac{\partial V_i^2(e)}{\partial e_i^2} &= p(p-1)|e_i|^{p-2}\mathrm{sgn}\,(e_i), \end{aligned} \tag{23}$$

where $\mathrm{sgn}\,(\cdot)$ is the sign function. Thus, by conditions (A_1), (A_3) and Lemma 2.2, we have

$$LV_i(e) \leq h(t)\left\{ \sum_{j=1}^{n} \hat{p}_{ij}V_j(e) + \sum_{j=1}^{n} \widehat{q}_{ij}V_j(e(t-\tau(t)))\right\}. \tag{24}$$

So, from condition (A_4), we know that inequality (11) holds.

For the initial condition $\phi \in PC^b_{\mathscr{F}_0}[[-\tau, 0], R^n]$, we can get

(25)

$$\mathbf{E}V_i(e(t)) \leq k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds}, \quad k_0 = \frac{\|\phi\|}{\min_{1\leq i\leq n}\{z_i\}}, \; t \in [t_0 - \tau, t_0].$$

From Lemma 2.1 and $z = (z_1, \dots, z_n)^T \in \Omega_M(\widehat{D})$, we have $k_0 z \in \Omega_M(\widehat{D})$. Then, all conditions of Theorem 3.1 are satisfied by (24), (25) and (A_4), so

$$\mathbf{E}V_i(e(t)) \le k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds}, \quad t \in [t_0, t_1),\ i \in \mathcal{N}. \tag{26}$$

Suppose that, for all $m = 1, \dots, k$, the inequalities

$$\mathbf{E}V_i(e(t)) \le \gamma_0 \cdots \gamma_{m-1} k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds}, \quad t \in [t_{m-1}, t_m),\ i \in \mathcal{N}, \tag{27}$$

hold, where $\rho(\widehat{M_0}) = 1$. Then from (A_5), (A_6), (27) and Lemma 2.3, we have

$$\begin{aligned}
\mathbf{E}V_i(e(t_k)) &= \mathbf{E}|e_i(t_k)|^p \\
&= \mathbf{E}|I_{ik}(e(t_k^-))|^p \le \mathbf{E}\Bigg(\sum_{j=1}^{n} M_{k_{ij}} |e_j(t_k^-)|\Bigg)^p \\
&\le \mathbf{E}\Bigg[\Bigg(\sum_{j=1}^{n} M_{k_{ij}}^{p/(p-1)}\Bigg)^{(p-1)} \sum_{j=1}^{n} |e_j(t_k^-)|^p\Bigg] \\
&\le \sum_{j=1}^{n} \widehat{M}_{k_{ij}} \mathbf{E}V_j(e(t_k^-)) \qquad (28) \\
&\le \gamma_0 \cdots \gamma_{k-1} k_0\, e^{-\lambda \int_{t_0}^{t_k} h(s)\,ds} \sum_{j=1}^{n} \widehat{M}_{k_{ij}} z_j \\
&\le \gamma_0 \cdots \gamma_{k-1} k_0\, e^{-\lambda \int_{t_0}^{t_k} h(s)\,ds} \rho(\widehat{M}_k) z_i \\
&\le \gamma_0 \cdots \gamma_{k-1} \gamma_k k_0 z_i e^{-\lambda \int_{t_0}^{t_k} h(s)\,ds}, \quad i \in \mathcal{N}.
\end{aligned}$$

This, together with (27) and $\rho(\widehat{M}_k) \ge 1$, leads to

$$\mathbf{E}V_i(e(t)) \le \gamma_0 \cdots \gamma_{k-1} \gamma_k k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds}, \quad t \in [t_k - \tau, t_k],\ i \in \mathcal{N}. \tag{29}$$

By Lemma 2.1 again, the vector $\gamma_0 \cdots \gamma_{k-1} \gamma_k k_0 z \in \Omega_M(\widehat{D})$. It follows from (24), (29) and Theorem 3.1 that

$$\mathbf{E}V_i(e(t)) \le \gamma_0 \cdots \gamma_{k-1} \gamma_k k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds}, \quad t \in [t_k, t_{k+1}),\ i \in \mathcal{N}.$$

By induction, we conclude that

$$\mathbf{E}V_i(e(t)) \le \gamma_0 \cdots \gamma_{k-1} k_0 z_i e^{-\lambda \int_{t_0}^{t} h(s)\,ds} \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \dots . \tag{30}$$

From (21),

$$\gamma_{k-1} \le e^{\eta \int_{t_{k-2}}^{t_{k-1}} h(s)\,ds}, \qquad \gamma_0 \cdots \gamma_{k-1} \le e^{\eta \int_{t_0}^{t_{k-1}} h(s)\,ds}, \tag{31}$$

and (30), we conclude that

$$\mathbf{E}V_i(e(t)) \le k_0 z_i e^{\eta \int_{t_0}^{t} h(s)\,ds} e^{-\lambda \int_{t_0}^{t} h(s)\,ds} \le k_0 z_i e^{-(\lambda-\eta) \int_{t_0}^{t} h(s)\,ds}, \quad i \in \mathcal{N},\ t \in [t_{k-1}, t_k). \tag{32}$$

That is,

$$\mathrm{col}\,\{\mathbf{E}|e_i(t)|^p\}_n \le k_0 z e^{-(\lambda-\eta) \int_{t_0}^{t} h(s)\,ds}, \quad t \in [t_0, t_k),\ k = 1, 2, \dots . \tag{33}$$

Thus, the proof is complete. □

Corollary 4.1. *If $h(t) \equiv 1$, the zero solution of* (4) *is exponentially p-stable, implying that the two systems,* (1) *and* (3), *are exponentially synchronized.*

Remark 4.2. In Theorem 4.1, we may properly choose the matrices $\widehat{M}_k$ in the condition (A_6) to guarantee $\Omega \neq \varnothing$. In particular, when $\widehat{M}_k = \xi_k E$ (ξ_k are nonnegative constants), Ω is certainly nonempty and $\Omega = \Omega_M(\widehat{D})$.

Remark 4.3. If $H_k = 0$, (4) becomes the stochastic delay dynamical system without impulses,

$$\begin{aligned} de(t) = h(t)[(A_0 - W)e(t) + A(f(y(t)) - f(x(t))) \\ + B(g(y(t-\tau(t))) - g(x(t-\tau(t))))]\,dt \\ + \sigma(t, e(t), e(t-\tau(t)))\,dw(t), \quad t \ge t_0. \end{aligned} \tag{34}$$

For the system (34), our results are also true. By Theorem 4.1, it is easy to get the following corollary.

Corollary 4.2. *Assume that* (A_1)–(A_4) *hold, then the zero solution of* (34) *is globally asymptotically p-stable, implying that the two systems,* (1) *and* (3)*, with* $H_k = 0$ *are globally asymptotically synchronized.*

Proof. Since $I_k = E$ $(k = 1, 2, \dots)$, $M_k = E$ in condition (A_5) and $\widehat{M}_k = E$ in condition (A_6). By Remark 4.2, $\Omega = \Omega_M(\widehat{D}) \neq \varnothing$. Then, $\rho(\widehat{R}_k) = 1$ and $\gamma_k = 1$ in condition (A_7). So, $\eta = 0$ satisfies (21). Thus, applying Theorem 4.1 to system (34), we can obtain Corollary 4.2. □

Remark 4.4. In [**7**], Li and Cao discussed the the stability of (34) with $h(t) \equiv 1$ and $f = g$ by using the LaSalle-type invariance principle for stochastic differential delay equations. The assumptions that $\dot{\tau}(t) < 1$ and f are monotone are required in [**7**]. However, Corollary 4.2 does not require these assumptions.

Remark 4.5. In [**10, 14, 15**], the authors give some stochastic synchronization results. The methods in their papers are all based on Lyapunov function. However, the construction of Lyapunov function is skillful and complicated. Therefore, the effectiveness of the conditions provided is difficult to check in practice. Our method in this paper is different from the methods mentioned in the above papers. Without referring to any Lyapunov function, we obtain some sufficient conditions ensuring the globally asymptotical p-stability of the error dynamical system. And the proposed globally asymptotical p-stability criteria is easily verified in practice.

5. Example. The following illustrative example will demonstrate the effectiveness of our results.

Example 5.1. Consider the following time-delayed chaotic system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = h(t)\left\{ A_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + A \begin{pmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{pmatrix} \right. \\ \left. + B \begin{pmatrix} g_1(x_1(t-\tau(t))) \\ g_2(x_2(t-\tau(t))) \end{pmatrix} \right\}. \tag{35}$$

If taking $h(t) = 6 + \sin t$, $f_i(x_i) = \tanh(x_i)$, $g_i(x_i) = 1/2(|x_i + 1| - |x_i - 1|)$, $i = 1, 2$, $\tau(t) = \sin^2 t \leq 1 \triangleq \tau$.

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -0.1 \\ -0.4 & 1.5 \end{pmatrix}, \quad B = \begin{pmatrix} -0.5 & -0.8 \\ -0.5 & -2 \end{pmatrix}.$$

We obtain that system (35) satisfies condition (A_1) with

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

And (A_2) is satisfied with $\widehat{K} = \sup_{t\geq t_0} \int_{t-\tau}^{t} h(s)\,ds < 7 < \infty$, $\lim_{t\to\infty} \int_{t_0}^{t} h(s)\,ds = \infty$.

For drive system (35) , construct a corresponding response system as follows:

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} &= h(t)\Bigg\{ A_0 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + A \begin{pmatrix} f_1(y_1(t)) \\ f_2(y_2(t)) \end{pmatrix} \\ &\qquad + B \begin{pmatrix} g_1(y_1(t-\tau(t))) \\ g_2(y_2(t-\tau(t))) \end{pmatrix} - W \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix} \Bigg\} \\ &\quad + \begin{pmatrix} \sigma_1(t, e(t), e(t-\tau(t))) \\ \sigma_2(t, e(t), e(t-\tau(t))) \end{pmatrix} dw(t), \\ \Delta y(t) &= -H_k e(t^-), \quad t = t_k, \end{aligned} \tag{36}$$

where $t_0 = 0$ and $t_k = t_{k-1} + 0.2k$ for $k = 1, 2, \dots$.

The controller gain matrix W, stochastic perturbation $\sigma(t, e(t), e(t-\tau(t)))$, and the controller impulse matrix H_k are chosen as follows:

$$\begin{aligned} W &= \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}, \\ \sigma &= \sqrt{6 + \sin t} \begin{pmatrix} e_1(t) & e_2(t-\tau(t)) \\ e_2(t) & e_1(t-\tau(t)) \end{pmatrix}, \\ H_k &= \begin{pmatrix} -\alpha_{1k} & -\beta_{1k} \\ -\beta_{2k} & -\alpha_{2k} \end{pmatrix}, \end{aligned}$$

where α_{ik} and β_{ik} are nonnegative constants, yielding

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Letting $p = 3$, we have

$$\widehat{P} = \begin{pmatrix} -3.5 & -0.2 \\ -0.5 & -3 \end{pmatrix}, \qquad \widehat{Q} = \begin{pmatrix} -0.5 & 1.2 \\ 1.5 & -2 \end{pmatrix}.$$

One can easily verify that $\widehat{D} = -(\widehat{P} + \widehat{Q})$ is a nonsingular M-matrix. By simple calculation, we have

$$\Omega_M(\widehat{D}) = \left\{(z_1, z_2)^T > 0 \mid \frac{1}{5}z_1 < z_2 < 4z_1\right\}. \tag{37}$$

Now, we discuss the asymptotic behavior for the synchronization error of two coupled chaotic systems as follows:

Case 5.1. Let $\alpha_{1k} = 0.5e^{0.01k}$, $\alpha_{2k} = 0.7e^{0.01k}$, $\beta_{1k} = 0.2e^{0.01k}$, $\beta_{2k} = 0.6e^{0.01k}$. Then we get

$$\begin{aligned} I_k = E + H_k &= \begin{pmatrix} 1 - 0.5e^{0.01k} & -0.2e^{0.01k} \\ -0.6e^{0.01k} & 1 - 0.7e^{0.01k} \end{pmatrix}, \\ M_k &= 0.5e^{0.01k} \begin{pmatrix} 1 & 0.4 \\ 1.2 & 0.6 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \widehat{M_k} &= 0.25e^{0.03k} \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \\ \rho(\widehat{M_k}) &= e^{0.03k}, \\ \Omega_\rho(\widehat{M_k}) &= \left\{(z_1, z_2)^T \mid z_2 = 2z_1\right\}. \end{aligned}$$

Thus $\widehat{M}_{k_{ij}}$ satisfies (19) and $\Omega = \{(z_1, z_2)^T > 0 \mid z_2 = 2z_1\}$ is nonempty. Letting $z = (1, 2)^T \in \Omega$ and $\gamma_k = e^{0.03k}$, we can obtain that for $k = 1, 2 \ldots,$

$$\begin{aligned} \frac{\ln \gamma_k}{\int_{t_{k-1}}^{t_k} h(s)\, ds} &= \frac{\ln e^{0.03k}}{\int_{t_{k-1}}^{t_k} (6 + \sin s)\, ds} \le \frac{\ln e^{0.03k}}{\int_{t_{k-1}}^{t_k} 5\, ds} \\ &= \frac{0.03k}{k} = 0.03 = \eta < \lambda = 0.1. \end{aligned} \tag{38}$$

So, by Theorem 4.1, the synchronization error of two coupled chaotic systems is globally asymptotically 3-stable and the convergence rate is equal to 0.07.

Case 5.2. Let $\alpha_{1k} = \alpha_{2k} = \beta_{1k} = \beta_{2k} = 0$. Then $I_k = E$. In addition, the error dynamical system between (35) and (36) becomes the delayed stochastic chaotic system without impulse. So, by Corollary 4.2, the synchronization error of two coupled chaotic systems is globally asymptotically 3-stable for sufficiently small $\varepsilon > 0$.

Remark 5.1. Since $\tau(t) = \sin^2(t)$ doesn't always satisfy the condition that $\dot{\tau}(t) < 1$, the methods in [**7**] are ineffective for studying the synchronization of the two coupled chaotic systems (35) and (36).

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