ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 1, 2012

BEST APPROXIMATION FORMULAS FOR THE DUNKL L^2-MULTIPLIER OPERATORS ON $\mathbf{R}^d$

FETHI SOLTANI

ABSTRACT. We study the Dunkl L^2-multiplier operators on $\mathbf{R}^d$, and we give for them Calderón's reproducing formulas and best approximation formulas using the theory of Dunkl transform and reproducing kernels.

1. Introduction. The Dunkl operators $\mathcal{D}_j$; $j = 1, \dots, d$, on $\mathbf{R}^d$, are parameterized differential-difference operators [**2**], acting on some Euclidean space. These operators extend the usual partial derivatives by additional reflection terms and give rise to generalizations of many multi-variable analytic structures like the exponential function, the Fourier transform and the standard convolution [**3, 4, 7, 15**]. During the last decade, such operators have found considerable attention in various areas of mathematics and mathematical physics [**3, 4, 7, 9**]. They allow the development of Dunkl L^2-multiplier operators on $\mathbf{R}^d$ from classical theory of Fourier analysis (see [**6, 11, 12, 17**]).

The Dunkl analysis, with respect to the multiplicity function k, concerns the Dunkl operators $\mathcal{D}_j$, Dunkl transform $\mathcal{F}_k$ and Dunkl convolution $*_k$ on $\mathbf{R}^d$. In the limit case $k = 0$; $\mathcal{D}_j$, $\mathcal{F}_k$ and $*_k$ agree with the partial derivatives ∂_j, Fourier transform $\mathcal{F}$ and standard convolution $*$, respectively.

Let m be a function in the Lebesgue space $L^2(\mathbf{R}^d, w_k(x)dx)$, where w_k is a positive weight function on $\mathbf{R}^d$ which will be defined later in Section 2. We define the Dunkl L^2-multiplier operators on $\mathbf{R}^d$, for regular functions f, by

$$T_{k,m,a}f(x) := \mathcal{F}_k^{-1}[m_a\mathcal{F}_k(f)](x), \quad a > 0,$$

where m_a is the function given by

$$m_a(x) = m(ax).$$

2010 AMS *Mathematics subject classification.* Primary 42C15, 44A15.
Keywords and phrases. Dunkl operators, L^2-multiplier operators, Calderón's reproducing formulas, best approximation formulas.
Received by the editors on March 31, 2009, and in revised form on June 24, 2009.
DOI:10.1216/RMJ-2012-42-1-305

The purpose of this paper is to study the multiplier operators $T_{k,m,a}$. Especially, we give Calderón's reproducing formula using the theory of Dunkl transform and Dunkl convolution; and we use the theory of reproducing kernels to give best approximation of these operators and a Calderón's reproducing formula of the associated extremal function.

In the one-dimensional case the Dunkl L^p-multiplier operators are studied by [**20**] in which the author gives some applications.

The contents of the paper are as follows. In Section 2, we recall some basic harmonic analysis results related to the Dunkl operators on $\mathbf{R}^d$. In particular, we list some basic properties of the Dunkl transform $\mathcal{F}_k$ and the Dunkl convolution product $*_k$ (Plancherel theorem, inversion formula, etc.).

In Section 3, we study the Dunkl L^2-multiplier operators $T_{k,m,a}$, and we give for them a Plancherel formula and pointwise reproducing inversion formula.

Next, we use the theory of Dunkl transform to give Calderón's reproducing formula. Let $m \in L^2 \cap L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ satisfy the admissibility condition:

$$\int_0^\infty |m(ax)|^2 \frac{da}{a} = 1, \quad \text{a.e. } x \in \mathbf{R}^d.$$

Then for $f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ and $0 < \varepsilon < \delta < \infty$, the function $f_{\varepsilon,\delta}$ given by

$$f_{\varepsilon,\delta}(x) := \int_\varepsilon^\delta \Big[T_{k,m,a} f *_k \mathcal{F}_k^{-1}(\overline{m_a})\Big](x)\,\frac{da}{a}, \quad x \in \mathbf{R}^d,$$

belongs to $L^2(\mathbf{R}^d, w_k(x)dx)$ and satisfies:

$$\lim_{\varepsilon\to 0, \delta\to\infty} \|f_{\varepsilon,\delta} - f\|_{L^2_k} = 0.$$

The last section of this paper is devoted to giving best approximation of the operators $T_{k,m,a}$, for m in $L^\infty(\mathbf{R}^d, w_k(x)\,dx)$.

For ρ a positive function on $\mathbf{R}^d$ satisfying the conditions:

$$\rho(z) \geq 1, \quad \int_{\mathbf{R}^d} \frac{w_k(z)}{\rho(z)}\,dz < \infty,$$

we consider the Sobolev type space $\mathcal{H}_\rho(\mathbf{R}^d)$ consisting of functions $f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ such that $\sqrt{\rho}\,\mathcal{F}_k(f) \in L^2(\mathbf{R}^d, w_k(x)\,dx)$.

The space $\mathcal{H}_\rho(\mathbf{R}^d)$ is a Hilbert space when endowed with the inner product

$$\langle f, g\rangle_\rho := \int_{\mathbf{R}^d} \rho(z)\mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g)(z)}w_k(z)\,dz.$$

Using the properties of the Dunkl transform $\mathcal{F}_k$, for $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, the operators $T_{k,m,a}$ are bounded from $\mathcal{H}_\rho(\mathbf{R}^d)$ into $L^2(\mathbf{R}^d, w_k(x)\,dx)$.

Next, for $\lambda > 0$, we define on the space $\mathcal{H}_\rho(\mathbf{R}^d)$, the new inner product by setting

$$\langle f, g\rangle_{\rho,\lambda} = \lambda\langle f, g\rangle_\rho + \frac{1}{d_k}\langle T_{k,m,a}f, T_{k,m,a}g\rangle_{L^2_k},$$

where d_k is the Mehta-type constant which will be defined later in Section 2. We show that $\mathcal{H}_\rho(\mathbf{R}^d)$ is a Hilbert space when equipped with the inner product $\langle\cdot,\cdot\rangle_{\rho,\lambda}$, and we exhibit its reproducing kernel $\mathcal{K}_{\rho,\lambda}$.

Building on the ideas of Saitoh, Matsuura and Yamada [**10, 16, 19, 23**], and using the theory of reproducing kernels [**1**], we give best approximation of the operators $T_{k,m,a}$ and nice estimates of the associated extremal function. More precisely, for all $\lambda > 0$, $h \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ and $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, the infimum

$$\inf_{f\in\mathcal{H}_\rho}\left\{\lambda\|f\|^2_\rho + \frac{1}{d_k}\|h - T_{k,m,a}f\|^2_{L^2_k}\right\},$$

is attained at one function $f^*_{\lambda,h,a}$, called the extremal function.

In particular, for $f \in \mathcal{H}_\rho(\mathbf{R}^d)$ and $h = T_{k,m,a}f$, the corresponding extremal function $f^*_{\lambda,a} = f^*_{\lambda,h,a}$ satisfies the following Calderón's reproducing formula:

$$\lim_{\lambda\to0^+}\|f^*_{\lambda,a} - f\|_\rho = 0.$$

Moreover, $\{f^*_{\lambda,a}\}_{\lambda>0}$ converges uniformly to f as $\lambda \to 0^+$.

2. The Dunkl harmonic analysis on $\mathbf{R}^d$. We consider $\mathbf{R}^d$ with the Euclidean inner product $\langle\cdot,\cdot\rangle$ and norm $\|y\| := \sqrt{\langle y, y\rangle}$.

For $\alpha \in \mathbf{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbf{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbf{R}^d\backslash\{0\}$ is called a root system, if $R \cap \mathbf{R}\cdot\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We assume that it is normalized by $\|\alpha\|^2 = 2$ for all $\alpha \in R$.

For a root system R, the reflections σ_α, $\alpha \in R$ generate a finite group $G \subset O(d)$, the reflection group associated with R. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbf{R}^d\backslash \cup_{\alpha\in R} H_\alpha$, we fix the positive subsystem:

$$R_+ := \{\alpha \in R \ / \ \langle \alpha, \beta\rangle > 0\}.$$

Then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

Let $k : R \to \mathbf{C}$ be a multiplicity function on R (i.e., a function which is constant on the orbits under the action of G). For abbreviation, we introduce the index:

$$\gamma = \gamma(k) := \sum_{\alpha\in R_+} k(\alpha).$$

Moreover, let w_k denote the weight function:

$$w_k(y) := \prod_{\alpha\in R_+} |\langle \alpha, y\rangle|^{2k(\alpha)}, \quad y \in \mathbf{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

We introduce the Mehta-type constant c_k, by

$$c_k := \left[\int_{\mathbf{R}^d} e^{-\|y\|^2} w_k(y)\, dy\right]^{-1}.$$

Etingof [**5**] has recently calculated this integral with a method valid for all reflection groups.

We denote by d_k the modified square of the Mehta-type constant c_k:

$$d_k := 2^{-2\gamma-d} c_k^2.$$

The Dunkl operators $\mathcal{D}_j$, $j = 1, \dots, d$, on $\mathbf{R}^d$ associated with the finite reflection group G and multiplicity function k are given for a function f of class C^1 on $\mathbf{R}^d$, by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y\rangle}.$$

For $y \in \mathbf{R}^d$, the initial problem $\mathcal{D}_j u(x, \cdot)(y) = x_j u(x, y)$, $j = 1, \dots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbf{R}^d$, which will be denoted by $E_k(x, y)$ and called the Dunkl kernel [**3, 7**].

This kernel has the Laplace-type representation (see [**14**]):

$$E_k(x, z) = \int_{\mathbf{R}^d} e^{\langle y, z\rangle} d\Gamma_x(y); \quad x \in \mathbf{R}^d,\ z \in \mathbf{C}^d, \tag{1}$$

where $\langle y, z\rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbf{R}^d$ such that

$$\operatorname{supp}(\Gamma_x) \subset \{y \in \mathbf{R}^d \ / \ \|y\| \leq \|x\|\}.$$

Therefore, we obtain

$$|E_k(-ix, y)| \leq 1, \quad \text{for every } x, y \in \mathbf{R}^d. \tag{2}$$

In particular, if $d = 1$ and $G = \mathbf{Z}_2$, we have

$$E_\gamma(xz) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} \int_{-1}^{1} e^{xzt}(1 - t^2)^{\gamma - 1}(1 + t)\, dt; \quad x, z \in \mathbf{R}.$$

The Dunkl kernel gives rise to an integral transform, called the Dunkl transform on $\mathbf{R}^d$, which was introduced by Dunkl [**4**], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu [**7**]. The Dunkl transform of a function f in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$, is given by

$$\mathcal{F}_k(f)(x) := \int_{\mathbf{R}^d} E_k(-ix, y) f(y) w_k(y)\, dy, \quad x \in \mathbf{R}^d.$$

We notice that $\mathcal{F}_0$ agrees with the Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(x) := \int_{\mathbf{R}^d} e^{-i\langle x, y\rangle} f(y)\, dy, \quad x \in \mathbf{R}^d.$$

We denote by $L^p(\mathbf{R}^d, w_k(x)\,dx)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbf{R}^d$, such that

$$\|f\|_{L_k^p} := \left[\int_{\mathbf{R}^d} |f(x)|^p w_k(x)\,dx\right]^{1/p} < \infty, \quad p \in [1,\infty[\,,$$
$$\|f\|_{L_k^\infty} := \operatorname{ess}\sup_{x\in\mathbf{R}^d} |f(x)| < \infty,$$

and by $L^p_{\mathrm{rad}}(\mathbf{R}^d, w_k(x)\,dx)$ the subspace of $L^p(\mathbf{R}^d, w_k(x)\,dx)$ consisting of radial functions.

For the Dunkl transform $\mathcal{F}_k$, de Jeu [**7**] proved the following properties.

Proposition 1. (i) *For all* $f \in L^1(\mathbf{R}^d, w_k(x)\,dx)$, *then* $\mathcal{F}_k(f) \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, *and we have*

$$\|\mathcal{F}_k(f)\|_{L_k^\infty} \le \|f\|_{L_k^1}.$$

(ii) $\mathcal{F}_k(\mathcal{D}_j f)(x) = ix_j\mathcal{F}_k(f)(x)$, *for every* $f \in \mathcal{S}(\mathbf{R}^d)$ *and* $x \in \mathbf{R}^d$.

(iii) $\mathcal{F}_k$ *is a topological isomorphism from* $\mathcal{S}(\mathbf{R}^d)$ *onto itself, and*

$$f(x) = d_k \int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)w_k(y)\,dy, \quad x \in \mathbf{R}^d.$$

The Dunkl transform enjoys properties similar to those of the classical Fourier transform.

Theorem 1 (see [**7**]). (i) **Plancherel formula**: *For all* $f \in \mathcal{S}(\mathbf{R}^d)$, *we have*

$$\|f\|^2_{L_k^2} = d_k\|\mathcal{F}_k(f)\|^2_{L_k^2}.$$

(ii) **Plancherel theorem**: *The normalized Dunkl transform* $\sqrt{d_k}\,\mathcal{F}_k$ *extends uniquely to an isometric isomorphism of* $L^2(\mathbf{R}^d, w_k(x)\,dx)$ *onto itself.*

(iii) **Inversion formula**: *Let f be a function in $L^1(\mathbf{R}^d, w_k(x)\,dx)$, such that $\mathcal{F}_k(f) \in L^1(\mathbf{R}^d, w_k(x)\,dx)$. Then*

$$f(x) = d_k \int_{\mathbf{R}^d} E_k(ix, y)\mathcal{F}_k(f)(y)w_k(y)\,dy, \quad a.e.\ x \in \mathbf{R}^d.$$

In [**22**], Thangavelu and Xu give the following definition of Dunkl translation operators.

Definition 1. The Dunkl translation operators τ_x, $x \in \mathbf{R}^d$, are defined on $L^2(\mathbf{R}^d, w_k(x)\,dx)$ by the equation:

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y), \quad y \in \mathbf{R}^d.$$

Note that from Theorem 2 (ii) and relation (2), the definition makes sense, and we have

$$\|\tau_x f\|_{L_k^2} \leq \|f\|_{L_k^2}, \quad f \in L^2(\mathbf{R}^d, w_k(x)\,dx). \tag{3}$$

When the function f is in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$, we obtain

$$\tau_x f(y) = d_k \int_{\mathbf{R}^d} E_k(ix, z)E_k(iy, z)\mathcal{F}_k(f)(z)w_k(z)\,dz; \quad x, y \in \mathbf{R}^d.$$

In the one-dimensional case, Rösler [**13**] established an explicit formula for Dunkl translation operators.

Proposition 2 (see [**13**]). *If $d = 1$ and $G = \mathbf{Z}_2$, then for $f \in L^2(\mathbf{R}, |x|^{2\gamma}dx)$ and $x, y \in \mathbf{R}$ such that $(x, y) \neq (0, 0)$, we have*

$$\tau_x f(y) = \int_0^\pi \left[f_e((x, y)_\theta) + f_0((x, y)_\theta)\frac{x + y}{(x, y)_\theta} \right] d\nu_{x,y}^\gamma(\theta),$$

where

$$d\nu_{x,y}^\gamma(\theta) := \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\,\Gamma(\gamma)}[1 - \operatorname{sgn}(xy)\cos\theta]\sin^{2\gamma-1}\theta\,d\theta,$$
$$(x, y)_\theta := \sqrt{x^2 + y^2 - 2|xy|\cos\theta},$$

and

$$f_e(z) = \frac{f(z) + f(-z)}{2}, \qquad f_0(z) = \frac{f(z) - f(-z)}{2}.$$

In paper [**15**] Rösler proved a modified radial product formula, involving the ordinary Bessel function, for the Dunkl kernel in d dimensions, and hence introduced Dunkl translation operators for radial functions.

Proposition 3 (see [**15**]). *If $f \in L^2_{\mathrm{rad}}(\mathbf{R}^d, w_k(x)\,dx)$, then we have*

$$\tau_x f(y) = \int_{\mathbf{R}^d} F\Big(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle y, z\rangle}\Big)\, d\Gamma_x(z), \quad x, y \in \mathbf{R}^d,$$

where $f(x) = F(\|x\|)$ and Γ_x is the representing measure given by (1).

Definition 2. The Dunkl convolution product $*_k$ of two functions f and g in $L^2(\mathbf{R}^d, w_k(x)\,dx)$ is defined by

$$f *_k g(x) := \int_{\mathbf{R}^d} \tau_x f(-y) g(y) w_k(y)\,dy, \quad x \in \mathbf{R}^d.$$

Note that, as $\tau_x f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$, the above convolution is well defined, and from (3) we have

$$\|f *_k g\|_{L^\infty_k} \le \|f\|_{L^2_k} \|g\|_{L^2_k}.$$

When functions f and g are in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$, we obtain

$$f *_k g(x) = d_k \int_{\mathbf{R}^d} E_k(ix, z) \mathcal{F}_k(f)(z) \mathcal{F}_k(g)(z) w_k(z)\,dz, \quad x \in \mathbf{R}^d.$$

Note that $*_0$ agrees with the standard convolution $*$ on $\mathbf{R}^d$:

$$f * g(x) := \int_{\mathbf{R}^d} f(x-y) g(y)\,dy, \quad x \in \mathbf{R}^d.$$

By using the same methods as in [**11**, page 238], we show for the Dunkl convolution $*_k$ the following properties.

Proposition 4. (i) *Let* $f, g \in L^2(\mathbf{R}^d, w_k(x)\,dx)$. *Then*

$$f *_k g(x) = \mathcal{F}_k^{-1}[\mathcal{F}_k(f)\mathcal{F}_k(g)].$$

(ii) *Let* $f, g \in L^2(\mathbf{R}^d, w_k(x)\,dx)$. *Then* $f *_k g$ *belongs to* $L^2(\mathbf{R}^d, w_k(x)\,dx)$ *if and only if* $\mathcal{F}_k(f)\mathcal{F}_k(g)$ *belongs to* $L^2(\mathbf{R}^d, w_k(x)\,dx)$ *and we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g), \quad \textit{in the } L^2\textit{-case.}$$

(iii) *Let* $f, g \in L^2(\mathbf{R}^d, w_k(x)\,dx)$. *Then*

$$\|f *_k g\|_{L_k^2}^2 = d_k \|\mathcal{F}_k(f)\mathcal{F}_k(g)\|_{L_k^2}^2,$$

where both sides are finite or infinite.

3. The Dunkl L^2-multiplier operators on $\mathbf{R}^d$. In this section, we study the Dunkl L^2-multiplier operators on $\mathbf{R}^d$ and, for these operators, we establish Calderón's reproducing formulas.

Definition 3. Let m be a function in $L^2(\mathbf{R}^d, w_k(x)dx)$. The Dunkl L^2-multiplier operator $T_{k,m,a}$, $a > 0$, is defined for regular functions f on $\mathbf{R}^d$, by

$$T_{k,m,a}f(x) := \mathcal{F}_k^{-1}[m_a \mathcal{F}_k(f)](x), \quad x \in \mathbf{R}^d,$$

where m_a is the function given by

$$m_a(x) = m(ax).$$

Remark 1. Let $a > 0$ and $m \in L^2(\mathbf{R}^d, w_k(x)\,dx)$. According to Proposition 4 (i) we can write the operator $T_{k,m,a}$ as:

$$T_{k,m,a}f(x) = \mathcal{F}_k^{-1}(m_a) *_k f(x), \quad x \in \mathbf{R}^d, \tag{4}$$

with

$$\mathcal{F}_k^{-1}(m_a)(x) = \frac{1}{a^{2\gamma+d}} \mathcal{F}_k^{-1}(m)\left(\frac{x}{a}\right).$$

In particular, from Propositions 2 and 3, we obtain the following results.

Proposition 5. (i) *If $d=1$, $G=\mathbf{Z}_2$ and $m\in L^2(\mathbf{R},|x|^{2\gamma}dx)$. Then for $f\in L^2(\mathbf{R},|x|^{2\gamma}dx)$, we have*

$$T_{\gamma,m,a}f(x)=T_{\gamma,m_e,a}f(x)+T_{\gamma,m_0,a}f(x),$$

where

$$T_{\gamma,m_e,a}f(x)=\frac{a_\gamma}{a^{2\gamma+1}}\int_{\mathbf{R}}\left[\int_0^\pi \mathcal{F}_\gamma^{-1}(m_e)\left(\frac{(x,y)_\theta}{a}\right)d\nu_{x,-y}^\gamma(\theta)\right]f(y)|y|^{2\gamma}dy$$

and

$$\begin{aligned}&T_{\gamma,m_0,a}f(x)\\&=\frac{a_\gamma}{a^{2\gamma+1}}\int_{\mathbf{R}}\left[\int_0^\pi \frac{x-y}{(x,y)_\theta}\mathcal{F}_\gamma^{-1}(m_0)\left(\frac{(x,y)_\theta}{a}\right)d\nu_{x,-y}^\gamma(\theta)\right]f(y)|y|^{2\gamma}dy.\end{aligned}$$

(ii) *If $m\in L^2_{\mathrm{rad}}(\mathbf{R}^d,w_k(x)dx)$ and $f\in L^2(\mathbf{R}^d,w_k(x)\,dx)$, then*

$$\begin{aligned}&T_{k,m,a}f(x)\\&=\frac{1}{a^{2\gamma+d}}\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}M\left(\frac{1}{a}\sqrt{\|x\|^2+\|y\|^2-2\langle y,z\rangle}\right)f(y)w_k(y)\,d\Gamma_x(z)\,dy,\end{aligned}$$

where $\mathcal{F}_k^{-1}(m)(x)=M(\|x\|)$.

The operators $T_{k,m,a}$ satisfy the following Young's inequalities.

Lemma 1. (i) *If $m\in L^2(\mathbf{R}^d,w_k(x)\,dx)$ and $f\in L^1(\mathbf{R}^d,w_k(x)\,dx)$, then $T_{k,m,a}f\in L^2(\mathbf{R}^d,w_k(x)\,dx)$, and we have*

$$\|T_{k,m,a}f\|_{L^2_k}\le\sqrt{\frac{d_k}{a^{2\gamma+d}}}\|m\|_{L^2_k}\|f\|_{L^1_k}.$$

(ii) *If $m\in L^\infty(\mathbf{R}^d,w_k(x)\,dx)$ and $f\in L^2(\mathbf{R}^d,w_k(x)\,dx)$, then $T_{k,m,a}f\in L^2(\mathbf{R}^d,w_k(x)dx)$, and we have*

$$\|T_{k,m,a}f\|_{L^2_k}\le\|m\|_{L^\infty_k}\|f\|_{L^2_k}.$$

(iii) *If* $m \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ *and* $f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$, *then* $T_{k,m,a}f \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, *and we have*

$$T_{k,m,a}f(x) = d_k \int_{\mathbf{R}^d} m(az)\mathcal{F}_k(f)(z)E_k(ix,z)w_k(z)\,dz, \quad x \in \mathbf{R}^d,$$

and

$$\|T_{k,m,a}f\|_{L_k^\infty} \le \frac{1}{\sqrt{a^{2\gamma+d}}}\|m\|_{L_k^2}\|f\|_{L_k^2}.$$

Proof. (i) If $m \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ and $f \in L^1(\mathbf{R}^d, w_k(x)\,dx)$, from Definition 3, Proposition 1 (i) and Theorem 1 (ii), the function $T_{k,m,a}f$ belongs to $L^2(\mathbf{R}^d, w_k(x)dx)$, and we have

$$\begin{aligned}\|T_{k,m,a}f\|_{L_k^2}^2 &= d_k \int_{\mathbf{R}^d} |m(az)|^2|\mathcal{F}_k(f)(z)|^2 w_k(z)\,dz \\ &\le \frac{d_k}{a^{2\gamma+d}}\|m\|_{L_k^2}^2\|\mathcal{F}_k(f)\|_{L_k^\infty}^2 \\ &\le \frac{d_k}{a^{2\gamma+d}}\|m\|_{L_k^2}^2\|f\|_{L_k^1}^2.\end{aligned}$$

Part (ii) follows from Definition 3 and Theorem 1 (ii), and part (iii) follows from Definition 3 and Theorem 1 (iii) using Hölder's inequality. □

In the following, we give Plancherel and pointwise reproducing inversion formulas for the operators $T_{k,m,a}$.

Theorem 2. *Let* m *be a function in* $L^2(\mathbf{R}^d, w_k(x)\,dx)$ *satisfying the admissibility condition*:

$$\int_0^\infty |m(ax)|^2\frac{da}{a} = 1, \quad a.e.\ x \in \mathbf{R}^d. \tag{5}$$

(i) **Plancherel formula**: *For* $f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$, *we have*

$$\int_{\mathbf{R}^d} |f(x)|^2 w_k(x)\,dx = \int_0^\infty \|T_{k,m,a}f\|_{L_k^2}^2\frac{da}{a}.$$

(ii) **First Calderón's formula**: *For* $f \in L^1(\mathbf{R}^d, w_k(x)\,dx)$ *such that* $\mathcal{F}_k(f) \in L^1(\mathbf{R}^d, w_k(x)\,dx)$, *we have*

$$f(x) = \int_0^\infty \Big[T_{k,m,a}f *_k \mathcal{F}_k^{-1}(\overline{m_a})\Big](x)\frac{da}{a}, \quad a.e.\ x \in \mathbf{R}^d.$$

Proof. (i) From (4) and Proposition 4 (iii), we obtain

$$\begin{aligned}\int_0^\infty \|T_{k,m,a}f\|^2_{L^2_k}\frac{da}{a} &= \int_0^\infty \int_{\mathbf{R}^d} \Big|\mathcal{F}_k^{-1}(m_a) *_k f(x)\Big|^2 w_k(x)\,dx\,\frac{da}{a} \\ &= d_k \int_{\mathbf{R}^d} |\mathcal{F}_k(f)(x)|^2\Big[\int_0^\infty |m(ax)|^2\frac{da}{a}\Big]w_k(x)\,dx.\end{aligned}$$

Then, the result follows from (5) and Theorem 1 (ii).

(ii) Let $f \in L^1(\mathbf{R}^d, w_k(x)\,dx)$. From Lemma 1 (i), relation (3) and Theorem 1 (ii) we have

$$\begin{aligned}\int_0^\infty \Big[T_{k,m,a}f *_k \mathcal{F}_k^{-1}(\overline{m_a})\Big](x)\frac{da}{a} &= \int_0^\infty \Big[\int_{\mathbf{R}^d} T_{k,m,a}f(y)\overline{\tau_{-x}[\mathcal{F}_k^{-1}(m_a)](y)}w_k(y)\,dy\Big]\frac{da}{a} \\ &= d_k \int_0^\infty \Big[\int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)|m(ay)|^2 w_k(y)\,dy\Big]\frac{da}{a}.\end{aligned}$$

Since

$$\int_0^\infty \Big[\int_{\mathbf{R}^d} |E_k(ix,y)\mathcal{F}_k(f)(y)||m(ay)|^2 w_k(y)\,dy\Big]\frac{da}{a} \le \|\mathcal{F}_k(f)\|_{L^1_k} < \infty.$$

Then, from Fubini's theorem, we have

$$\begin{aligned}\int_0^\infty \Big[T_{k,m,a}f *_k \mathcal{F}_k^{-1}(\overline{m_a})\Big](x)\frac{da}{a} &= d_k \int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)\Big[\int_0^\infty |m(ay)|^2\frac{da}{a}\Big]w_k(y)\,dy.\end{aligned}$$

We obtain (ii) from (5) and Theorem 1 (iii). □

To establish Calderón's reproducing formula for the operators $T_{k,m,a}$, we need the following lemma.

Lemma 2. *Let $m \in L^2 \cap L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ satisfy the admissibility condition* (5). *For $0 < \varepsilon < \delta < \infty$, we put*

$$K_{\varepsilon,\delta}(x) := \int_\varepsilon^\delta |m(ax)|^2 \frac{da}{a}.$$

Then

$$K_{\varepsilon,\delta} \in L^2 \cap L^\infty(\mathbf{R}^d, w_k(x)\,dx).$$

Proof. Using Hölder's inequality for the measure da/a we obtain

$$|K_{\varepsilon,\delta}(x)|^2 \le \ln\left(\frac{\delta}{\varepsilon}\right) \int_\varepsilon^\delta |m(ax)|^4 \frac{da}{a}, \quad x \in \mathbf{R}^d.$$

Therefore,

$$\begin{aligned}
\|K_{\varepsilon,\delta}\|_{L_k^2}^2 &\le \ln(\delta/\varepsilon) \int_\varepsilon^\delta \left[\int_{\mathbf{R}^d} |m(ax)|^4 w_k(x)\,dx\right] \frac{da}{a} \\
&\le \ln(\delta/\varepsilon) \int_\varepsilon^\delta \left[\int_{\mathbf{R}^d} |m(x)|^4 w_k(x)\,dx\right] \frac{da}{a^{2\gamma+d+1}} \\
&\le \frac{[\varepsilon^{-2\gamma-d} - \delta^{-2\gamma-d}]}{2\gamma+d} \ln\left(\frac{\delta}{\varepsilon}\right) \|m\|_{L_k^\infty}^2 \|m\|_{L_k^2}^2 < \infty.
\end{aligned}$$

On the other hand, from (5) we have

$$\|K_{\varepsilon,\delta}\|_{L_k^\infty} \le 1,$$

which completes the proof of the lemma. □

The previous pointwise reproducing inversion formula can be interpreted in the L^2-sense as follows.

Theorem 3 (second Calderón's formula). *Let $m \in L^2 \cap L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ satisfy the admissibility condition* (5). *Then, for $f \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ and $0 < \varepsilon < \delta < \infty$, the function $f_{\varepsilon,\delta}$ given by*

$$f_{\varepsilon,\delta}(x) := \int_\varepsilon^\delta \left[T_{k,m,a} f *_k \mathcal{F}_k^{-1}(\overline{m_a})\right](x) \frac{da}{a}, \quad x \in \mathbf{R}^d,$$

belongs to $L^2(\mathbf{R}^d, w_k(x)dx)$ *and satisfies*

$$\lim_{\substack{\varepsilon\to 0,\\ \delta\to\infty}} \|f_{\varepsilon,\delta} - f\|_{L^2_k} = 0. \tag{6}$$

Proof. From Lemma 1 (ii), relation (3) and Theorem 1 (ii) we have

$$\begin{aligned} f_{\varepsilon,\delta}(x) &= \int_\varepsilon^\delta \left[\int_{\mathbf{R}^d} T_{k,m,a}f(y)\overline{\tau_{-x}[\mathcal{F}_k^{-1}(m_a)](y)}w_k(y)\,dy\right]\frac{da}{a} \\ &= d_k \int_\varepsilon^\delta \left[\int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)|m(ay)|^2 w_k(y)\,dy\right]\frac{da}{a}. \end{aligned}$$

But from Fubini-Tonnelli's theorem, Hölder's inequality and Lemma 2 we have

$$\begin{aligned} \int_\varepsilon^\delta \int_{\mathbf{R}^d} |E_k(ix,y)\mathcal{F}_k(f)(y)||m(ay)|^2 w_k(y)\,dy\frac{da}{a} \\ &\le \int_{\mathbf{R}^d} |\mathcal{F}_k(f)(y)|K_{\varepsilon,\delta}(y)w_k(y)\,dy \\ &\le \frac{1}{\sqrt{d_k}}\|f\|_{L^2_k}\|K_{\varepsilon,\delta}\|_{L^2_k} < \infty. \end{aligned}$$

Then, from Fubini's theorem and Theorem 1 (iii), we get

$$\begin{aligned} f_{\varepsilon,\delta}(x) &= d_k \int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)\left[\int_\varepsilon^\delta |m(ay)|^2\frac{da}{a}\right]w_k(y)\,dy \\ &= d_k \int_{\mathbf{R}^d} E_k(ix,y)\mathcal{F}_k(f)(y)K_{\varepsilon,\delta}(y)w_k(y)\,dy \\ &= \mathcal{F}_k^{-1}[\mathcal{F}_k(f)K_{\varepsilon,\delta}](x). \end{aligned}$$

Thus, using the fact that $K_{\varepsilon,\delta} \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ we prove that $f_{\varepsilon,\delta} \in L^2(\mathbf{R}^d, w_k(x)\,dx)$, and by Proposition 4 (ii) we obtain

$$\mathcal{F}_k(f_{\varepsilon,\delta}) = \mathcal{F}_k(f)K_{\varepsilon,\delta}.$$

From this relation and Theorem 1 (ii), we obtain

$$\|f_{\varepsilon,\delta} - f\|^2_{L^2_k} = d_k \int_{\mathbf{R}^d} |\mathcal{F}_k(f)(y)|^2[1 - K_{\varepsilon,\delta}(y)]^2 w_k(y)\,dy.$$

On the other hand, (5) leads to

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} K_{\varepsilon,\delta}(y) = 1, \quad \text{a.e. } y \in \mathbf{R}^d,$$

and

$$|\mathcal{F}_k(f)(y)|^2[1 - K_{\varepsilon,\delta}(y)]^2 \leq |\mathcal{F}_k(f)(y)|^2.$$

Hence, (6) follows from the dominated convergence theorem. □

As an application, we give the following example.

Example 1. The function m_t, $t > 0$, defined by

$$m_t(x) := \sqrt{8}t\|x\|^2 e^{-t\|x\|^2}, \quad x \in \mathbf{R}^d,$$

belongs to $L^2 \cap L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, and satisfies the admissibility condition (5). Then the associated operator $T_{k,m_t,a}$ given by

$$T_{k,m_t,a}f(x) = -\frac{\sqrt{8}t}{a^{2\gamma+d}} \int_{\mathbf{R}^d} \frac{d}{dt}\left[\Gamma_k\left(\frac{x}{a}, \frac{y}{a}, t\right)\right] f(y)w_k(y)\,dy,$$

where $\Gamma_k(x,y,t)$ is the Dunkl-type heat kernel [**15, 21**] given by

$$\Gamma_k(x,y,t) := \frac{\sqrt{d_k}}{(2t)^{\gamma+d/2}} e^{-(\|x\|^2+\|y\|^2)/4t} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

satisfies Calderón's reproducing formula (6).

Note that, for $a = 1$, the operator $T_{k,m_t,1}f(x)$ is

$$T_{k,m_t,1}f(x) = -\sqrt{8}t\frac{d}{dt}\left[L_{t,k}f(x)\right],$$

where

$$L_{t,k}f(x) := \int_{\mathbf{R}^d} \Gamma_k(x,y,t)f(y)w_k(y)\,dy, \tag{7}$$

is the Dunkl-type Weierstrass transform studied in [**21**].

4. The extremal function related to Dunkl L^2-multiplier operators. This section contains the second main result of this paper, that is, the existence and unicity of the extremal function related to Dunkl L^2-multiplier operators on $\mathbf{R}^d$ studied in the previous section.

Definition 4. Let ρ be a positive function on $\mathbf{R}^d$ satisfying the conditions:

$$\rho(z) \geq 1, \quad z \in \mathbf{R}^d, \tag{8}$$

$$\|1/\rho\|_{L^1_k} = \int_{\mathbf{R}^d} \frac{w_k(z)}{\rho(z)}\,dz < \infty. \tag{9}$$

We define the space $\mathcal{H}_\rho(\mathbf{R}^d)$ by

$$\mathcal{H}_\rho(\mathbf{R}^d) := \left\{ f \in L^2(\mathbf{R}^d, w_k(x)\,dx) : \sqrt{\rho}\,\mathcal{F}_k(f) \in L^2(\mathbf{R}^d, w_k(x)\,dx) \right\}.$$

The space $\mathcal{H}_\rho(\mathbf{R}^d)$ provided with inner product

$$\langle f, g\rangle_\rho = \int_{\mathbf{R}^d} \rho(z)\mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g)(z)}w_k(z)\,dz,$$

and the norm $\|f\|_\rho = \sqrt{\langle f, f\rangle_\rho}$, is a Hilbert space.

Remark 2. The function $1/\sqrt{\rho}$ belongs to $L^2(\mathbf{R}^d, w_k(x)\,dx)$. Hence, for all $f \in \mathcal{H}_\rho(\mathbf{R}^d)$, the function $\mathcal{F}_k(f)$ belongs to $L^1(\mathbf{R}^d, w_k(x)\,dx)$, and we have

$$f(x) = d_k \int_{\mathbf{R}^d} E_k(ix, y)\mathcal{F}_k(f)(y)w_k(y)\,dy, \quad \text{a.e. } x \in \mathbf{R}^d.$$

Proposition 6. *Let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$. The operators $T_{k,m,a}$ given by Definition 3 are bounded linear operators from $\mathcal{H}_\rho(\mathbf{R}^d)$ into $L^2(\mathbf{R}^d, w_k(x)\,dx)$, and we have*

$$\|T_{k,m,a}f\|_{L^2_k} \leq \sqrt{d_k}\|m\|_{L^\infty_k}\|f\|_\rho, \quad f \in \mathcal{H}_\rho(\mathbf{R}^d).$$

Proof. Let $f \in \mathcal{H}_\rho(\mathbf{R}^d)$. By Lemma 1 (ii), the operator $T_{k,m,a}f$ belongs to $L^2(\mathbf{R}^d, w_k(x)\,dx)$, and

$$\|T_{k,m,a}f\|_{L^2_k} \leq \|m\|_{L^\infty_k}\|f\|_{L^2_k}.$$

But, by condition (8), we have $\|f\|_{L^2_k} \leq \sqrt{d_k}\|f\|_\rho$, which gives the result. □

Definition 5. Let $\lambda > 0$, and let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$. We denote by $\langle\cdot,\cdot\rangle_{\rho,\lambda}$ the inner product defined on the space $\mathcal{H}_\rho(\mathbf{R}^d)$ by

$$\langle f, g\rangle_{\rho,\lambda} := \int_{\mathbf{R}^d} (\lambda\rho(z) + |m(az)|^2)\mathcal{F}_k(f)(z)\overline{\mathcal{F}_k(g)(z)}w_k(z)\,dz, \tag{10}$$

and the norm $\|f\|_{\rho,\lambda} := \sqrt{\langle f, f\rangle_{\rho,\lambda}}$.

Proposition 7. *Let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$, and let $f \in \mathcal{H}_\rho(\mathbf{R}^d)$.*

(i) *The norm $\|\cdot\|_{\rho,\lambda}$ satisfies:*

$$\|f\|^2_{\rho,\lambda} = \lambda\|f\|^2_\rho + \frac{1}{d_k}\|T_{k,m,a}f\|^2_{L^2_k}.$$

(ii) *The two norms $\|\cdot\|_{\rho,\lambda}$ and $\|\cdot\|_\rho$ are equivalent, and*

$$\sqrt{\lambda}\,\|f\|_\rho \leq \|f\|_{\rho,\lambda} \leq \sqrt{\lambda + \|m\|^2_{L^\infty_k}}\,\|f\|_\rho.$$

Proof. (i) follows from Definition 3 and Theorem 1 (ii), and (ii) follows from assertion (i) and Proposition 6. □

Lemma 3. *Let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$. Then the Hilbert space $(\mathcal{H}_\rho(\mathbf{R}^d), \langle\cdot,\cdot\rangle_{\rho,\lambda})$ possesses the following reproducing kernel:*

$$\mathcal{K}_{\rho,\lambda}(x,y) = (d_k)^2 \int_{\mathbf{R}^d} \frac{E_k(ix,z)E_k(-iy,z)}{\lambda\rho(z) + |m(az)|^2} w_k(z)\,dz, \tag{11}$$

that is,

(i) *For all $y \in \mathbf{R}^d$, the function $x \to \mathcal{K}_{\rho,\lambda}(x,y)$ belongs to $\mathcal{H}_\rho(\mathbf{R}^d)$.*

(ii) *The reproducing property: for all* $f \in \mathcal{H}_\rho(\mathbf{R}^d)$ *and* $y \in \mathbf{R}^d$,

$$\langle f, \mathcal{K}_{\rho,\lambda}(\cdot, y)\rangle_{\rho,\lambda} = f(y).$$

Proof. (i) Let $y \in \mathbf{R}^d$. From (2), (8) and (9), the function $\Phi_y : z \to d_k(E_k(-iy, z))/(\lambda\rho(z) + |m(az)|^2)$ belongs to $L^1 \cap L^2(\mathbf{R}^d, w_k(z)\,dz)$. Then, the function $\mathcal{K}_{\rho,\lambda}$ is well defined and by Theorem 1 (iii), we have

$$\mathcal{K}_{\rho,\lambda}(x, y) = \mathcal{F}_k^{-1}(\Phi_y)(x), \quad x \in \mathbf{R}^d.$$

From Theorem 1 (ii), it follows that $\mathcal{K}_{\rho,\lambda}(\cdot, y)$ belongs to $L^2(\mathbf{R}^d, w_k(x)\,dx)$, and we have

$$\mathcal{F}_k[\mathcal{K}_{\rho,\lambda}(\cdot, y)](z) = d_k \frac{E_k(-iy, z)}{\lambda\rho(z) + |m(az)|^2}, \quad z \in \mathbf{R}^d. \tag{12}$$

Then by (2), we obtain

$$\Big|\mathcal{F}_k[\mathcal{K}_{\rho,\lambda}(\cdot, y)](z)\Big| \leq \frac{d_k}{\lambda\rho(z)},$$

and

$$\|\mathcal{K}_{\rho,\lambda}(\cdot, y)\|_\rho^2 \leq \left(\frac{d_k}{\lambda}\right)^2 \|1/\rho\|_{L_k^1} < \infty.$$

This proves that, for all $y \in \mathbf{R}^d$, the function $\mathcal{K}_{\rho,\lambda}(\cdot, y)$ belongs to $\mathcal{H}_\rho(\mathbf{R}^d)$.

(ii) Let $f \in \mathcal{H}_\rho(\mathbf{R}^d)$ and $y \in \mathbf{R}^d$. From (10) and (12), we have

$$\langle f, \mathcal{K}_{\rho,\lambda}(\cdot, y)\rangle_{\rho,\lambda} = d_k \int_{\mathbf{R}^d} E_k(iy, z)\mathcal{F}_k(f)(z) w_k(z)\,dz,$$

and from Remark 2, we obtain the reproducing property:

$$\langle f, \mathcal{K}_{\rho,\lambda}(\cdot, y)\rangle_{\rho,\lambda} = f(y).$$

This completes the proof of the lemma. □

If $m = 0$ and $\lambda = 1$, we obtain the following remark.

Remark 3. The Hilbert space $(\mathcal{H}_\rho(\mathbf{R}^d), \langle\cdot,\cdot\rangle_\rho)$ possesses the following reproducing kernel:

$$\mathcal{K}_\rho(x,y) = (d_k)^2 \int_{\mathbf{R}^d} \frac{E_k(ix,z)E_k(-iy,z)}{\rho(z)} w_k(z)\,dz; \quad x,y \in \mathbf{R}^d.$$

The main result of this section can then be stated as follows.

Theorem 4. *Let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ and $a > 0$. For any $h \in L^2(\mathbf{R}^d, w_k(x)\,dx)$ and for any $\lambda > 0$, there exists a unique function $f^*_{\lambda,h,a}$, where the infimum*

$$\inf_{f\in\mathcal{H}_\rho} \left\{ \lambda \|f\|^2_\rho + \frac{1}{d_k}\|h - T_{k,m,a}f\|^2_{L^2_k} \right\} \tag{13}$$

is attained. Moreover, the extremal function $f^*_{\lambda,h,a}$ is given by

$$f^*_{\lambda,h,a}(y) = \int_{\mathbf{R}^d} h(x)\overline{V_{\rho,\lambda}(x,y)}w_k(x)\,dx, \tag{14}$$

where

$$V_{\rho,\lambda}(x,y) = d_k \int_{\mathbf{R}^d} \frac{m(az)E_k(-iy,z)}{\lambda\rho(z) + |m(az)|^2} E_k(ix,z)w_k(z)\,dz.$$

Proof. The existence and unicity of the extremal function $f^*_{\lambda,h,a}$ satisfying (13) is given by [**8, 10, 18**]. On the other hand, from Lemma 3 we have

$$f^*_{\lambda,h,a}(y) = \frac{1}{d_k}\langle h, T_{k,m,a}[\mathcal{K}_{\rho,\lambda}(\cdot,y)]\rangle_{L^2_k},$$

where $\mathcal{K}_{\rho,\lambda}$ is the kernel given by (11).

From Lemma 1 (iii) and (12), we obtain

$$\begin{aligned} V_{\rho,\lambda}(x,y) &= \frac{1}{d_k}T_{k,m,a}[\mathcal{K}_{\rho,\lambda}(\cdot,y)](x) \\ &= d_k \int_{\mathbf{R}^d} \frac{m(az)E_k(-iy,z)}{\lambda\rho(z) + |m(az)|^2} E_k(ix,z)w_k(z)\,dz \end{aligned}$$

This clearly yields the result. □

Some properties of the extremal function $f^*_{\lambda,h,a}$ are checked in the following.

Theorem 5. *Let* $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ *and* $h \in L^2(\mathbf{R}^d, w_k(x)\,dx)$. *The extremal function* $f^*_{\lambda,h,a}$, *given by* (14), *satisfies*:

(i)

$$\mathcal{F}_k(f^*_{\lambda,h,a})(z) = \frac{\overline{m(az)}}{\lambda\rho(z) + |m(az)|^2}\mathcal{F}_k(h)(z), \quad z \in \mathbf{R}^d.$$

(ii)

$$\|f^*_{\lambda,h,a}\|^2_\rho \leq \frac{1}{4\lambda d_k}\|h\|^2_{L^2_k}.$$

Proof. (i) Let $y \in \mathbf{R}^d$. The function $\Psi_y : z \to (m(az)E_k(-iy,z))/(\lambda\rho(z)+|m(az)|^2)$ belongs to $L^1 \cap L^2(\mathbf{R}^d, w_k(z)\,dz)$. Then by Theorem 1 (iii), we have

$$V_{\rho,\lambda}(x,y) = \mathcal{F}_k^{-1}(\Psi_y)(x), \quad x \in \mathbf{R}^d.$$

From Theorem 1 (ii), it follows that $V_{\rho,\lambda}(\cdot,y)$ belongs to $L^2(\mathbf{R}^d, w_k(x)dx)$, and

$$\begin{aligned} f^*_{\lambda,h,a}(y) &= d_k \int_{\mathbf{R}^d} \mathcal{F}_k(h)(z)\overline{\Psi_y(z)}w_k(z)\,dz \\ &= d_k \int_{\mathbf{R}^d} \frac{\mathcal{F}_k(h)(z)\overline{m(az)}}{\lambda\rho(z)+|m(az)|^2}E_k(iy,z)w_k(z)\,dz. \end{aligned}$$

The function $F : z \to (\mathcal{F}_k(h)(z)\overline{m(az)})/(\lambda\rho(z)+|m(az)|^2)$ belongs to $L^1 \cap L^2(\mathbf{R}^d, w_k(z)\,dz)$. Then by Theorem 1 (iii), we have

$$f^*_{\lambda,h,a}(y) = \mathcal{F}_k^{-1}(F)(y).$$

From Theorem 1 (ii), it follows that $f^*_{\lambda,h,a}$ belongs to $L^2(\mathbf{R}^d, w_k(y)\,dy)$, and

$$\mathcal{F}_k(f^*_{\lambda,h,a})(z) = \frac{\overline{m(az)}}{\lambda\rho(z) + |m(az)|^2}\mathcal{F}_k(h)(z), \quad z \in \mathbf{R}^d.$$

(ii) From assertion (i) and using the inequality $(x+y)^2 \geq 4xy$, we obtain

$$\rho(z)|\mathcal{F}_k(f^*_{\lambda,h,a})(z)|^2 \leq \frac{1}{4\lambda}|\mathcal{F}_k(h)(z)|^2.$$

Thus, from Theorem 1 (ii), we obtain

$$\|f^*_{\lambda,h,a}\|^2_\rho \leq \frac{1}{4\lambda}\|\mathcal{F}_k(h)\|^2_{L^2_k} \leq \frac{1}{4\lambda d_k}\|h\|^2_{L^2_k}. \qquad \square$$

If we take $h = T_{k,m,a}f$ in (14), where $f \in \mathcal{H}_\rho(\mathbf{R}^d)$, we obtain the following Calderón's reproducing formula.

Theorem 6 (**third Calderón's formula**). *Let $m \in L^\infty(\mathbf{R}^d, w_k(x)\,dx)$ and $f \in \mathcal{H}_\rho(\mathbf{R}^d)$. The extremal function $f^*_{\lambda,a}$ given by*

$$f^*_{\lambda,a}(y) = \int_{\mathbf{R}^d} T_{k,m,a}f(x)\overline{V_{\rho,\lambda}(x,y)}w_k(x)\,dx$$

satisfies

$$\lim_{\lambda\to 0^+}\|f^*_{\lambda,a} - f\|_\rho = 0. \tag{15}$$

Moreover, $\{f^*_{\lambda,a}\}_{\lambda>0}$ converges uniformly to f as $\lambda \to 0^+$.

Proof. Let $f \in \mathcal{H}_\rho(\mathbf{R}^d)$, $h = T_{k,m,a}f$ and $f^*_{\lambda,a} = f^*_{\lambda,h,a}$. From Proposition 6, function h belongs to $L^2(\mathbf{R}^d, w_k(x)\,dx)$.

Applying Definition 3 and Theorem 5 (i), we obtain

$$\mathcal{F}_k(f^*_{\lambda,a})(z) = \frac{|m(az)|^2}{\lambda\rho(z) + |m(az)|^2}\mathcal{F}_k(f)(z).$$

Thus, it follows that

$$\mathcal{F}_k(f^*_{\lambda,a} - f)(z) = \frac{-\lambda\rho(z)}{\lambda\rho(z) + |m(az)|^2}\mathcal{F}_k(f)(z), \quad z \in \mathbf{R}. \tag{16}$$

Consequently,

$$\|f^*_{\lambda,a} - f\|^2_\rho = \int_{\mathbf{R}^d} \frac{\lambda^2\rho^3(z)|\mathcal{F}_k(f)(z)|^2}{[\lambda\rho(z) + |m(az)|^2]^2}w_k(z)\,dz.$$

Using the dominated convergence theorem and the fact that

$$\frac{\lambda^2\rho^3(z)|\mathcal{F}_k(f)(z)|^2}{[\lambda\rho(z)+|m(az)|^2]^2} \leq \rho(z)|\mathcal{F}_k(f)(z)|^2,$$

we deduce that

$$\lim_{\lambda\to 0^+} \|f^*_{\lambda,a} - f\|^2_\rho = 0.$$

On the other hand, from Remark 2, the function $\mathcal{F}_k(f) \in L^1 \cap L^2(\mathbf{R}^d, w_k(z)\,dz)$. Then by (16) and Theorem 1 (iii), we get

$$f^*_{\lambda,a}(y) - f(y) = d_k \int_{\mathbf{R}^d} \frac{-\lambda\rho(z)\mathcal{F}_k(f)(z)}{\lambda\rho(z)+|m(az)|^2} E_k(iy,z)w_k(z)\,dz.$$

So

$$\sup_{y\in\mathbf{R}^d} |f^*_{\lambda,a}(y) - f(y)| \leq d_k \int_{\mathbf{R}^d} \frac{\lambda\rho(z)|\mathcal{F}_k(f)(z)|}{\lambda\rho(z)+|m(az)|^2} w_k(z)\,dz.$$

Again, by the dominated convergence theorem and the fact that

$$\frac{\lambda\rho(z)|\mathcal{F}_k(f)(z)|}{\lambda\rho(z)+|m(az)|^2} \leq |\mathcal{F}_k(f)(z)|,$$

we deduce that

$$\lim_{\lambda\to 0^+} \sup_{y\in\mathbf{R}^d} |f^*_{\lambda,a}(y) - f(y)| = 0,$$

which ends the proof. □

As an application, we give the following example.

Example 2. If $\rho(z) = (1+\|z\|^2)^s$, $s > \gamma + d/2$ and m_t, $t > 0$ the function is defined by

$$m_t(z) := e^{-t\|z\|^2}, \quad z \in \mathbf{R}^d.$$

Then, the extremal function $f^*_{\lambda,a}$ given by

$$f^*_{\lambda,a}(y) = \int_{\mathbf{R}^d} T_{k,m_t,a}f(x)\overline{V_{\rho,\lambda}(x,y)}w_k(x)\,dx,$$

where

$$V_{\rho,\lambda}(x,y) = d_k \int_{\mathbf{R}^d} \frac{e^{-ta^2\|z\|^2} E_k(-iy,z)}{\lambda(1+\|z\|^2)^s + e^{-2ta^2\|z\|^2}} E_k(ix,z) w_k(z)\, dz,$$

satisfies Calderón's reproducing formula (15).

Note that, for $a = 1$, the operator $T_{k,m_t,1}f(x)$ is the Dunkl-type Weierstrass transform $L_{t,k}f(x)$ given by (7).

Acknowledgments. The author would like to express his deep thanks to the referees for their careful reading and editing of the manuscript.

REFERENCES

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1948), 337–404.

2. C.F. Dunkl, *Differential-difference operators associated with reflections groups*, Trans. Amer. Math. Soc. **311** (1989), 167–183.

3. ———, *Integral kernels with reflection group invariance*, Canadian J. Math. **43** (1991), 1213–1227.

4. ———, *Hankel transforms associated to finite reflection groups*, Contemp. Math. **138** (1992), 123–138.

5. P. Etingof, *A uniform proof of the Macdonald-Mehta-Opdam identity for finite Coxeter groups*, arXiv: 0903.5084v1 [math.RT].

6. A. Grossmann and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape*, SIAM **15** (1984), 723–736.

7. M.F.E. de Jeu, *The Dunkl transform*, Inv. Math. **113** (1993), 147–162.

8. G.S. Kimeldorf and G. Wahba, *Some results on Tchebycheffian spline functions*, J. Math. Anal. Appl. **33** (1971), 82–95.

9. L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, Comm. Math. Phys. **178** (1996), 425–452.

10. T. Matsuura, S. Saitoh and D.D. Trong, *Inversion formulas in heat conduction multidimensional spaces*, J. Inverse Ill-posed Problems **13** (2005), 479–493.

11. M.A. Mourou and K. Trimèche, *Calderón's reproducing formula for a singular differential operator and inversion of the generalized Abel transform*, J. Fourier Anal. Appl. **4** (1998), 229–245.

12. R.S. Pathak and G. Pandey, *Calderón's reproducing formula for Hankel convolution*, Inter. J. Math. Math. Sci. **2006** (2006), 1–7.

13. M. Rösler, *Bessel-type signed hypergroups on* **R**, in , Proc XI, *Probability measures on groups and related structures*, H. Heyer and A. Mukherjea, eds., World Scientific, Singapore, 1995.

14. M. Rösler, *Positivity of Dunkl's intertwining operator*, Duke Math. J. **98** (1999), 445–463.

15. ———, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc. **355** (2003), 2413–2438.

16. S. Saitoh, *The Weierstrass transform and an isometry in the heat equation*, Appl. Anal. **16** (1983), 1–6.

17. ———, *One approach to some general integral transforms and its applications*, Int. Trans. Spec. Funct. **3** (1995), 49–84.

18. ———, *Approximate real inversion formulas of the Gaussian convolution*, Appl. Anal. **83** (2004), 727–733.

19. ———, *Applications of Tikhonov regularization to inverse problems using reproducing kernels*, J. Phys.: Conf. Ser. **88** (2007), 012019.

20. F. Soltani, *L^p-Fourier multipliers for the Dunkl operator on the real line*, J. Functional Anal. **209** (2004), 16–35.

21. ———, *Inversion formulas in the Dunkl-type heat conduction on $\mathbf{R}^d$*, Appl. Anal. **84** (2005), 541–553.

22. S. Thangavelu and Y. Xu, *Convolution operator and maximal function for Dunkl transform*, J. Anal. Math. **97** (2005), 25–56.

23. M. Yamada, T. Matsuura and S. Saitoh, *Representations of inverse functions by the integral transform with the sign kernel*, Frac. Calc. Appl. Anal. **2** (2007), 161–168.

Higher College of Technology and Informatics, Street of the Employers 45, Charguia 2 – 2035 Tunis, Tunisia
Email address: fethisoltani05@yahoo.fr