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REMARKS ON GENERALIZED TRIGONOMETRIC FUNCTIONS

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ABSTRACT. A natural generalization of the sine function occurs as an eigenfunction of the Dirichlet problem for the one-dimensional p-Laplacian. Our study of the properties of p-trigonometric functions and their connection with classical analysis leads to a variety of new identities and inequalities and to the basis properties of the p-eigenfunctions.

1. Introduction. The spectral properties of the Dirichlet Laplacian on the unit interval of the real line are familiar and simple: the problem

$$-u'' = \lambda u \text{ on } (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

has eigenvalues $(n\pi)^2$ and corresponding eigenvectors u_n, $u_n(t) = \sin(n\pi t)$ $(n \in \mathbf{N})$. It is a remarkable fact (see, for example, [**6**]) that the corresponding problem for the one-dimensional p-Laplacian Δ_p $(1 < p < \infty)$, namely,

$$\begin{gathered} -\Delta_p u := -\left(|u'|^{p-2}\, u'\right)' = \lambda\, |u|^{p-2}\, u \quad \text{on } (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.2}$$

has eigenfunctions expressible in terms of functions similar to the sine function. In fact, (1.2) has eigenvalues

$$\lambda_n = (p-1)(n\pi_p)^p, \quad \text{where } \pi_p = \frac{2\pi}{p\sin(\pi/p)},$$

and associated eigenfunctions $\sin_p(n\pi_p t)$ $(n \in \mathbf{N})$. Here $\sin_p$ is the function defined on $[0, \pi_p/2]$ to be the inverse of the function F_p : $[0,1] \to \mathbf{R}$ given by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt,$$

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and extended to the whole of the real line in a natural way so as to be $2\pi_p$-periodic. This is clearly similar to the ordinary sine function, which corresponds to $p = 2$. It is easy to make p-analogues of the other trigonometric functions. For example, $\cos_p$ is defined to be the derivative of $\sin_p$, from which it follows readily that

$$|\sin_p x|^p + |\cos_p x|^p = 1 \quad \text{for all } x \in \mathbf{R}.$$

However, the special nature which the case $p = 2$ has here, as in many other parts of analysis, is underlined by the fact that if $p \neq 2$, the derivative of $\cos_p$ is not $-\sin_p$.

The literature on these p-trigonometric functions is now quite extensive: we refer in particular to Lindqvist ([**11, 12**]) and Lindqvist and Peetre ([**13, 14**]); [**14**] contains a fascinating account of the history of such work. Despite this, however, it seems to us that the rich vein of striking formulae and identities stemming from these functions is far from being exhausted. Moreover, while there are genuine difficulties in working with these p-functions, notably the lack of reasonable kinds of addition formulae, nevertheless there are enough identities and inequalities available for them to form a significant addition to the analyst's toolkit. In this paper we seek to justify this point of view, making full use of connections with functions from classical analysis: for example,

$$F_p(x) = xF(1/p, 1/p; 1 + 1/p; x^p) \quad (0 < x < 1),$$

where on the right-hand side F denotes the usual hypergeometric function. We begin with what might be called the p-calculus and give a variety of results involving derivatives and integrals of the $p-$trigonometric functions, followed by identities that seem to us to be of interest. For example, it turns out that

$$\sum_{k=0}^{\infty} \frac{\Gamma(k + 2/p)}{(kp + 1)^2 \Gamma(k + 1)} = \frac{\pi \Gamma^2(1/p)}{2p^2 \sin(\pi/p)}.$$

When $p = 2$ this reduces to the familiar identity

$$\pi^2/8 = \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^2}.$$

We also obtain a new representation of Catalan's constant G: recall that this is defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{1.3}$$

and that it appears in estimates of certain combinatorial functions and in various definite integrals (see [**8, 15**]). First we establish the identity

$$\int_0^{\pi_p/2} \frac{x}{\sin_p x}\,dx = \frac{\pi_p}{2} \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+1/p)}{k!\Gamma(1/p)} \right)^2 \frac{1}{(kp+1)}.$$

When $p = 2$ the value of the integral on the left-hand side is known to be $2G$, and so we have the representation

$$G = \frac{\pi}{4} \sum_{k=0}^{\infty} \left(\frac{(2k)!}{2^{2k}(k!)^2} \right)^2 \frac{1}{(2k+1)},$$

which we have been unable to find in the extensive literature on this topic.

The integral

$$I_p := p \int_0^1 \log \sin_p \left(\frac{\pi_p \theta}{2} \right) d\theta$$

also claims our interest because of the different forms it can take. We show, for example, that

$$I_p = \frac{\Gamma'(1/p)}{\Gamma(1/p)} + \gamma = -\frac{2p}{\pi_p} \int_0^1 \frac{\sin_p^{-1} x}{x}\,dx = -\frac{p}{\pi_p} \int_0^{\infty} (\cot_p^{-1} y)^2\,dy,$$

where γ is Euler's constant and $\cot_p x = \cos_p x / \sin_p x$. The value of I_p for rational p is given in [**2**].

We conclude by discussing the Fourier sine coefficients of the functions $\sin_p(n\pi_p t)$ on $(0,1)$. With the kth such coefficient of $\sin_p(\pi_p t)$ denoted by $\tau_k(p)$, we show that for all $p \in (1,\infty)$, $|\tau_k(p)| = 0(k^{-2})$ as $k \to \infty$, while $|\tau_k(p)| = 0(k^{-3})$ if $1 < p < 2$. We obtain a variety of identities of Parseval type involving these coefficients, valid for all $p \in (1,\infty)$, such as a formula for

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tau_{2k+1}(p)$$

which, when $p \to 1$, gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.$$

Information about $\tau_k(p)$ is useful in dealing with the basis problem for $\sin_p$ functions. To explain this, we recall that it is a standard fact that the functions $\sin(n\pi t)$ form a basis in $L_q(0,1)$ for every $q \in (1,\infty)$. The main object of a recent paper by Binding et al. [**4**] is to show that there is a number $p_0 \in (1,2)$ such that, for each $p \in (p_0,\infty)$, the functions $\sin_p(n\pi_p t)$ also form a basis in every such $L_q(0,1)$. The idea of their ingenious proof is to show that if p is not too close to 1, then there is a homeomorphism of $L_q(0,1)$ onto itself that maps $\sin(n\pi t)$ to $\sin_p(n\pi_p t)$ for every $n \in \mathbf{N}$; to establish this they rely in part on the claim that, for each $t \in (0,1)$, $\sin_p(\pi_p t)$ decreases as p increases. We give a proof of this last assertion in Corollary 4.4 below since the argument given in [**4**] is incomplete. We also discuss how the result may be sharpened a little and derive a lower bound p_1 (> 1) for the value of p_0 obtainable by this method. Our original motivation for the study of the p-calculus was to obtain improved estimates for the Fourier coefficients of $\sin_p$ and thereby show that the basis property for these functions holds for all $p \in (1,\infty)$. We were somewhat surprised to find, as shown in Section 4, that the method of proof must fail for $p \in (1,p_1)$. It is tempting to conjecture that the basis property does hold even when $p \in (1,p_1)$, but to establish this another approach to the problem is needed.

2. Definitions and basic properties. Throughout we shall assume that $1 < p < \infty$ and write $p' = p/(p-1)$. Define $F_p : [0,1] \to \mathbf{R}$ by

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt, \quad x \in [0,1]. \tag{2.1}$$

Plainly $F_2 = \sin^{-1}$. As F_p is strictly increasing it has an inverse, which we denote by $\sin_p$ to emphasize the connection with the usual sine function. This is defined on the interval $[0, \pi_p/2]$, where

$$\begin{aligned}\pi_p/2 = \sin_p^{-1}(1) &= \int_0^1 (1-t^p)^{-1/p} dt \\ &= p^{-1} \int_0^1 (1-s)^{-1/p} s^{-1/p'} ds \\ &= p^{-1} B(1/p', 1/p),\end{aligned}$$

where B is the beta function. Hence

$$\pi_p = \frac{2\pi}{p \sin(\pi/p)}. \tag{2.2}$$

Note that $\pi_2 = \pi$ and

$$p\pi_p = 2\Gamma(1/p')\Gamma(1/p) = p'\pi_{p'}. \tag{2.3}$$

Moreover, π_p decreases as p increases, and

$$\lim_{p\to 1} \pi_p = \infty, \lim_{p\to\infty} \pi_p = 2, \lim_{p\to 1}(p-1)\pi_p = \lim_{p\to 1} \pi_{p'} = 2. \tag{2.4}$$

We see that $\sin_p$ is strictly increasing on $[0, \pi_p/2]$, $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. It may be extended to $[0, \pi_p]$ by defining $\sin_p x = \sin_p(\pi_p - x)$ for $x \in [\pi_p/2, \pi_p]$; further extension to $[-\pi_p, \pi_p]$ is made by oddness, and finally $\sin_p$ is extended to the whole of $\mathbf{R}$ by $2\pi_p$-periodicity. Note that in some other works (see [**12**], for example) $\sin_p$ is defined slightly differently, and that care should therefore be used when comparing the results given here with those provided elsewhere.

Define $\cos_p : \mathbf{R} \to \mathbf{R}$ by

$$\cos_p x = \frac{d}{dx} \sin_p x, \quad x \in \mathbf{R}. \tag{2.5}$$

Clearly $\cos_p$ is even, $2\pi_p$-periodic and odd about $\pi_p/2$. If $x \in [0, \pi_p/2]$ and we put $y = \sin_p x$, then

$$\cos_p x = (1-y^p)^{1/p} = (1-(\sin_p x)^p)^{1/p}. \tag{2.6}$$

Hence $\cos_p$ is strictly decreasing on $[0, \pi_p/2]$, $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$; moreover,

$$|\sin_p x|^p + |\cos_p x|^p = 1 \quad (x \in \mathbf{R}). \tag{2.7}$$

This is clear from (2.6) if $x \in [0, \pi_p/2]$ and follows for all $x \in \mathbf{R}$ by symmetry and periodicity. Analogues of the other trigonometric functions may now be given in the obvious way: thus $\tan_p$ is defined by

$$\tan_p x = \frac{\sin_p x}{\cos_p x} \tag{2.8}$$

at those points x for which $\cos_p x \neq 0$, that is, for all $x \in \mathbf{R}$ except for the points $(k+1/2)\pi_p$ $(k \in \mathbf{Z})$. Plainly $\tan_p$ is odd and π_p−periodic; $\tan_p(0) = 0$. The analogy with the classical trigonometric functions is less than complete, however. For example, while the extended $\sin_p$ function does belong to $C^1(\mathbf{R})$, it is far from being real analytic on $\mathbf{R}$ if $p \neq 2$. To see this, note that with the aid of (2.7) its second derivative at x can be seen to be $-h(\sin_p x)$, where

$$h(y) = (1-y^p)^{(2/p)-1} y^{p-1},$$

and so is not continuous at $\pi_p/2$ if $2 < p < \infty$. Nevertheless, $\sin_p$ is real analytic on $[0, \pi_p/2)$.

While we have restricted p to the open interval $(1, \infty)$, the limit values 1 and ∞ may be allowed in a natural way. Thus we may define

$$\sin_1^{-1} x = -\log(1-x), \qquad \sin_\infty^{-1} x = x \quad (0 \leq x < 1)$$

so that

$$\sin_1 x = 1 - e^{-x}, \qquad \sin_\infty x = x$$

on appropriate intervals. We shall not pursue this further here.

In the next proposition we record some basic facts concerning the derivatives of p-trigonometric functions. These follow immediately from the definitions and (2.7).

Proposition 2.1. *For all $x \in [0, \pi_p/2)$,*

$$\frac{d}{dx}\cos_p x = -\sin_p^{p-1} x \cos_p^{2-p} x, \tag{2.9}$$

$$\frac{d}{dx}\tan_p x = 1 + \tan_p^p x,$$

$$\frac{d}{dx}\cos_p^{p-1} x = -(p-1)\sin_p^{p-1} x, \tag{2.10}$$

$$\frac{d}{dx}\sin_p^{p-1} x = (p-1)\sin_p^{p-2} x \cos_p x.$$

The Appendix contains further examples of this sort.

Now we provide some elementary, but useful, identities.

Proposition 2.2. *For all* $y \in [0,1]$*,*

$$\cos_p^{-1} y = \sin_p^{-1}(1-y^p)^{1/p}, \qquad \sin_p^{-1} y = \cos_p^{-1}(1-y^p)^{1/p} \tag{2.11}$$

and

$$\begin{gathered} \frac{2}{\pi_p}\sin_p^{-1} y^{1/p} + \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-y)^{1/p'} = 1, \\ \cos_p^p(\pi_p y/2) = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2). \end{gathered} \tag{2.12}$$

Proof. The first two claims follow directly from (2.7). For the third, note that

$$\sin_{p'}^{-1}(1-y^p)^{1/p'} = \int_0^{(1-y^p)^{1/p'}} (1-t^{p'})^{-1/p'}dt,$$

and that the change of variable $s = (1-t^{p'})^{1/p}$ transforms this integral into

$$\begin{aligned} \frac{p}{p'}\int_y^1 (1-s^p)^{-1/p}ds &= \frac{p}{p'}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right) \\ &= \frac{\pi_{p'}}{\pi_p}\left(\frac{\pi_p}{2} - \sin_p^{-1} y\right), \end{aligned}$$

the final step following from (2.3). Lastly, to obtain the fourth identity, write

$$\cos_p^p(\pi_p y/2) = 1 - \sin_p^p(\pi_p y/2) := 1-x$$

and observe that, in view of the third identity,

$$y = \frac{2}{\pi_p}\sin_p^{-1} x^{1/p} = 1 - \frac{2}{\pi_{p'}}\sin_{p'}^{-1}(1-x)^{1/p'},$$

which gives

$$1-x = \sin_{p'}^{p'}(\pi_{p'}(1-y)/2). \qquad \square$$

Further identities of this sort are given in the Appendix.

We also have a p-analogue of the classical Jordan inequality.

Proposition 2.3. *For all* $\theta \in (0, \pi_p/2]$,

$$\frac{2}{\pi_p} \leq \frac{\sin_p \theta}{\theta} < 1. \tag{2.13}$$

Proof. A natural change of variable shows that

$$\sin_p^{-1} x = x \int_0^1 (1 - x^p s^p)^{-1/p} ds,$$

and so

$$\theta = (\sin_p \theta) \int_0^1 (1 - (\sin_p \theta)^p \, s^p)^{-1/p} ds.$$

As

$$1 \leq \int_0^1 (1 - (\sin_p \theta)^p \, s^p)^{-1/p} ds \leq \frac{\pi_p}{2}$$

for all $\theta \in (0, \pi_p/2]$, the result follows. □

There are connections between the generalized trigonometric functions and those of classical analysis. Since

$$\sin_p^{-1} x = \frac{x}{p} \int_0^1 t^{-1/p'} (1 - x^p t)^{-1/p} dt,$$

we have the representations

(2.14)

$$\sin_p^{-1} x = xF\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right) = x(1 - x^p)^{1/p'} F\left(1, 1; 1 + \frac{1}{p}; x^p\right)$$
$$(0 \leq x < 1),$$

where F is the hypergeometric function (see [**2**, Theorems 2.2.1 and 2.2.5]). This leads naturally to an expression in terms of the incomplete beta function $I(a, b; \cdot)$ defined, for any positive a and b, by

$$I(a, b; x) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1 - t)^{b-1} dt, \quad x \in [0, 1].$$

In fact, we have

$$\sin_p^{-1} x = \frac{\pi_p}{2} I\left(\frac{1}{p}, \frac{1}{p'}; x^p\right), \quad x \in [0,1], \tag{2.15}$$

which may be rewritten as

$$(\sin_p x)^p = I^{-1}\left(\frac{1}{p}, \frac{1}{p'}; \frac{2}{\pi_p} x\right), \quad x \in [0, \pi_p/2]. \tag{2.16}$$

Since

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{x^n}{n!},$$

we have from (2.14) the power series expansion of $\sin_p^{-1} x$ as

$$\sin_p^{-1} x = x \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{\Gamma(1/p)(np+1)} \frac{x^{np}}{n!} \quad (0 \le x < 1). \tag{2.17}$$

From this an expansion of $\sin_p x$ may be obtained, the first three terms being given below:

$$\sin_p x = x - \frac{1}{p(p+1)} x^{p+1} - \frac{(p^2 - 2p - 1)}{2p^2(p+1)(2p+1)} x^{2p+1} + \cdots \quad (0 \le x < \pi_p/2). \tag{2.18}$$

Subsequent terms have very complicated coefficients, no regular pattern being discernible for general p.

Further inequalities, complementing Proposition 2.3, can be obtained by exploiting the hypergeometric connection, as we next show.

Theorem 2.4. (i) *The function* $x \mapsto x^{-1} \sin_p^{-1} x$ *is strictly increasing on* (0,1) *and maps this interval onto* $(1, \pi_p/2)$.

(ii) *The function*

$$x \longmapsto \frac{x^{-1}(1-x^p)^{-1/p'} \sin_p^{-1} x - 1}{(1-x^p)^{-1/p'} - 1}$$

is strictly increasing on $(0,1)$ *and maps this interval onto* $(p^2/(p^2-1),$ $\pi_p/2)$.

Proof. (i) This is immediate from (2.17).

(ii) From (2.17) we have

$$\begin{aligned}(2.19)\qquad \sin_p^{-1} x - x(1-x^p)^{1/p'} &= xF(1/p,1/p;1+1/p;x^p) - x(1-x^p)^{-1/p'} \\ &= \sum_{k=1}^{\infty} \frac{x^{kp+1}}{k!\Gamma(1/p)}\left\{\frac{\Gamma(k+1/p)}{(kp+1)} + \frac{(p-1)\Gamma(k-1+1/p)}{p}\right\} \\ &= \sum_{k=1}^{\infty} \frac{kp^2\Gamma(k+1/p)}{k!\Gamma(1/p)(kp+1)((k-1)p+1)}\, x^{kp+1}.\end{aligned}$$

Moreover,

$$(2.20)\qquad x - x(1-x^p)^{1/p'} = \frac{p-1}{p}\sum_{k=1}^{\infty}\frac{\Gamma(k-1+1/p)}{k!\Gamma(1/p)}x^{kp+1}.$$

Hence

$$\begin{aligned}(2.21)\qquad &\left\{\sin_p^{-1} x - x(1-x^p)^{1/p'}\right\} - \frac{p^2}{p^2-1}\left\{x - x(1-x^p)^{1/p'}\right\} \\ &\qquad = \sum_{k=1}^{\infty}\frac{\Gamma(k-1+1/p)\,p(k-1)}{k!\Gamma(1/p)(p+1)(kp+1)}x^{kp+1},\end{aligned}$$

from which the result follows. □

In fact, (ii) can be deduced from the results of [**16**], but we gave a direct proof for the reader's convenience.

Corollary 2.5. *For all* $x \in (0,1)$,

(i) $x < \sin_p^{-1} x < \pi_p x/2$,

(ii) $x^{-1}\sin_p^{-1} x < (\pi_p/2) - ((\pi_p/2)-1)(1-x^p)^{1/p'}$ *and*

(iii) $x^{-1}\sin_p^{-1} x > \{p^2 - (1-x^p)^{1/p'}\}/(p^2-1)$.

Proof. Parts (i) and (ii) are immediate from (i) and (ii) respectively of Theorem 2.4. As for (iii), from (2.21) we see that

$$\begin{aligned}\sin_p^{-1} x = {} & \frac{p^2 x}{p^2-1} - \frac{x}{p^2-1}(1-x^p)^{1/p'} \\ & + \sum_{k=2}^{\infty} \frac{\Gamma\left(k-1+(1/p)\right)}{k!\Gamma\left(1/p\right)} \frac{p(k-1)}{(p+1)(kp+1)} x^{kp+1},\end{aligned} \tag{2.22}$$

and (iii) follows. □

Note that (i) is simply the p-Jordan inequality, Proposition 2.3. Moreover, (ii) and (iii) can be rewritten to give

$$1 + \frac{1}{p^2-1}\left(1-\cos_p^{p-1}\theta\right) < \frac{\theta}{\sin_p \theta} < \frac{\pi_p}{2} - \left(\frac{\pi_p}{2}-1\right)\cos_p^{p-1}\theta,$$
$$0 < \theta \leq \frac{\pi_p}{2},$$

which is an improvement of that inequality; when $p=2$ it becomes

$$1 + \frac{2}{3}\sin^2\frac{\theta}{2} < \frac{\theta}{\sin\theta} < \frac{\pi}{2} - \left(\frac{\pi}{2}-1\right)\cos\theta, \quad 0<\theta\leq\frac{\pi}{2}.$$

We conclude this section by giving some integrals involving the p-trigonometric functions.

Proposition 2.6. *For all* $x \in (0, \pi_p/2)$,

$$\int \cos_p x\,dx = \sin_p x, \;\; p\int \cos_p^p x\,dx = (p-1)x + \sin_p x \cos_p^{p-1} x,$$
$$(p-1)\int \sin_p^{p-1} x\,dx = -\cos_p^{p-1} x, \;\; \int \tan_p^p x\,dx = \tan_p x - x$$

and

$$\int \sin_p x dx = \frac{1}{2}\sin_p^2 x \; F\left(\frac{1}{p}, \frac{2}{p}; 1+\frac{2}{p}; \sin_p^p x\right). \tag{2.23}$$

Apart from (2.23), these follow directly from the definitions. To obtain (2.23), make the substitution $u = \sin_p x$, note that

$$\int \sin_p x\,dx = \int u(1-u^p)^{-1/p}\,du = \int u \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{\Gamma(1/p)} \frac{u^{pn}}{n!}\,du,$$

integrate, and then write the resulting series in terms of the hypergeometric function.

Further examples are listed in the Appendix.

3. Definite integrals and infinite sums.

3.1. Diverse integrals. We begin with some elementary observations.

Proposition 3.1. *Let $k, l > 0$. Then*

$$\begin{aligned} \int_0^{\pi_p/2} \sin_p^k x\,dx &= \frac{1}{p} B\left(\frac{k+1}{p}, \frac{1}{p'}\right), \\ \int_0^{\pi_p/2} \cos_p^k x\,dx &= \frac{1}{p} B\left(\frac{1}{p}, 1 + \frac{k-1}{p}\right) \end{aligned} \tag{3.1}$$

and

$$\int_0^{\pi_p/2} \sin_p^k x \cos_p^l x\,dx = \frac{1}{p} B\left(\frac{k+1}{p}, 1 + \frac{l-1}{p}\right). \tag{3.2}$$

These follow directly by making natural substitutions: for example, in the first integral we put $y = \sin_p x$ and then $t = y^p$. Note that the conditions on k and l can be weakened: in the first and third equality it is enough to require that $k > -1$, while in the remaining cases the conditions $k, l > 1 - p$ will do.

Proposition 3.2. *Let $a, b > 0$. Then*

$$\int_0^{\infty} \frac{\cos_p ax - \cos_p bx}{x}\,dx = \log \frac{b}{a}.$$

Proof. Given any $K > 0$, change of variables shows that

$$\int_0^{K/a} \frac{\cos_p ax - 1}{x}\, dx = \int_0^{K/b} \frac{\cos_p bx - 1}{x}\, dx.$$

Hence,

$$\begin{aligned}\int_0^{K/a} \frac{\cos_p ax - \cos_p bx}{x}\, dx &= \int_{K/a}^{K/b} \frac{\cos_p bx - 1}{x}\, dx \\ &= \left[\frac{\sin_p bx}{bx} - \log x\right]_{K/a}^{K/b} + \frac{1}{b}\int_{K/a}^{K/b} \frac{\sin_p bx}{x^2}\, dx \\ &= \log\frac{b}{a} + \frac{\sin_p K}{K} - \frac{a\sin_p(bK/a)}{Kb} \\ &\quad + \frac{1}{bK}\int_{1/a}^{1/b} \frac{\sin_p Kbx}{x^2}\, dx.\end{aligned}$$

Now let $K \to \infty$. $\square$

The p-analogue of the unit circle in the plane is the curve $|x|^p + |y|^p = 1$, which can be parameterized by writing $x = \cos_p \theta$, $y = \sin_p \theta$ $(0 \le \theta \le 2\pi_p)$. This can be used in connection with integrals of the form

$$I = \int_0^{\pi_p/2} f(\cos_p \theta, \sin_p \theta)\, d\theta. \tag{3.3}$$

Given $\theta \in [0, \pi_p/2]$, there is a unique $\phi \in [0, \pi/2]$ such that $\cos_p \theta = \cos^{2/p}\phi$ and $\sin_p \theta = \sin^{2/p}\phi$. Then $\tan^2\phi = \tan_p^p \theta$, $\theta = \tan_p^{-1}(\tan\phi)^{2/p}$ and $(d/d\phi)\tan\phi = (d/d\theta)\tan_p\theta$. Thus the integral I above can be expressed as

$$I = \frac{2}{p}\int_0^{\pi/2} f(\cos^{2/p}\phi, \sin^{2/p}\phi)\tan^{(2-p)/p}\phi\, d\phi. \tag{3.4}$$

The reverse procedure is also possible, of course, so that integrals of the form

$$J = \int_0^{\pi/2} g(\cos\phi, \sin\phi)\, d\phi$$

can be written as

$$J = \frac{p}{2}\int_0^{\pi_p/2} g(\cos_p^{p/2}\theta, \sin_p^{p/2}\theta)(\tan_p\theta)^{(p-2)/2}\,d\theta.$$

As an illustration of this technique, we recall that (see [**9**, 3.687]) given any $\mu, \nu > 0$, with $\mu + \nu < 2$,

$$\frac{\cos\left((\nu-\mu/4)\pi\right)}{2\cos\left((\nu+\mu/4)\pi\right)} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) = \int_0^{\pi/2} \frac{\sin^{\mu-1}x + \sin^{\nu-1}x}{\cos^{\mu+\nu-1}x}\,dx,$$

from which we obtain, on taking $\nu = \mu$ and changing to p-trigonometric functions,

$$\frac{B(\mu/2,\mu/2)}{\cos(\mu\pi/2)} = 2p\int_0^{\pi_p/2} \frac{(\sin_p\theta)^{(p\mu-2)/2}}{(\cos_p\theta)^{p\mu-1}}\,d\theta.$$

The choice $\mu = 1/p$ now gives

$$\int_0^{\pi_p/2} (\sin_p\theta)^{-1/2} d\theta = \frac{\Gamma\left(1/2p\right)^2}{2p\Gamma\left(1/p\right)\cos\left(\pi/2p\right)},$$

which we could also have obtained from the note following Proposition 3.1.

To conclude this subsection we remark that there is a p-analogue of the Riemann-Lebesgue lemma: if $f \in L_1(\mathbf{R})$ and $p \in (1,\infty)$, then

$$\lim_{n\to\infty}\int_{\mathbf{R}} f(t)c(nt)\,dt = 0,$$

where $c(nt)$ stands for either $\sin_p(nt)$ or $\cos_p(nt)$. To prove this, note that since the step-functions are dense in $L_1(\mathbf{R})$ it is enough to suppose that f is the characteristic function $\chi_{(a,b)}$ of any bounded interval (a,b) in $\mathbf{R}$. When $c_n(t) = \cos_p(nt)$, the argument is immediate:

$$\left|\int_a^b \cos_p(nt)\,dt\right| = \frac{1}{n}\left|\left[\sin_p(nt)\right]_a^b\right| \le 2/n \longrightarrow 0 \quad \text{as } n \to \infty.$$

If $c_n(t) = \sin_p(nt)$, first suppose that $c > 0$ and, given $n \in \mathbf{N}$, let M be the unique non-negative integer such that $nc = 2M\pi_p + \delta$, where

$0 \leq \delta < 2\pi_p$. In view of the antisymmetry and periodicity properties of $\sin_p$,

$$\int_0^{2M\pi_p} \sin_p t\, dt = M \int_0^{2\pi_p} \sin_p t\, dt = 0.$$

Hence,

$$\left| \int_0^c \sin_p(nt)\, dt \right| = \frac{1}{n} \left| \int_0^{nc} \sin_p t\, dt \right| = \frac{1}{n} \left| \int_{2M\pi_p}^{2M\pi_p+\delta} \sin_p t\, dt \right| \leq \delta/n.$$

Thus,

$$\begin{aligned} \left| \int_a^b \sin_p(nt)\, dt \right| &\leq \left| \int_0^{|b|} \sin_p(nt)\, dt \right| \\ &\quad + \left| \int_0^{|a|} \sin_p(nt)\, dt \right| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and the proof is complete. Encouraging though this result may be, theorems of Riesz-Fischer or Hausdorff-Young type, involving the 'p-Fourier coefficients,' are not to be expected because of the absence of orthogonality relations.

3.2. The Catalan constant. The result mentioned in the Introduction concerning Catalan's constant is a consequence of the power series expansion (2.17) of $\sin_p^{-1} x$, which leads to the representation

$$x = \sin_p x \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{\Gamma(1/p)(np+1)} \frac{(\sin_p x)^{np}}{n!}, \quad 0 < x < \frac{\pi_p}{2}. \tag{3.5}$$

Hence, with the use of (3.1), we have

$$\int_0^{\pi_p/2} \frac{x}{\sin_p x}\, dx = \frac{\pi_p}{2} \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/p)}{n!\Gamma(1/p)} \right)^2 \frac{1}{np+1}.$$

However, it is known that (see, for example, [**8**, 1.7.4])

$$\int_0^{\pi/2} \frac{x}{\sin x}\, dx = 2G,$$

where G is the Catalan constant defined by (1.3). We therefore have

Proposition 3.3. *The Catalan constant is expressible as*

$$G = \frac{\pi}{4} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{(n!)^2 2^{2n}} \right)^2 \frac{1}{2n+1}. \tag{3.6}$$

3.3. Various series. Consideration of inverse p-trigonometric functions can give interesting identities. For example,

$$\tan_p^{-1}(1) = \int_0^1 (1+s^p)^{-1} ds = \sum_{n=0}^{\infty} \frac{(-1)^n}{np+1}.$$

Moreover, if we write $\tan_p^{-1}(1) = \theta$ so that $\sin_p \theta = \cos_p \theta$ and $\sin_p^p \theta = 1 - \sin_p^p \theta$, then $\theta = \sin_p^{-1} 2^{-1/p}$, so that

$$\theta = 2^{-1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; \frac{1}{2}\right) = \frac{1}{2p}\left\{\psi\left(\frac{1}{2} + \frac{1}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right\},$$

where ψ is the digamma function, $\psi(x) = \Gamma'(x)/\Gamma(x)$ (see [**1**, 15.1.28]). We thus have

$$\tan_p^{-1}(1) = \frac{1}{2p}\left\{\psi\left(\frac{1}{2} + \frac{1}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{np+1}. \tag{3.7}$$

Such series are considered in Ramanujan's notebooks (see [**3**, pages 184–190]), where the values of the integral representing $\tan_p^{-1}(1)$ can be found for $p = 3, 4, 5, 6, 8, 10$: for example,

$$\tan_3^{-1}(1) = \frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}$$

and

$$\tan_4^{-1}(1) = \frac{1}{4\sqrt{2}}\log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + \frac{\pi}{4\sqrt{3}}.$$

Hence,

$$\frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$$

and

$$\frac{1}{4\sqrt{2}} \log\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + \frac{\pi}{4\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1}.$$

Also, because

$$\sin_p^{-1}(1) = F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; 1\right)$$

we see that

$$\frac{\pi\Gamma(1/p)}{p\sin(\pi/p)} = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{n!(np+1)}. \tag{3.8}$$

Another series representation stems from the fact that

$$\frac{1}{2}(\sin_p^{-1} 1)^2 = \int_0^1 \frac{\sin_p^{-1} s}{(1-s^p)^{1/p}}\, ds;$$

expanding the denominator in the integral and then integrating we obtain

$$\frac{\pi\Gamma^2(1/p)}{2p^2\sin(\pi/p)} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2/p)}{(np+1)^2\Gamma(n+1)}. \tag{3.9}$$

When $p=2$ this gives the familiar formula

$$\pi^2/8 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2};$$

indeed, this was how Euler originally obtained this result. When $p=3$ we have

$$\frac{\pi\Gamma^2(1/3)}{9\sqrt{3}\Gamma(2/3)} = 1 + \sum_{n=1}^{\infty} \frac{2\cdot 5\cdot 8\cdots(3n-1)}{n!3^n(3n+1)^2}.$$

3.4. The integral I_p. The integral

$$I_p := p\int_0^1 \log \sin_p(\pi_p\theta/2)\, d\theta \tag{3.10}$$

seems to us to warrant study in view of its diverse representations. First observe that the substitution $x = \sin_p(\pi_p\theta/2)$ leads to

$$
\begin{aligned}
I_p &= \frac{2p}{\pi_p}\int_0^1 \frac{\log x}{(1-x^p)^{1/p}}\,dx \\
&= \frac{2p}{\pi_p}\int_0^1 \log x \frac{d}{dx}\sin_p^{-1} x\,dx \qquad (3.11)\\
&= -\frac{2p}{\pi_p}\int_0^1 \frac{\sin_p^{-1} x}{x}\,dx.
\end{aligned}
$$

With (2.14) this gives

$$
\begin{aligned}
I_p &= -\frac{2p}{\pi_p}\int_0^1 F(1/p,1/p;1+1/p;x^p)\,dx \\
&= -\frac{2}{\pi_p}\int_0^1 t^{-1/p'}F(1/p,1/p;1+1/p;t)\,dt \\
&= -\frac{2}{\pi_p}\int_0^1 t^{-1/p'}(1-t)^{1/p'}F(1,1;1+1/p;t)\,dt,
\end{aligned}
$$

the last equality following from (2.14) again. Expansion of $F(1,1;1+1/p;t)$ produces

$$
\begin{aligned}
I_p &= -\frac{2}{\pi_p}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(1+1/p)}{\Gamma(n+1+1/p)}\cdot\int_0^1 t^{n-1/p'}(1-t)^{1/p'}\,dt \\
&= -\frac{2}{\pi_p}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(1+1/p)}{\Gamma(n+1+1/p)}\cdot\frac{\Gamma(n+1/p)\Gamma(2-1/p)}{\Gamma(n+2)} \\
&= -\frac{2}{pp'\pi_p}\Gamma(1/p)\Gamma(1/p')\sum_{n=0}^{\infty}\frac{1}{(n+1/p)(n+1)} \\
&= -\frac{1}{p'}\sum_{n=0}^{\infty}\frac{1}{(n+1/p)(n+1)} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+1/p}\right) = \frac{\Gamma'(1/p)}{\Gamma(1/p)}+\gamma,
\end{aligned}
$$

where γ is Euler's constant; the final equality comes from [**2**, Theorem 1.2.5]. Summarizing, we have

$$
I_p = \frac{\Gamma'(1/p)}{\Gamma(1/p)}+\gamma. \qquad (3.12)
$$

Note that when $p = 2$ we obtain, with obvious notation,

$$\begin{aligned} I_2 &= -\sum_{n=0}^{\infty} \frac{1}{(2n+1)(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{2}{2n-1} \right) \\ &= \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - 2\sum_{n=1}^{2N} \frac{1}{n} + \sum_{n=1}^{N} \frac{1}{n} \right) \\ &= \lim_{N\to\infty} (2\log N - 2\log 2N + 2\gamma_N - 2\gamma_{2N}) \\ &= -2\log 2, \end{aligned}$$

a familiar result. Observe also that, when m, n are positive integers, the value of $I_{m/n}$ is known (see [**2**, (1.2.19)]) to be

$$\begin{aligned} I_{m/n} &= -\frac{\pi}{2} \cot \frac{n\pi}{m} - \log m \\ &\quad + \frac{1}{2} \sum_{k=1}^{m-1} \cos\left(\frac{2\pi k n}{m} \right) \log\left(2 - 2\cos\left(\frac{2\pi k}{m} \right) \right). \end{aligned}$$

In view of (3.11) we see that

$$\int_0^1 \frac{\sin_p^{-1} x}{x} dx = -\frac{\pi_p}{2p} \left(\frac{\Gamma'(1/p)}{\Gamma(1/p)} + \gamma \right). \tag{3.13}$$

To provide other forms of I_p we start from (3.11) and put $x = \sin_p \theta$ to obtain

$$I_p = -\frac{2p}{\pi_p} \int_0^{\pi_p/2} \theta \cot_p \theta \, d\theta. \tag{3.14}$$

Integration by parts and then use of the substitution $\theta = \cot_p^{-1} y$ gives

$$I_p = \frac{p}{\pi_p} \int_0^{\pi_p/2} \theta^2 \frac{\cos_p^{2-p} \theta}{\sin_p^2 \theta} \, d\theta = -\frac{p}{\pi_p} \int_0^{\infty} (\cot_p^{-1} y)^2 dy. \tag{3.15}$$

Finally, we note that the representation (3.5) of x as a series of powers of $\sin_p x$ leads to

$$\int_0^{\pi_p/2} x \cot_p x \, dx = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/p)}{k!\Gamma(1/p)(kp+1)^2}.$$

Together with (3.14) and the fact that $I_2 = -2\log 2$ gives

$$\frac{\pi^{3/2}}{2}\log 2 = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{k!(2k+1)^2}. \tag{3.16}$$

3.5. The addition formula question. In the Introduction we remarked on the lack of usable addition formulae for the $p-$trigonometric functions. Here we illustrate the problem by consideration of the function $\tan_p$. Suppose that

$$\phi(u,v) = \tan_p\left(\tan_p^{-1} u + \tan_p^{-1} v\right).$$

Then

$$\int_0^{\phi} \frac{ds}{1+s^p} = \int_0^u \frac{ds}{1+s^p} + \int_0^v \frac{ds}{1+s^p}.$$

These integrals can be evaluated for particular integer values of p: for example, [**3**, pages 189–190] gives expressions for them when $p = 3, 4, 5, 6, 8, 10$. However, these rapidly become complicated. For example, corresponding to $p = 3$ we have

$$\int_0^u \frac{ds}{1+s^3} = \frac{1}{6}\log\frac{(1+u)^3}{1+u^3} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u\sqrt{3}}{2-u}\right),$$

from which it seems impossible to derive an explicit formula for ϕ. The existence of reasonable addition formulae for any of the p-functions seems to be very unlikely.

4. Fourier coefficients and the basis problem. We begin by considering the Fourier sine coefficients of the functions $\sin_p(n\pi_p t)$. Given $p \in (1,\infty)$ and $n \in \mathbf{N}$, for simplicity we write $f_{n,p}(t) = \sin_p(n\pi_p t)$ $(t \in \mathbf{R})$ and set $e_n = f_{n,2}$, so that $e_n(t) = \sin(n\pi t)$. Since each $f_{n,p}$ is continuous on $[0,1]$ it has a Fourier sine expansion:

$$f_{n,p}(t) = \sum_{k=1}^{\infty} \widehat{f_{n,p}}(k)\sin(k\pi t),\ \ \widehat{f_{n,p}}(k) = 2\int_0^1 f_{n,p}(t)\sin(k\pi t)\,dt.$$

The symmetry of $f_{1,p}$ about $t = 1/2$ means that $\widehat{f_{1,p}}(k) = 0$ when k is even and that

$$\begin{aligned}\widehat{f_{n,p}}(k) &= 2\int_0^1 f_{1,p}(nt)\sin(k\pi t)\,dt \\ &= 2\sum_{m=1}^{\infty}\widehat{f_{1,p}}(m)\int_0^1 \sin(k\pi t)\sin(mn\pi t)\,dt \\ &= \begin{cases}\widehat{f_{1,p}}(m) & \text{if } mn = k \text{ for some odd } m, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

For brevity we shall put $\tau_m(p) = \widehat{f_{1,p}}(m)$. As all the Fourier coefficients of the $f_{n,p}$ may be expressed in terms of the $\tau_m(p)$, we focus our attention on the behavior of these terms, beginning with their decay properties as $m \to \infty$.

For even m, $\tau_m(p) = 0$. For odd m, say $m = 2k+1$,

$$\begin{aligned}\tau_{2k+1}(p) &= 4\int_0^{1/2} \sin_p(\pi_p t)\sin\left((2k+1)\pi t\right)dt \\ &= \frac{4\pi_p}{(2k+1)\pi}\int_0^{1/2}\cos_p(\pi_p t)\cos\left((2k+1)\pi t\right)dt \\ &= -\frac{4\pi_p}{(2k+1)^2\pi^2}\int_0^{1/2}\sin\left((2k+1)\pi t\right)\frac{d}{dt}\cos_p(\pi_p t)\,dt \\ &= \frac{4\pi_p}{(2k+1)^2\pi^2}\int_0^1 \sin\left(\frac{(2k+1)\pi}{\pi_p}\cos_p^{-1}x\right)dx.\end{aligned} \tag{4.1}$$

In a similar way we have

$$\tau_{2k+1}(p) = \frac{4}{(2k+1)\pi}\int_0^1 \cos\left(\frac{(2k+1)\pi}{\pi_p}\sin_p^{-1}x\right)dx. \tag{4.2}$$

From (4.1) we immediately have the estimate

$$|\tau_{2k+1}(p)| \le \frac{4\pi_p}{(2k+1)^2\pi^2} \quad (k \in \mathbf{N}). \tag{4.3}$$

This is obtained in [**4**] by a slightly less direct procedure.

In fact, when $1 < p < 2$ (which as we shall see is the case of most interest) the decay of the $\tau_{2k+1}(p)$ is faster than indicated above. To establish this we start from the representation (4.1). Put $t = \cos_p^{-1} x := \phi(x)$ so that $x = \cos_p t$. Then

$$\begin{aligned}\frac{dx}{dt} &= -x^{2-p}(1-x^p)^{1-1/p},\\ \phi'(x) = \frac{dt}{dx} &= -x^{-(2-p)}(1-x^p)^{-(1-1/p)}\end{aligned}$$

and so $|\phi'(x)| \geq 1$ in $(0,1)$. Moreover,

$$\begin{aligned}\phi''(x) &= -(p-2)x^{p-3}(1-x^p)^{-1+1/p}\\ &\quad + (1-1/p)x^{p-2}(1-x^p)^{-2+1/p)} \cdot -px^{p-1}\\ &= x^{p-3}(1-x^p)^{-(2-1/p)}\left\{(2-p)(1-x^p)-(p-1)x^p\right\}\\ &= x^{p-3}(1-x^p)^{-(2-1/p)}(2-p-x^p).\end{aligned}$$

Hence $\phi'(x)$ is increasing in $(0,(2-p)^{1/p})$ and decreasing in $((2-p)^{1/p},1)$. The minimum of $|\phi'(x)|$ on $(0,1)$ is

$$\left\{(2-p)^{-(2-p)}(p-1)^{-(p-1)}\right\}^{1/p} := m_p,$$

attained at $(2-p)^{1/p}$. We now use an estimate of van der Corput type to deal with the oscillatory integral in (4.1). Put $b = (2-p)^{1/p}$ and temporarily set $a = (2k+1)\pi/\pi_p$. Integration by parts gives

$$\begin{aligned}\int_0^b \sin\left(a\cos_p^{-1} x\right)\, dx &= \int_0^b \sin\left(a\phi(x)\right) a\phi'(x) \cdot \frac{1}{a\phi'(x)}\, dx\\ &= \left[-\cos\left(a\phi(x)\right)\cdot\frac{1}{a\phi'(x)}\right]_0^b\\ &\quad + \frac{1}{a}\int_0^b \cos\left(a\phi(x)\right)\frac{d}{dx}\left\{1/\phi'(x)\right\}\, dx.\end{aligned}$$

Together, with the monotonicity of $\phi'(x)$, this gives

$$\left|\int_0^b \sin\left(a\cos_p^{-1} x\right)\, dx\right| \leq \frac{1}{am_p} + \frac{1}{a}\int_0^b \left|\frac{d}{dx}\left\{1/\phi'(x)\right\}\right| dx$$

$$= \frac{1}{am_p} + \frac{1}{a}\left| \int_0^b \frac{d}{dx}\left\{1/\phi'(x)\right\}\, dx \right|$$
$$= \frac{2}{am_p}.$$

In the same way we see that

$$\left| \int_b^1 \sin\left(a \cos_p^{-1} x\right)\, dx \right| \le \frac{2}{am_p}.$$

It follows that

$$\left| \int_0^1 \sin\left(\frac{(2k+1)\pi}{\pi_p} \cos_p^{-1} s\right) ds \right| \le \frac{4\pi_p}{(2k+1)\pi m_p}.$$

Combined with (4.1) this shows that, if $1 < p < 2$,

$$|\tau_{2k+1}(p)| \le \frac{16\pi_p^2}{m_p(2k+1)^3\pi^3} \quad (k \in \mathbf{N}). \tag{4.4}$$

Interesting formulae may be obtained by use of the Euler polynomials E_n, which have the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi) \tag{4.5}$$

(see [**1**, Chapter 23] or [**5**]). Recursive computation of E_n is possible by means of the formulae

$$E_n(x) + \sum_{k=0}^{n} {}^nC_k E_k(x) = 2x^n \quad (n \in \mathbf{N}).$$

The first three Euler polynomials are

$$E_0(x) = 1, \qquad E_1(x) = x - 1/2, \qquad E_2(x) = x^2 - x.$$

The Fourier expansions of the E_n are

$$E_{2n}(x) = \frac{4(-1)^n(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}},$$
$$E_{2n-1}(x) = \frac{4(-1)^n(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}},$$

for $n \in \mathbf{N}$ and $0 \le x \le 1$. Hence

$$\int_0^{1/2} \sin_p(\pi_p x) E_n(x) dx = \frac{(-1)^n (2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\tau_{2k+1}(p)}{(2k+1)^{2n+1}},$$

that is,

$$\sum_{k=0}^{\infty} \frac{\tau_{2k+1}(p)}{(2k+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1}}{(2n)!} \int_0^{1/2} \sin_p(\pi_p x) E_n(x)\, dx \quad (n \in \mathbf{N} \cup \{\mathbf{0}\}). \tag{4.6}$$

To supplement this, we have from Parseval's formula,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tau_{2k+1}(p) = \int_0^{\pi/2} x \sin_p(\pi_p x/\pi)\, dx \tag{4.7}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \tau_{2k+1}(p) = -\frac{2}{\pi} \int_0^{\pi/2} \sin_p\left(\frac{\pi_p x}{\pi}\right) \left(\int_0^x \log\tan(u/2)\, du \right) dx. \tag{4.8}$$

After appropriate changes of variable, the cases $n = 0, 1$ of (4.6) and (4.7) give

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \tau_{2k+1}(p) = (\pi/\pi_p) \int_0^{\pi_p/2} \sin_p x\, dx, \tag{4.9}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \tau_{2k+1}(p) = \frac{1}{2} (\pi/\pi_p)^3 \int_0^{\pi_p/2} x(\pi_p - x) \sin_p x\, dx \tag{4.10}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tau_{2k+1}(p) = \left(\frac{\pi}{\pi_p}\right)^2 \int_0^{\pi_p/2} x \sin_p x\, dx. \tag{4.11}$$

We remark that (4.8) and (4.11), in which the summations involve an even power of $2k+1$, may be thought of as interpolating between the results containing odd powers of $2k+1$. Note that, in view of Proposition 3.1, (4.9) may be written as

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \tau_{2k+1}(p) = \frac{\pi \Gamma(2/p)}{2\Gamma(1/p)\Gamma(1+1/p)}. \tag{4.12}$$

Moreover, Proposition 3.1 together with (3.5), which represents x as a sum of multiples of powers of $\sin_p x$, enables us to evaluate the integral in (4.11). Observe also that the formulae above give familiar results when $p \to 1$. Thus, (4.9), (4.10) and (4.11) give

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}, \qquad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32},$$

respectively.

It is possible, and even desirable as we shall see later, to obtain an integral representation of $\sum_{k=0}^{\infty} \tau_{2k+1}(p)$. To achieve this we define

$$\phi(r,p) = \sum_{k=0}^{\infty} r^{2k+1} \tau_{2k+1}(p) \tag{4.13}$$

for every $r \in [0,1]$, Then, if $0 \le r < 1$,

$$\begin{aligned} \phi(r,p) &= 4 \int_0^{1/2} \sin_p(\pi_p x) \sum_{k=0}^{\infty} r^{2k+1} \sin(2k+1)\pi x \, dx \\ &= 2 \int_0^{1/2} \sin_p(\pi_p x) \frac{\lambda \sin \pi x}{1 - \lambda^2 \cos^2 \pi x} \, dx, \end{aligned}$$

where $\lambda = 2r/(1+r^2)$. In view of the decay property (4.3) of $\tau_{2k+1}(p)$ it is clear that $\phi(1,p) < \infty$. Hence, by Abel's limit theorem, $\phi(1,p) = \lim_{r \to 1} \phi(r,p)$; and, since

$$\left| \sin_p(\pi_p x) \frac{\lambda \sin \pi x}{1 - \lambda^2 \cos^2 \pi x} \right| \le \left| \frac{\sin_p(\pi_p x)}{\lambda \sin \pi x} \right|$$

we see from (2.13) and the dominated convergence theorem that

$$\begin{aligned}\sum_{k=0}^{\infty}\tau_{2k+1}(p) &= \lim_{r\to 1}\phi(r,p) = 2\int_0^{1/2}\frac{\sin_p(\pi_p x)}{\sin \pi x}\,dx \\ &= \frac{2}{\pi_p}\int_0^{\pi_p/2}\frac{\sin_p x}{\sin(\pi x/\pi_p)}\,dx.\end{aligned} \tag{4.14}$$

The substitution $u=\sin_p x$ and integration by parts lead to

$$\sum_{k=0}^{\infty}\tau_{2k+1}(p) = \frac{2}{\pi}\int_0^1 \log\left\{\cot\frac{1}{2}\left(\frac{\pi}{\pi_p}\sin_p^{-1}u\right)\right\}du. \tag{4.15}$$

In terms of the incomplete beta function (see (2.15)) this becomes

$$\sum_{k=0}^{\infty}\tau_{2k+1}(p) = \frac{2}{\pi}\int_0^1 \log\left\{\cot\left(\frac{\pi}{4}I\left(\frac{1}{p},\frac{1}{p'};u^p\right)\right)\right\}du. \tag{4.16}$$

We now turn to a basis question. It is well known that $(\exp(in\pi x))_{n\in\mathbf{N}}$ is a basis in $L_q(-1,1)$ for every $q\in(1,\infty)$; see, for example, [**7**, 12.10.1]. Given any element of $L_q(0,1)$, its odd extension to $L_q(-1,1)$ has a unique representation in terms of the functions $\sin(n\pi x)$, which means that $(\sin(n\pi x))$ is a basis of $L_q(0,1)$. The object of a striking recent paper [**4**] was to show that the functions $\sin_p(n\pi_p x)$ have a similar property, provided that p is not too close to 1, $p\geq p_0$, say, where $p_0\in(1,2)$. Here we analyze this paper and clarify the limitations of the method of proof that were pointed out in it.

Given any function f on $[0,1]$, extend it to a function $\widetilde{f}$ on $[0,\infty)$ by setting $\widetilde{f}(t) = -\widetilde{f}(2k-t)$ for $t\in[k,k+1]$, $k\in\mathbf{N}$; define $M_m : L_q(0,1)\to L_q(0,1)$ by $M_m g(t)=\widetilde{g}(mt)$ ($m\in\mathbf{N}$, $1<q<\infty$) and note that $M_m e_n = e_{mn}$: recall that $e_n(t)=\sin(n\pi t)$. In [**4**] it is shown that M_m is a linear isometry and that the map T defined by $Tg(t)=\sum_{m=1}^{\infty}\tau_m(p)M_m g(t)$ is a bounded linear map of $L_q(0,1)$ to itself with the property that, for all $n\in\mathbf{N}$, $Te_n=f_{n,p}$, where $f_{n,p}(t)=\sin_p(n\pi t)$. If it can be shown that T is a homeomorphism, then it will follows from standard considerations (see, for example, [**10**, page 75]) that the $f_{n,p}$ inherit from the e_n the property of forming a basis in $L_q(0,1)$ for every $q \in (1,\infty)$. It follows from [**4**] that T is a homeomorphism if

$$\sum_{k=1}^{\infty} |\tau_{2k+1}(p)| < |\tau_1(p)|, \tag{4.17}$$

and it is in trying to satisfy this inequality that the restriction on p appears. Because of (4.3) we have

$$\sum_{k=1}^{\infty} |\tau_{2k+1}(p)| \leq \frac{4\pi_p}{\pi^2}\left(\frac{\pi^2}{8} - 1\right), \tag{4.18}$$

and it remains to estimate $|\tau_1(p)|$ from below. In [**4**] this is done by claiming that, for each $t \in (0,1)$, $f_{1,p}(t) = \sin_p(\pi_p t)$ is strictly decreasing in p, and this claim in turn rests on the assertion that as p increases, both π_p^{-1} and $(1-t^p)^{-1/p}$ are strictly increasing. However, while it is true that π_p^{-1} is strictly increasing, a routine differentiation shows that in fact $(1-t^p)^{-1/p}$ is strictly decreasing as p increases, and so this assertion is false. Nevertheless, as we show below in Corollary 4.4, the claim is true. First, however, we show that a lower bound for $|\tau_1(p)|$ may be obtained by using the following elementary consequence of the p-Jordan inequality.

Lemma 4.1. *For all $p \in (1,\infty)$ and all $t \in (0,1/2)$,*

$$\sin_p(\pi_p t) > 2t.$$

Proof. By Proposition 2.3, $\sin_p \theta > 2\theta/\pi_p$ if $0 < \theta < \pi_p/2$. Now put $\theta = \pi_p t$. □

Proposition 4.2. *For all $p \in (1,\infty)$, $\tau_1(p) > 8/\pi^2$.*

Proof. Simply observe that, by the last lemma,

$$\tau_1(p) = 4\int_0^{1/2} f_{1,p}(t)\sin(\pi t)\,dt > 4\int_0^{1/2} 2t\sin(\pi t)\,dt = 8/\pi^2. \qquad \square$$

Together with (4.18) and the fact that π_p decreases as p increases, this shows that the functions $\sin_p(n\pi_p x)$ form a basis in $L_q(0,1)$ for every $q \in (1,\infty)$ if $2 \le p < \infty$.

Now we deal with the dependence of $\sin_p(\pi_p t)$ on p.

Proposition 4.3. *Suppose that $1 < p < q < \infty$. Then the function f defined by*

$$f(x) = \frac{\sin_q^{-1} x}{\sin_p^{-1} x}$$

is strictly decreasing on $(0,1)$.

Proof. Let

$$g(x) = \frac{(1-x^q)^{1/q}}{(1-x^p)^{1/p}} \quad (0 < x < 1).$$

Then, for all $x \in (0,1)$,

$$g'(x) = g(x)\left\{\frac{-x^{q-1}}{1-x^q} + \frac{x^{p-1}}{1-x^p}\right\} = \frac{(x^p - x^q)g(x)}{x(1-x^q)(1-x^p)} > 0.$$

Now let

$$G(x) = \sin_p^{-1} x - g(x)\sin_q^{-1} x;$$

then

$$G'(x) = -(\sin_q^{-1} x)g'(x) < 0 \quad \text{in } (0,1).$$

It follows that $G(x) < 0$ in $(0,1)$, and hence that

$$f'(x) = \frac{G(x)}{(\sin_p^{-1} x)^2(1-x^q)^{1/q}} < 0 \quad \text{in } (0,1). \qquad \square$$

As an immediate consequence we have

Corollary 4.4. (i) *If $1 < p < q < \infty$, then*

$$1 > \frac{\sin_q^{-1} x}{\sin_p^{-1} x} \ge \frac{\pi_q}{\pi_p} \quad \textit{in } (0,1].$$

(ii) *If* $1 < p \le q < \infty$, *then*

$$\sin_p^{-1} x \ge \sin_q^{-1} x \quad \text{and} \quad \frac{1}{\pi_q} \sin_q^{-1} x \ge \frac{1}{\pi_p} \sin_p^{-1} x \quad \text{in } [0,1].$$

(iii) *If* $1 < p \le q < \infty$, *then*

$$\sin_p(\pi_p x) \ge \sin_q(\pi_q x) \quad \text{in } [0,1/2].$$

The delicacy of the inequalities involved here is underlined by the opposing nature of the two parts of (ii).

Corollary 4.4 (iii) seals the gap in the proof of [**4**], and from this point onwards the argument for the basis property is much as given in that paper: first, observe that if $1 < p < 2$,

$$\tau_1(p) = 4\int_0^{1/2} \sin_p(\pi_p t)\sin(\pi t)\,dt > 4\int_0^{1/2} \sin^2(\pi t)\,dt = 1,$$

which, for this range of values of p, improves the estimate given in Proposition 4.2; second, use this estimate together with (4.17) and (4.18) to obtain their result:

Theorem 4.5. *The functions* $\sin_p(n\pi_p x)$ *form a basis in* $L_q(0,1)$ *for every* $q \in (1,\infty)$ *if* $p_0 < p < \infty$, *where* p_0 *is defined by the equation*

$$\pi_{p_0} = \frac{2\pi^2}{\pi^2 - 8}. \tag{4.19}$$

Proof. In view of (4.17), this follows immediately from (4.18) and Proposition 4.2, together with the fact that π_p decreases as p increases. □

Numerical solution of (4.19) shows that p_0 is approximately equal to 1.05.

Some improvement of this result can be obtained by using the estimate

$$|\tau_{2k+1}(p)| \leq \frac{16\pi_p^2}{m_p(2k+1)^3\pi^3}$$

given in (4.4) and valid for $1 < p < 2$. While for fixed p this gives a faster rate of decay, as k increases, than the inequality $|\tau_{2k+1}(p)| \leq 4\pi_p/((2k+1)\pi)^2$ that we have been using (given in (4.3)), unfortunately it does not immediately produce a better result than that already derived because of the π_p^2 factor. To obtain any sharpening it is necessary to use (4.4) for large values of k and (4.3) for smaller values. However, Theorem 4.5 is already close to the limit of what can be established by the technique represented by satisfaction of (4.17), as we now show.

From

$$\tau_1(p) = \frac{4}{\pi}\int_0^1 \cos\left(\frac{\pi}{\pi_p}\sin_p^{-1} x\right) dx = \frac{4}{\pi}\int_0^1 \cos\left(\frac{\pi}{2} I\left(\frac{1}{p}, \frac{1}{p'}; x^p\right)\right) dx$$

(see (4.2)) and (4.16) we see that the equation

$$\sum_{k=0}^{\infty} \tau_{2k+1}(p) = 2\tau_1(p)$$

is equivalent to

$$\int_0^1 \log\left\{\cot\left(\frac{\pi}{4} I\left(\frac{1}{p}, \frac{1}{p'}; u^p\right)\right)\right\} du = 4\int_0^1 \cos\left(\frac{\pi}{2} I\left(\frac{1}{p}, \frac{1}{p'}; x^p\right)\right) dx. \tag{4.20}$$

In this form the equation is suitable for numerical calculation, and we are indebted to Simon Eveson for showing numerically that (4.20) has a solution, p_1 say, approximately equal to 1.0439898. Since

$$\left|\sum_{k=0}^{\infty} \tau_{2k+1}(p)\right| \leq \sum_{k=0}^{\infty} |\tau_{2k+1}(p)|,$$

for all $p \in (1, \infty)$, it follows that (4.17) cannot hold if $p = p_1$, and so a lower bound for the validity of the technique of proof used to establish the basis property is given by $p = p_1$.

APPENDIX

5.1. Derivatives. The following table gives further examples of the derivatives of p-trigonometric functions, all of which can be verified directly. In each case the variable x is supposed to belong to $[0, \pi_p/2)$.

$f(x)$	$f'(x)$
$(\sin_p x \cos_p x)^{p-1}$	$(p-1)(\cos_p^p x - \sin_p^p x)\sin_p^{p-2} x$
$\cos_p^p x - \sin_p^p x$	$-2p \sin_p^{p-1} x \cos_p x$
$\cot_p x$	$-(\cos_p^{2-p} x)/\sin_p^2 x$
$\sec_p x$	$\tan_p^{p-1} x \sec_p x$
$\mathrm{cosec}_p x$	$-\mathrm{cosec}_p x \cot_p x$

Note also that

$$-\Delta_p(\sin_p x) = (p-1)\,|\sin_p x|^{p-2} \sin_p x \quad (x \in \mathbf{R}),$$

where Δ_p is the one-dimensional p-Laplacian (see (1.2)).

5.2. Trigonometric identities. Here we list further identities of the same type as those given in Proposition 2.2, and valid for all $x \in [0,1]$.

(i) $(2/\pi_p)\cos_p^{-1}(1-x^{p'})^{1/p} + (2/\pi_{p'})\cos_{p'}^{-1} x = 1.$

(ii) $(2/\pi_p)\sin_p^{-1} x + (2/\pi_{p'})\cos_{p'}^{-1} x^{p-1} = 1.$

(iii) $\sin_p^p(\pi_p x/2) = \cos_{p'}^{p'}(\pi_{p'}(1-x)/2).$

(iv) $\cos_p^{p-2}(\pi_p x/2) = \sin_{p'}^{2-p'}(\pi_{p'}(1-x)/2).$

(v) $\sin_p^{p-2}(\pi_p x/2) = \cos_{p'}^{2-p'}(\pi_{p'}(1-x)/2).$

5.3. Integrals. The following table gives examples of indefinite integrals additional to those provided in Proposition 2.6. These can be verified by differentiation, for example.

$f(x)$	$\int f(x)\,dx$
$p \sin_p^p x$	$x - \sin_p x \cos_p^{p-1} x$
$\tan_p^{p-1} x$	$-\log \cos_p x$
$\cos_p^{p+1} x$	$\sin_p x - \frac{1}{p+1} \sin_p^{p+1} x$
$\sin_p^{p-1} x \cos_p^k x$	$-\frac{1}{p+k-1} \cos_p^{p+k-1} x$
$1/(\sin_p x \cos_p^{p-1} x)$	$\log \tan_p x$
$\sin_p^{p-1} x \log \cos_p x$	$\frac{1}{(p-1)^2} \cos_p^{p-1} x\{1 - (p-1) \log \cos_p x\}$

5.4. The p-calculus. Some care is needed when working with $p-$functions outside the interval $[0, \pi_p/2]$. For example, the extension of $\sin_p$ to $[-\pi_p/2, 0]$ that would be obtained by use of the power series for the inverse of F_p is not the periodic extension of $\sin_p$: indeed, this extension would be an odd function only if p is an even integer.

We have not been able to find a p-analogue of the exponential function or the gamma function, and the existence of addition formulae or a p-version of the Euler reflection formula seems unlikely.

An integral involving only p-functions, with the same p occurring in all functions, can be transformed into an integral of standard functions in the way described in Section 3. The evaluation in finite form of other types of integrals presents some difficulties. In connection with the basis problem, it would be extremely useful to find simple expressions for integrals such as

$$\int \sin_p \alpha x \sin_q \beta x\, dx \quad (p \neq q).$$

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