ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

ON GENERALIZED COMMUTATIVITY DEGREE OF A FINITE GROUP

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ABSTRACT. Commutativity degree of a finite group is the probability that the commutator of two arbitrarily chosen group elements equals the identity element of the group. The object of this paper is to study the probability that the generalized commutator of an arbitrarily chosen n-tuple of group elements equals a given group element.

1. Introduction. Throughout this paper G denotes a finite group, g an element of the commutator subgroup G', and $n \geq 2$ a positive integer. Recall that the generalized commutator of an n-tuple $(x_1, x_2, \dots, x_n) \in G^n$, the n-fold product of G with itself, is given by

$$[x_1, x_2, \dots, x_n] = x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \in G'.$$

In [**12**], Pournaki et al. have considered the probability $\mathrm{Pr}_g(G)$ that the commutator of an arbitrarily chosen pair of group elements equals g and extended the work of Rusin [**14**]. The main objective of this paper is to study the ratio

$$\mathrm{Pr}_g^n(G) = \frac{|\{(x_1, x_2, \dots, x_n) \in G^n : [x_1, x_2, \dots, x_n] = g\}|}{|G|^n}$$

and further extend the results obtained by Pournaki et al. At the same time, we obtain some new results as well. It may be mentioned here that some of the works that have been carried out in [**3, 5, 11**] are related to the present problem.

Note that $\mathrm{Pr}_g^2(G) = \mathrm{Pr}_g(G)$, which coincides with the usual commutativity degree $\mathrm{Pr}(G)$ of G if we take $g = 1$, the identity element of G. It may be recalled (see, for example, [**6**]) that $\mathrm{Pr}(G) = \frac{k(G)}{|G|}$ where $k(G)$ denotes the number of conjugacy classes of G.

2010 AMS *Mathematics subject classification.* Primary 20C15, 20D60, 20P05.
Keywords and phrases. Finite groups, group characters, commutativity degree.
The second author is the corresponding author.
Received by the editors on November 25, 2008, and in revised form on February 11, 2009.
DOI:10.1216/RMJ-2011-41-6-1987

In [**15**, Theorem 1], Tambour has generalized a classical result of Frobenius [**4**] to prove that the number of n-tuples $(x_1, x_2, \dots, x_n) \in G^n$ satisfying $[x_1, x_2, \dots, x_n] = g$ is

$$\sum_{\chi \in \mathrm{Irr}\,(G)} \frac{|G|^{n-1}}{\chi(1)^{n-\varepsilon_n}} \chi(g)$$

where ε_n is 1 or 2 according as n is even or odd; $\mathrm{Irr}\,(G)$ denotes the set of all irreducible complex characters of G. As such, we have

$$\mathrm{Pr}_g^n(G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}\,(G)} \frac{\chi(g)}{\chi(1)^{n-\varepsilon_n}}. \tag{1}$$

It is clear, from the definition of $\mathrm{Pr}_g^n(G)$, that $0 \leq \mathrm{Pr}_g^n(G) \leq 1$, and $\mathrm{Pr}_1^n(G) = 1$ if and only if G is abelian. Also, from (1), we have $\mathrm{Pr}_g^n(G) = \mathrm{Pr}_g^{n+1}(G)$ if n is even. In view of these facts, we always assume that G is nonabelian and n is even (whence $\varepsilon_n = 1$).

2. Some preliminaries. In this section we derive some results which also serve as prerequisites for the forthcoming sections. We begin by observing that $\mathrm{Pr}_g^n(G)$ can be thought of as a completely multiplicative arithmetic function (see [**1**]) of finite groups in the following sense.

Proposition 2.1. *For any two finite groups H and K with $(h, k) \in H' \times K'$,*

$$\mathrm{Pr}_{(h,k)}^n(H \times K) = \mathrm{Pr}_h^n(H)\mathrm{Pr}_k^n(K).$$

Proof. It is enough to note that the n-tuples of pairs $((x_1, y_1), \dots, (x_n, y_n)) \in (H \times K)^n$ satisfying $[((x_1, y_1), \dots, (x_n, y_n))] = (h, k)$ are in a one-to-one correspondence with the pairs of n-tuples $((x_1, \dots, x_n), (y_1, \dots, y_n)) \in H^n \times K^n$ satisfying $[x_1, \dots, x_n] = h$ and $[y_1, \dots, y_n] = k$. $\square$

As in [**7**, page 133], G is said to be *isoclinic* to a group H if there are isomorphisms $\phi : G/Z(G) \longrightarrow H/Z(H)$ and $\psi : G' \longrightarrow H'$ such that

$a_H \circ (\phi \times \phi) = \psi \circ a_G$ where the maps $a_G : G/Z(G) \times G/Z(G) \longrightarrow G'$ and $a_H : H/Z(H) \times H/Z(H) \longrightarrow H'$ are induced by the commutator maps of G and H respectively. The following proposition shows that $\mathrm{Pr}_g^n(G)$ is an invariant under isoclinism of finite groups.

Proposition 2.2. *Let G be isoclinic to a finite group H. Then, with notations as above,*

$$\mathrm{Pr}_g^n(G) = \mathrm{Pr}_{\psi(g)}^n(H).$$

Proof. Note that the generalized commutator map from G^n to G', given by $(x_1, x_2, \dots, x_n) \longmapsto [x_1, x_2, \dots, x_n]$, factors through the quotient group $G^n/Z(G^n) = (G/Z(G))^n$. Therefore, it follows that

$$\begin{aligned}(2)\quad &|\{(x_1, x_2, \dots, x_n) \in G^n : [x_1, x_2, \dots, x_n] = g\}| \\ &\quad = |\{(x_1Z(G), x_2Z(G), \dots, x_nZ(G)) \in (G/Z(G))^n : \\ &\qquad\qquad [x_1, x_2, \dots, x_n] = g\}| \times |Z(G)|^n.\end{aligned}$$

Consider now an n-tuple $(x_1Z(G), x_2Z(G), \dots, x_nZ(G)) \in (G/Z(G))^n$ such that $[x_1, x_2, \dots, x_n] = g$. It is a routine matter to see that

$$(3)\qquad [x_1, x_2, \dots, x_n] = [x_1,\ x_2 \cdots x_n][x_2,\ x_3 \cdots x_n] \cdots [x_{n-1},\ x_n]$$

where the right hand side is a product of commutators. Therefore, from the definition of isoclinism, we have

$$\begin{aligned}\psi(g) &= \psi([x_1,\ x_2 \cdots x_n])\psi([x_2,\ x_3 \cdots x_n]) \cdots \psi([x_{n-1},\ x_n]) \\ &= [y_1,\ y_2 \cdots y_n][y_2,\ y_3 \cdots y_n] \cdots [y_{n-1},\ y_n] \\ &= [y_1,\ y_2, \dots, y_n], \quad \text{(as in (3))},\end{aligned}$$

where $y_1, y_2, \dots, y_n \in H$ are given by $\phi(x_iZ(G)) = y_iZ(H)$, $1 \le i \le n$. Since $\phi : G/Z(G) \longrightarrow H/Z(H)$ and $\psi : G' \longrightarrow H'$ are isomorphisms, it follows that there is a one-to-one correspondence between the n-tuples $(x_1Z(G), x_2Z(G), \dots, x_nZ(G)) \in (G/Z(G))^n$ satisfying $[x_1, x_2, \dots, x_n] = g$ and the n-tuples $(y_1Z(H), y_2Z(H), \dots, y_nZ(H)) \in (H/Z(H))^n$ satisfying $[y_1, y_2, \dots, y_n] = \psi(g)$. Hence, by (2) and the corresponding equation for H, we have

$$\mathrm{Pr}_g^n(G) = \mathrm{Pr}_{\psi(g)}^n(H),$$

noting that $|G/Z(G)| = |H/Z(H)|$. □

It has also been observed in [**7**, page 136] that up to isoclinism, there exists precisely one nonabelian group of order p^3, and the groups which are isoclinic to a group of order p^3 are characterized by the property that their center has index p^2. As such, we have

Lemma 2.3. *Given a prime p, $|G : Z(G)| = p^2$ if and only if G is isoclinic to the group presented as*

$$\langle x, y \mid x^{p^2} = 1 = y^p,\ y^{-1}xy = x^{p+1}\rangle.$$

We know that finite nilpotent groups are the direct product of their Sylow subgroups. Therefore, it follows that if $G' = Z(G)$ and $|G'| = p$, a prime, then G is an extra-special p-group, and so, by [**13**, page 146], $|G| = p^{2k+1}$ for some positive integer k. In [**7**, page 135], it has been proved that every group is isoclinic to a group whose center is contained in the commutator subgroup. Hence, we have

Lemma 2.4. *Let $G' \subseteq Z(G)$ and $|G'| = p$, a prime. Then G is isoclinic to an extra-special p-group of order p^{2k+1} with $k = \frac{1}{2}\log_p |G : Z(G)|$.*

Finally, in this section, we have the following lemma which is used extensively in this paper.

Lemma 2.5. *Let $\sum_{i=1}^{n} r_i(a_i - 1) = 0$ where r_i's are positive rational numbers, and $a_i \in \mathbf{C}$, $|a_i| \le 1$ for all $i = 1, 2, \ldots, n$. Then $a_i = 1$ for all $i = 1, 2, \ldots, n$.*

Proof. Note that

$$0 = \sum_{i=1}^{n} \mathrm{Re}\,(r_i(a_i - 1)) = \sum_{i=1}^{n} r_i(\mathrm{Re}\,(a_i) - 1) \le \sum_{i=1}^{n} r_i(|a_i| - 1) \le 0.$$

Hence, for all $i = 1, 2, \ldots, n$, we have $\mathrm{Re}\,(a_i) = |a_i| = 1$ which means that $a_i = 1$. $\square$

3. Some nontrivial bounds for $\Pr_g^n(G)$. Since $G' \subseteq \ker(\chi)$ for all $\chi \in \mathrm{Irr}\,(G)$ with $\chi(1) = 1$, and also since $|G : G'|$ equals the number of linear characters of G, (1) can be rewritten as

$$\Pr_g^n(G) = \frac{1}{|G'|} + \frac{1}{|G|} \sum_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi(1) \neq 1}} \frac{\chi(g)}{\chi(1)^{n-1}}. \tag{4}$$

In particular, putting $g = 1$, we have

$$\frac{1}{|G'|} < \Pr_1^n(G) \leq \Pr_1^2(G) = \Pr(G), \tag{5}$$

noting that G has at least one nonlinear irreducible character. From (4), since $|\chi(g)| \leq \chi(1)$, we also have

$$\Pr_g^n(G) \leq \Pr_1^n(G), \tag{6}$$

and

$$\left| \Pr_g^n(G) - \frac{1}{|G'|} \right| \leq \frac{1}{|G|} \sum_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi(1) \neq 1}} \frac{1}{\chi(1)^{n-2}}. \tag{7}$$

Let $m = \min\{\chi(1) : \chi \in \mathrm{Irr}\,(G), \chi(1) \neq 1\}$. Let p be the smallest prime divisor of $|G|$. Clearly,

$$\chi(1) \geq m \geq p \geq 2 \quad \text{for all} \quad \chi \in \mathrm{Irr}\,(G) \text{ with } \chi(1) \neq 1.$$

Throughout this paper we write d to denote any one of 2, m and p. Then, from (7), we have

$$\left| \Pr_g^n(G) - \frac{1}{|G'|} \right| \leq \frac{1}{d^{n-2}} \left[\Pr(G) - \frac{1}{|G'|} \right], \tag{8}$$

noting that the number of nonlinear irreducible characters of G is given by $k(G) - |G : G'|$, and $\Pr(G) = \frac{k(G)}{|G|}$.

Since

$$|G| = \sum_{\chi \in \mathrm{Irr}\,(G)} \chi(1)^2 \geq |G : G'| + d^2[k(G) - |G : G'|], \tag{9}$$

we have

$$\text{Pr}(G) \leq \frac{1}{d^2}\left[1 + \frac{d^2 - 1}{|G'|}\right]. \tag{10}$$

Therefore, from (8), it follows that

$$\left|\text{Pr}_g^n(G) - \frac{1}{|G'|}\right| \leq \frac{1}{d^n}\left[1 - \frac{1}{|G'|}\right]. \tag{11}$$

In particular, putting $d = 2$ and noting that $|G'| \geq 2$, we have

$$\text{Pr}_g^n(G) \leq \frac{2^n + 1}{2^{n+1}}. \tag{12}$$

As an immediate consequence of (11), we also have

$$\lim_{n \longrightarrow \infty} \text{Pr}_g^n(G) = \frac{1}{|G'|}. \tag{13}$$

If G is a simple group then $G' = G$, $\text{Pr}(G) \leq \frac{1}{12}$ (see [**2**]), and also $m \geq 3$ (see [**8**, Proposition 6.8, page 72]). Hence, in this case, (8) gives

$$\left|\text{Pr}_g^n(G) - \frac{1}{|G|}\right| \leq \frac{1}{3^{n-2}}\left[\frac{1}{12} - \frac{1}{|G|}\right]. \tag{14}$$

Note here that A_5 is the smallest nonabelian simple group and its order is 60. So, we have

$$\text{Pr}_g^n(G) \leq \frac{3^{n-2} + 4}{3^{n-1} \times 20}.$$

4. Conditions for attaining upper bounds. In this section we derive some necessary and sufficient conditions so that $\text{Pr}_g^n(G)$ attains the upper bounds mentioned in Section 3. We begin with (6) to have

Proposition 4.1. $\text{Pr}_g^n(G) = \text{Pr}_1^n(G)$ *if and only if* $g = 1$.

Proof. By (1), we have

$$
\begin{aligned}
\Pr_g^n(G) &= \Pr_1^n(G) \\
&\iff \sum_{\chi \in \mathrm{Irr}\,(G)} \frac{1}{\chi(1)^{n-2}} \left(\frac{\chi(g)}{\chi(1)} - 1 \right) = 0 \\
&\iff \chi(g) = \chi(1)\ \forall\ \chi \in \mathrm{Irr}\,(G), \quad \text{(by Lemma 2.5)},
\end{aligned}
$$

which holds if and only if $g = 1$. □

Let $\mathrm{cd}\,(G) = \{\chi(1) : \chi \in \mathrm{Irr}\,(G)\}$. Then, corresponding to (8), we have

Proposition 4.2. *$\Pr_g^n(G) = \frac{1}{d^{n-2}}[\Pr(G) + \frac{d^{n-2}-1}{|G'|}]$ if and only if $g = 1$ and $\mathrm{cd}\,(G) = \{1, d\}$.*

Proof. Note that

$$
\begin{aligned}
&\Pr_1^n(G) - \frac{1}{|G'|} = \frac{1}{d^{n-2}} \left[\Pr(G) - \frac{1}{|G'|} \right] \\
&\iff \sum_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi(1) \neq 1}} \frac{1}{d^{n-2}} \left(\frac{d^{n-2}}{\chi(1)^{n-2}} - 1 \right) = 0, \quad \text{(using (4))} \\
&\iff d = \chi(1)\ \forall\ \chi \in \mathrm{Irr}\,(G) \text{ with } \chi(1) \neq 1, \ \text{(by Lemma 2.5)}, \\
&\iff \mathrm{cd}\,(G) = \{1, d\}.
\end{aligned}
$$

Hence, in view of (6), (8) and Proposition 4.1, the result follows. □

Note that the equality holds in (10) if and only if it holds in (9). As such, by Lemma 2.5, we have

Lemma 4.3. *$\Pr(G) = \frac{1}{d^2}[1 + \frac{d^2-1}{|G'|}]$ if and only if $\mathrm{cd}\,(G) = \{1, d\}$.*

More generally, corresponding to (11), we have

Proposition 4.4. *$\Pr_g^n(G) = \frac{1}{d^n}[1 + \frac{d^n-1}{|G'|}]$ if and only if $g = 1$ and $\mathrm{cd}\,(G) = \{1, d\}$.*

Proof. Using (8) and (10), we have

$$\Pr_g^n(G) - \frac{1}{|G'|} \leq \frac{1}{d^{n-2}}\left[\Pr(G) - \frac{1}{|G'|}\right] \leq \frac{1}{d^n}\left[1 - \frac{1}{|G'|}\right].$$

Hence, by Proposition 4.2 and Lemma 4.3, the result follows. □

Lemma 4.3 also gives, in particular, the following result.

Proposition 4.5. *With p denoting the smallest prime divisor of $|G|$,*

$$|G : Z(G)| = p^2 \iff \mathrm{cd}\,(G) = \{1, p\} \text{ and } |G'| = p.$$

Proof. By (10),

$$\Pr(G) \leq \frac{1}{p^2}\left[1 + \frac{p^2 - 1}{|G'|}\right] \leq \frac{1}{p^2}\left[1 + \frac{p^2 - 1}{p}\right].$$

So, by Lemma 4.3, we have

$$\Pr(G) = \frac{1}{p^2}\left[1 + \frac{p^2 - 1}{p}\right] \iff \mathrm{cd}\,(G) = \{1, p\} \text{ and } |G'| = p.$$

On the other hand, from [**10**, Theorem 3], we have

$$\Pr(G) = \frac{1}{p^2}\left[1 + \frac{p^2 - 1}{p}\right] \iff |G : Z(G)| = p^2.$$

Hence, the result follows. □

Corresponding to (11) and (12), we also have

Proposition 4.6. *Let p be the smallest prime divisor of $|G|$, or $p = 2$. Then,*

$$\Pr_g^n(G) = \frac{p^n + p - 1}{p^{n+1}}$$

if and only if $g = 1$, *and* G *is isoclinic to*

$$\langle x, y \mid x^{p^2} = 1 = y^p,\ y^{-1}xy = x^{p+1}\rangle.$$

In particular, $\mathrm{Pr}_g^n(G) = \frac{2^n+1}{2^{n+1}}$ *if and only if* $g = 1$, *and* G *is isoclinic to* D_8, *the dihedral group (i.e. to* Q_8, *the group of quaternions).*

Proof. By (11), We have

$$\mathrm{Pr}_g^n(G) \leq \frac{1}{p^n}\left[1 + \frac{p^n - 1}{|G'|}\right] \leq \frac{1}{p^n}\left[1 + \frac{p^n - 1}{p}\right].$$

So, by Proposition 4.4, we have

$$\mathrm{Pr}_g^n(G) = \frac{1}{p^n}\left[1 + \frac{p^n - 1}{p}\right] \Longleftrightarrow g = 1, \mathrm{cd}\,(G) = \{1, p\} \quad \text{and } |G'| = p.$$

Hence, the result follows from Proposition 4.5 and Lemma 2.3. □

5. Conditions for attaining lower bounds. In this section we discuss the conditions under which $\mathrm{Pr}_g^n(G)$ attains the lower bounds given in Section 3. It may be noted here that the lower bound occurs in equation (7) with

$$-\frac{1}{|G|}\sum_{\substack{\chi\in\mathrm{Irr}\,(G)\\ \chi(1)\neq 1}} \frac{1}{\chi(1)^{n-2}} \leq \mathrm{Pr}_g^n(G) - \frac{1}{|G'|}.$$

We begin with the following observation.

Proposition 5.1. *If* $\mathrm{cd}\,(G) = \{1, d\}$ *and* $g \neq 1$, *then*

$$\mathrm{Pr}_g^n(G) = \frac{1}{|G'|}\left[1 - \frac{1}{d^n}\right].$$

Proof. The proof is parallel to that of Theorem 2.2 of [**12**]. Using the second orthogonality relation (see [**9**, Theorem 2.18]), we have

$$\sum_{\substack{\chi\in\mathrm{Irr}\,(G)\\ \chi(1)\neq 1}} \chi(g) = -\frac{|G : G'|}{d}.$$

The result now follows from (4). □

Corresponding to (8), we have

Proposition 5.2. $\Pr_g^n(G) = \frac{1}{d^{n-2}}[-\Pr(G) + \frac{d^{n-2}+1}{|G'|}]$ *if and only if* $g \neq 1$, $\mathrm{cd}\,(G) = \{1, d\}$, $|G'| = 2$.

Proof. Suppose that the given formula for $\Pr_g^n(G)$ holds. Then,

$$\begin{aligned}\Pr_g^n(G) - \frac{1}{|G'|} &= -\frac{1}{d^{n-2}}\left[\Pr(G) - \frac{1}{|G'|}\right] \\ &\implies \sum_{\substack{\chi \in \mathrm{Irr}\,(G) \\ \chi(1) \neq 1}} \frac{1}{d^{n-2}}\left(\frac{-\chi(g)d^{n-2}}{\chi(1)^{n-1}} - 1\right) = 0, \quad \text{(using (4))} \\ &\implies -\chi(g)d^{n-2} = \chi(1)^{n-1} \\ &\qquad \forall\ \chi \in \mathrm{Irr}\,(G) \text{ with } \chi(1) \neq 1,\end{aligned}$$

by Lemma 2.5, noting that $|\chi(g)| \leq \chi(1)$ and $d \leq \chi(1)$ for all $\chi \in \mathrm{Irr}\,(G)$ with $\chi(1) \neq 1$. Thus, we have $g \neq 1$, $\chi(g)$ is real, and $-\chi(g) = \chi(1) = d$ for all $\chi \in \mathrm{Irr}\,(G)$ with $\chi(1) \neq 1$. In particular, we have $\mathrm{cd}\,(G) = \{1, d\}$. Also, by the second orthogonality relation, we have

$$\begin{aligned}\sum_{\chi \in \mathrm{Irr}\,(G)} \chi(g)\chi(1) = 0& \\ &\implies |G : G'| - d^2\,(k(G) - |G : G'|) = 0 \\ &\implies \Pr(G) = \frac{d^2+1}{d^2|G'|} \\ &\implies |G'| = 2, \quad \text{(by Lemma 4.3)}.\end{aligned}$$

Conversely, if $g \neq 1$, $\mathrm{cd}\,(G) = \{1, d\}$, $|G'| = 2$ then, by Lemma 4.3 and Proposition 5.1, we have

$$\frac{1}{d^{n-2}}\left[-\Pr(G) + \frac{d^{n-2}+1}{|G'|}\right] = \frac{1}{2}\left[1 - \frac{1}{d^n}\right] = \Pr_g^n(G).$$

This completes the proof. □

As an immediate consequence, we have, corresponding to (11),

Corollary 5.3. $\Pr_g^n(G) = \frac{1}{d^n}[-1 + \frac{d^n+1}{|G'|}]$ *if and only if* $g \neq 1$, $\mathrm{cd}\,(G) = \{1, d\}$, $|G'| = 2$.

Proof. Using (8) and (10), we have

$$\Pr_g^n(G) - \frac{1}{|G'|} \geq \frac{-1}{d^{n-2}}\left[\Pr(G) - \frac{1}{|G'|}\right] \geq \frac{-1}{d^n}\left[1 - \frac{1}{|G'|}\right].$$

Hence, using Lemma 4.3, the result follows. □

Remark 5.4. If G is a simple group, then the equality in (14) can never hold. Because the equality in (14) implies the equality in (8). This, in view of Propositions 4.2 and 5.2, implies that $|\mathrm{cd}\,(G)| = 2$. But, by [**9**, Corollary 12.6], this is impossible.

6. Some more bounds and equalities. It is well known (see [**9**, page 28]) that $|G : Z(G)|^{1/2}$ is an upper bound for $\mathrm{cd}\,(G)$. If this upper bound is attained then G is said to be of *central type*. Therefore, as an immediate consequence of Propositions 4.4 and 5.1, we have

Proposition 6.1. *Let* $|\mathrm{cd}\,(G)| = 2$. *Then,*

$$\Pr_1^n(G) \geq \frac{1}{|G'|}\left[1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}}\right], \quad \textit{and}$$

$$\Pr_g^n(G) \leq \frac{1}{|G'|}\left[1 - \frac{1}{|G : Z(G)|^{n/2}}\right] \quad \textit{if } g \neq 1.$$

Moreover, in both cases, equality holds if and only if G *is of central type.*

Since, G is nonabelian, $|G/Z(G)| \geq 4$. As such, we have

Corollary 6.2. *If* G *is of central type with* $|\mathrm{cd}\,(G)| = 2$ *then*

$$\Pr_1^n(G) \leq \frac{1}{|G'|}\left[1 + \frac{|G'| - 1}{2^n}\right], \quad \textit{and}$$

$$\Pr_g^n(G) \geq \frac{1}{|G'|}\left[1 - \frac{1}{2^n}\right] \quad \textit{if } g \neq 1.$$

From [**9**, page 31], we know that if $G' \subseteq Z(G)$ and G' is of prime order (for example, if G is an extra-special p-group) then G is of central type with $|\mathrm{cd}\,(G)| = 2$, i.e., $\mathrm{cd}\,(G) = \{1, |G : Z(G)|^{1/2}\}$. This fact is also observed in the proof of Proposition 3.1 of [**12**]. Therefore, in view of Proposition 2.2 and Lemma 2.4, it follows, from Proposition 6.1, that

Proposition 6.3. *Let $G' \subseteq Z(G)$ and $|G'| = p$ be a prime. Then*

$$\mathrm{Pr}_g^n(G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{p^{nk}}\right) & \text{if } g = 1 \\ \frac{1}{p}\left(1 - \frac{1}{p^{nk}}\right) & \text{if } g \neq 1 \end{cases}$$

where $k = \frac{1}{2}\log_p |G : Z(G)|$.

More generally, we have

Corollary 6.4. *Let $|G'|$ be square free and $G' \subseteq Z(G)$. Then*

$$\mathrm{Pr}_g^n(G) = \prod_{p||G'|} \left(\frac{1}{p}\left(1 - \frac{1}{p^{nk_p}}\right)\right)^{\delta_p} \left(\frac{1}{p}\left(1 + \frac{p-1}{p^{nk_p}}\right)\right)^{1-\delta_p}$$

where $k_p = \frac{1}{2}\log_p |G_p : Z(G_p)|$, G_p is the Sylow p-subgroup of G, and $\delta_p = 1$ or 0 according to whether p divides or does not divide the order of g.

Proof. Note that G, being nilpotent, is the direct product of its Sylow subgroups. Therefore, since $G' \subseteq Z(G)$ and $|G'|$ is squarefree, it follows that, for each prime divisor p of $|G'|$, the Sylow p-subgroup G_p of G satisfies the conditions ${G_p}' \subseteq Z(G_p)$ and $|{G_p}'| = p$. It also follows that, all other Sylow subgroups (if any) of G are abelian. Hence, G is isoclinic to the product $\prod G_p$ where p runs through the prime divisors of $|G'|$. Thus, using Propositions 2.1 and 2.2, the corollary follows. □

If $|G'|$ is a prime with $G' \cap Z(G) = \{1\}$ then $\mathrm{Pr}_g^n(G)$ can be computed in exactly the same manner as in [**12**, Section 4] in terms of the invariant number $\mathrm{i}(G)$ of G. More precisely, we have

Proposition 6.5. *If* $|G'| = p$, $G' \cap Z(G) = \{1\}$ *and* $\mathrm{i}(G) = r$, *then*

$$\Pr_g^n(G) = \begin{cases} \frac{r^n+p-1}{pr^n} & \text{if } g = 1 \\ \frac{r^n-1}{pr^n} & \text{if } g \neq 1. \end{cases}$$

The following two results generalize Propositions 5.2 and 5.3 of [**12**].

Proposition 6.6. *If* $g \neq 1$, *then* $\Pr_g^n(G) < \frac{1}{p}$ *where* p *is the smallest prime divisor of* $|G|$; *in particular, we have* $\Pr_g^n(G) < \frac{1}{2}$.

Proof. If $\Pr_g^n(G) \geq \frac{1}{p}$ then, from (5), (6) and Proposition 4.1, it follows that $\Pr(G) > \frac{1}{p}$. Therefore, from (10) with $d = p$, we have $|G'| < p + 1$, and so, $|G'| = p$. Hence, we have either $G' \subseteq Z(G)$ or $G' \cap Z(G) = \{1\}$. In both situations, we have, from Propositions 6.3 and 6.5, $\Pr_g^n(G) < \frac{1}{p}$. This contradiction proves the result. □

Proposition 6.7. *For each* $\varepsilon > 0$ *and for each prime* p, *there exists a* G *such that*

$$\left|\Pr_g^n(G) - \frac{1}{p}\right| < \varepsilon.$$

Proof. In view of Proposition 6.3, it is enough to choose a positive integer k such that $k > -\frac{1}{n}\log_p \varepsilon$, and consider G to be an extra-special p-group of order p^{2k+1}. □

We conclude this paper with the following observation.

Proposition 6.8. $\Pr_1^{n+2}(G) < \Pr_1^n(G)$. *On the other hand, for* $g \neq 1$ *and* $|\mathrm{cd}\,(G)| = 2$, $\Pr_g^{n+2}(G) > \Pr_g^n(G)$.

Proof. Follows immediately from (4) and Proposition 5.1. □

Acknowledgments. The authors are grateful for the helpful suggestions of the referee.

REFERENCES

1. A.K. Das, *On arithmetic functions of finite groups*, Bull. Austral. Math. Soc. **75** (2007), 45–58.

2. J.D. Dixon, *Problem* 176, Canad. Math. Bull. **16** (1973), 302.

3. A. Erfanian, R. Rezaei and P. Lescot, *On the relative commutativity degree of a subgroup of a finite group*, Comm. Algebra **35** (2007), 4183–4197.

4. F.G. Frobenius, *Über Gruppencharaktere*, Gesammelte Abhandlungen Band III, J.P. Serre, ed., Springer-Verlag, Berlin, 1968, 1–37.

5. R.M. Guralnick and G.R. Robinson, *On the commuting probability in finite groups*, J. Algebra **300** (2006), 509–528. See also *Addendum to paper*, "*On the commuting probability in finite groups*," by R.M. Guralnick and G.R. Robinson, J. Algebra **319** (2008), 1822.

6. W.H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly **80** (1973), 1031–1034.

7. P. Hall, *The classification of prime power groups*, J. reine Angew. Math. **182** (1940), 130–141.

8. B. Huppert, *Character theory of finite groups*, Walter de Gruyter, Berlin, 1998.

9. I.M. Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994.

10. D. MacHale, *How commutative can a non-commutative group be?*, Math. Gaz. **58** (1974), 199–202.

11. M.R.R. Moghaddam, A.R. Salemkar and K. Chiti, *n-isoclinism classes and n-nilpotency degree of finite groups*, Algebra Colloq. **12** (2005), 225–261.

12. M.R. Pournaki and R. Sobhani, *Probability that the commutator of two group elements is equal to a given element*, J. Pure Appl. Algebra **212** (2008), 727–734.

13. D.J.S. Robinson, *A course in the theory of groups*, second edition, Grad. Text Math. **80**, Springer, New York, 1996.

14. D.J. Rusin, *What is the probability that two elements of a finite group commute?*, Pacific J. Math. **82** (1979), 237–247.

15. T. Tambour, *The number of solutions of some equations in finite groups and a new proof of Itô's theorem*, Comm. Algebra **28** (2000), 5353–5361.

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