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ON REPRESENTATIONS AND DIFFERENCES OF STIELTJES COEFFICIENTS, AND OTHER RELATIONS

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ABSTRACT. The Stieltjes coefficients $\gamma_k(a)$ arise in the expansion of the Hurwitz zeta function $\zeta(s,a)$ about its single simple pole at $s=1$ and are of fundamental and long-standing importance in analytic number theory and other disciplines. We present an array of exact results for the Stieltjes coefficients, including series representations and summatory relations. Other integral representations provide the difference of Stieltjes coefficients at rational arguments. The presentation serves to link a variety of topics in analysis and special function and special number theory, including logarithmic series, integrals, and the derivatives of the Hurwitz zeta and Dirichlet L-functions at special points. The results have a wide range of application, both theoretical and computational.

1. Introduction and statement of results. The Stieltjes (or generalized Euler) constants $\gamma_k(a)$ appear as expansion coefficients in the Laurent series about $s=1$ for the Hurwitz zeta function $\zeta(s,a)$, one of the generalizations of the Riemann zeta function $\zeta(s)$. Elsewhere [**6**], we developed new summatory relations amongst the values $\gamma_k(a)$ as well as demonstrated one of the very recent conjectures put forward by Kreminski [**19**] on the relationship between $\gamma_k(a)$ and $-\gamma_k(a+1/2)$ as $k\to\infty$ [**5**]. New series representations of the Riemann and Hurwitz zeta functions, as well as series representations of $\gamma_1(a)$ and $\gamma_2(a)$ are given very recently in [**9**].

In this paper, we present an array of exact results for the Stieltjes constants. These include individual and summatory relations for paired differences of these coefficients for rational arguments. Our work provides a unification of several important topics of analysis and analytic number theory. These include certain logarithmic sums, integrals of

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analytic number theory, special functions, the derivatives of the Hurwitz zeta and Dirichlet L-functions at special points, and differences of the Stieltjes constants. As the corresponding logarithmic sums are slowly converging, our results provide useful complements and alternatives to numerical computation. In addition, our analytic results and accompanying representations of $\gamma_k(a)$ provide a basis for inequalities and monotonicity results for the Stieltjes constants.

We may stress the fundamental and long-standing nature of the Stieltjes coefficients. They arise in the expansion of the Hurwitz zeta function about its *unique* polar singularity. They can be used to write other important constants of analytic number theory, and they appear often in describing error terms and as a result of applying asymptotic analyses. As well, they can be expected to play a role in investigations of the nonvanishing of L-functions and their derivatives along the line $s = 1$ and elsewhere in the critical strip.

An expository paper of Vardi [**27**] discusses the evaluation of certain logarithmic integrals (or their equivalents through change of variable) and describes the underlying connection with Dirichlet L-series. However, the presentation is illustrative and no connection with logarithmic series or the Stieltjes coefficients is mentioned. Much more recently, Medina and Moll [**20**] have followed Vardi's approach and given a number of examples for integrands containing a rational function. Additionally, Adamchik [**2**] considered differences of the first derivative of the Hurwitz zeta function at rational arguments and related them to logarithmic integrals.

The Hurwitz zeta function, initially defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \operatorname{Re} s > 1, \tag{1.1}$$

can be analytically continued to the whole complex plane $\mathbf{C} - \{1\}$. The defining relation for the Stieltjes constants in terms of a Laurent expansion is

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(a)}{k!}(s-1)^k, \quad s \neq 1. \tag{1.2}$$

This equation reflects that $\zeta(s,a)$ has a simple pole at $s = 1$ with residue 1. By convention, γ_k represents $\gamma_k(1)$ and thus explicitly we have [**4, 15, 18, 21, 25, 29**]

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!}(s-1)^k, \quad s \neq 1. \tag{1.3}$$

As noted in [**29**] and again in [**30**], we have $\gamma_0(a) = -\psi(a)$, where $\psi = \Gamma'/\Gamma$ is the digamma function, so that $\gamma_0 = \gamma$, the Euler constant.

Our main results are presented in the following set of propositions.

Proposition 1. *We have the summatory relation*

$$\sum_{n=0}^{\infty} \frac{1}{n!}[\gamma_{n+1}(a) - \gamma_{n+1}(b)] = \ln\left[\frac{\Gamma(b)}{\Gamma(a)}\right], \quad \operatorname{Re} a > 0,\ \operatorname{Re} b > 0. \tag{1.4}$$

Let $L_{\pm k}$ be a Dirichlet series corresponding to a real character χ_k of modulus k with $\chi_k(k-1) = \pm 1$. Then we have

Proposition 2. *For χ_k a nonprincipal Dirichlet character we have*

(a)

$$\left.\frac{\partial}{\partial s} L_{\pm k}(s)\right|_{s=1} = -\sum_{n=2}^{\infty} \frac{\chi_k(n)\ln n}{n} \tag{1.5}$$

$$= k^{-1}\sum_{m=1}^{k} \chi_k(m)\left[\zeta'\left(1, \frac{m}{k}\right) - \ln k\ \zeta\left(1, \frac{m}{k}\right)\right] \tag{1.6}$$

$$= k^{-1}\sum_{m=1}^{k} \chi_k(m)\left[\gamma_1\left(\frac{m}{k}\right) + \ln k\ \psi\left(\frac{m}{k}\right)\right] \tag{1.7}$$

$$= \int_0^{\infty} \frac{(\ln u + \gamma)}{1 - e^{-ku}}\left(\sum_{m=1}^{k} \chi_k(m) e^{-mu}\right) du, \tag{1.8}$$

and

(b)

$$\left.\frac{\partial^2}{\partial s^2} L_{\pm k}(s)\right|_{s=1} = \sum_{n=2}^{\infty} \frac{\chi_k(n)\ln^2 n}{n} \tag{1.9}$$

$$= k^{-1} \sum_{m=1}^{k} \chi_k(m) \left[\zeta''\left(1, \frac{m}{k}\right) - 2 \ln k \ \zeta'\left(1, \frac{m}{k}\right) + \ln^2 k \ \zeta\left(1, \frac{m}{k}\right) \right] \tag{1.10}$$

$$= k^{-1} \sum_{m=1}^{k} \chi_k(m) \left[\gamma_2\left(1, \frac{m}{k}\right) - 2 \ln k \ \gamma_1\left(1, \frac{m}{k}\right) - \ln^2 k \ \psi\left(1, \frac{m}{k}\right) \right] \tag{1.11}$$

$$= \int_0^\infty \frac{[\ln^2 u + 2\gamma \ln u + \gamma^2 - \zeta(2)]}{1 - e^{-ku}} \left(\sum_{m=1}^{k} \chi_k(m) e^{-mu} \right) du. \tag{1.12}$$

Proposition 3. *Suppose that χ_k is a nonprincipal character and that $\chi_k(k-1) = -1$. Then*

(1.13)

$$\begin{aligned} -\sum_{m=1}^{k} \chi_k(m) \gamma_1\left(\frac{m}{k}\right) &= kL'_{-k}(1) - \ln k \ \sum_{m=1}^{k} \chi_k(m) \psi\left(\frac{m}{k}\right) \\ &= k \int_0^\infty \frac{(\ln u + \gamma)}{1 - e^{-ku}} \left(\sum_{m=1}^{k} \chi_k(m) e^{-mu} \right) du \\ &\quad - \ln k \ \sum_{m=1}^{k} \chi_k(m) \psi\left(\frac{m}{k}\right) \\ &= -\frac{\pi}{k^{1/2}} (\ln 2\pi + \gamma) \sum_{m=1}^{k} m \chi_k(m) \\ &\quad - \pi k^{1/2} \ln \prod_{m=1}^{k} \Gamma^{\chi_k(m)}\left(\frac{m}{k}\right) \\ &\quad - \ln k \ \sum_{m=1}^{k} \chi_k(m) \psi\left(\frac{m}{k}\right). \end{aligned}$$

Proposition 4. *Suppose that χ_k is a nonprincipal character and that $\chi_k(k-1) = +1$. Then*

(1.14)

$$\begin{aligned}
-\sum_{m=1}^{k} \chi_k(m)\gamma_1\left(\frac{m}{k}\right) &= kL'_{+k}(1) - \ln k \sum_{m=1}^{k} \chi_k(m)\psi\left(\frac{m}{k}\right) \\
&= k\int_0^\infty \frac{(\ln u+\gamma)}{1-e^{-ku}}\left(\sum_{m=1}^{k}\chi_k(m)e^{-mu}\right) du \\
&\quad - \ln k \sum_{m=1}^{k} \chi_k(m)\psi\left(\frac{m}{k}\right) \\
&= k^{1/2}\Bigg[2(\ln 2\pi+\gamma)\ln \prod_{m=1}^{k}\Gamma^{\chi_k(m)}\left(\frac{m}{k}\right) \\
&\qquad - \sum_{m=1}^{k}\chi_k(m)\zeta''\left(0,\frac{m}{k}\right)\Bigg] \\
&\quad - \ln k \sum_{m=1}^{k} \chi_k(m)\psi\left(\frac{m}{k}\right).
\end{aligned}$$

Proposition 5. *Let $\Gamma(s,t)$ be the incomplete Gamma function. Then for $0 < a \le 1$, $m, n = 0, 1, 2, \ldots,$ we have*

(1.15)

$$\begin{aligned}
\gamma_n(a) = \sum_{k=0}^{m} \frac{\ln^n(k+a)}{k+a} &- \frac{\ln^{n+1}(m+a)}{n+1} - \frac{\ln^n(m+a)}{2(m+a)} \\
+ \sum_{j=m}^{\infty}\Bigg\{ &\ln^n(j+a+1) - \ln^n(j+a) \\
&- \frac{1}{(n+1)}\left[\ln^{n+1}(j+a+1) - \ln^{n+1}(j+a)\right] \\
&- (a+j+1/2)\left[\Gamma[n,\ln(j+a)] - \Gamma[n,\ln(j+a+1)]\right. \\
&\qquad \left. -\Gamma[n+1,\ln(j+a)] + \Gamma[n+1,\ln(j+a+1)]\right]\Bigg\}.
\end{aligned}$$

Proposition 6. *Let $H_n = \sum_{k=1}^{n} 1/k$ be the nth harmonic number.*

Then we have for $\mathrm{Re}\, a > 0$

(1.16)
$$\gamma_1(a) = -\frac{1}{2}\ln^2(a+1) + \sum_{k=1}^\infty \frac{(-1)^k}{(k+1)}[(\zeta(k+1,a) - a^{-(k+1)})H_k + \zeta'(k+1,a)] + \ln a \ln\left(1+\frac{1}{a}\right).$$

Proposition 7. *Let the generalized harmonic number* $H_n^{(2)} = \sum_{k=1}^n 1/k^2 = \psi'(1) - \psi'(n+1) = \pi^2/6 - \psi'(n+1)$*. Then we have for* $\mathrm{Re}\, a > 1/2$

$$\gamma_1(a) = -\frac{1}{2}\ln^2\left(a-\frac{1}{2}\right) + \sum_{k=1}^\infty \frac{1}{4^k(2k+1)}\left[H_{2k}\zeta(2k+1,a) + \zeta'(2k+1,a)\right], \tag{1.17}$$

and

(1.18)
$$-\gamma_2(a) = \frac{1}{3}\ln^3\left(a-\frac{1}{2}\right) + \sum_{k=1}^\infty \frac{1}{4^k(2k+1)}\Big[(H_{2k}^2 - H_{2k}^{(2)})\zeta(2k+1,a) + 2H_{2k}\zeta'(2k+1,a) + \zeta''(2k+1,a)\Big].$$

More generally, for $k \geq 1$ *and* $n \geq 1$*, let*

$$r_n(k,m) \equiv (-1)^{m-n}\frac{s(2k+1, n-m+1)}{(n-m+1)_{m+2k-n}}, \quad 0 \leq m \leq n, \tag{1.19}$$

where $(z)_a = \Gamma(z+a)/\Gamma(z)$ *is the Pochhammer symbol and* $s(n,m)$ *are the Stirling numbers of the first kind. Then we have*

(1.20)
$$\gamma_n(a) = -\frac{1}{n+1}\ln^{n+1}\left(a-\frac{1}{2}\right) - (-1)^n\sum_{k=1}^\infty \frac{1}{4^k(2k+1)}\sum_{m=0}^n \binom{n}{m} r_n(k,m)\zeta^{(m)}(2k+1,a).$$

Proposition 8. *Let $s(n,m)$ denote the Stirling numbers of the first kind. Then we have for* $\mathrm{Re}\, a > 0$,

(i)

$$\gamma_2(a) = -\frac{1}{3}\ln^3(a+1) + \frac{\ln^2 a}{a} - \sum_{k=1}^{\infty} \frac{1}{(k+1)}\Big[(-1)^k \zeta''(k+1,a+1) - \frac{2}{k!}s(k+1,2)\zeta'(k+1,a+1) + \frac{2}{k!}s(k+1,3)\zeta(k+1,a+1)\Big], \tag{1.21}$$

and, for general $n \geq 2$,

(ii)
(1.22)

$$\gamma_n(a) = -\frac{1}{n+1}\ln^{n+1}(a+1) + \frac{\ln^n a}{a} - (-1)^n \sum_{k=1}^{\infty} \frac{1}{(k+1)} \times \Big[(-1)^k \zeta^{(n)}(k+1,a+1) - \frac{n!}{k!}\sum_{j=0}^{n-1} \frac{(-1)^j}{(n-j-1)!} \times s(k+1,j+2)\zeta^{(n-j-1)}(k+1,a+1)\Big].$$

Let $\{x\} = x - [x]$ denote the fractional part of x. It is known that

$$I_1 \equiv \int_0^1 \left\{\frac{1}{x}\right\} dx = 1 - \gamma, \tag{1.23}$$

a result that is reproved and discussed in the Appendix. We show herein

Proposition 9. *We have*

$$I_2 \equiv \int_0^1 \int_0^1 \left\{\frac{1}{xy}\right\} dx\, dy = \int_1^{\infty} \int_x^{\infty} \frac{\{y\}}{xy^2}\, dy\, dx = 1 - \gamma - \gamma_1. \tag{1.24}$$

Proposition 10. *Put*

$$I_n \equiv \int_0^1 \int_0^1 \cdots \int_0^1 \left\{ \frac{1}{x_1 x_2 \cdots x_n} \right\} dx_1\, dx_2 \cdots dx_n, \quad n \geq 1. \tag{1.25}$$

Then we have

$$I_n = 1 - \sum_{j=0}^{n-1} \frac{\gamma_j}{j!}. \tag{1.26}$$

The next section contains useful properties of special functions, mainly Dirichlet L series, needed for the proofs. The proofs as well as some Corollaries and Remarks are given in the succeeding section.

2. Special functions.

With $B_n(x)$ the Bernoulli polynomials, their periodic extension is denoted $P_n(x) \equiv B_n(x - [x])$. In particular, $P_1(x) = x - [x] - 1/2$.

The polygamma functions $\psi^{(j)}$ are connected to the Hurwitz zeta function via $\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x)$ for integers $n \geq 1$ [**12**]. Therefore, we obtain from equation (1.2) for the trigamma function

$$\psi'(a) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a), \tag{2.1}$$

and more generally,

$$\psi^{(n)}(a) = (-1)^{n+1} n! \left[\frac{1}{n} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a) n^k \right], \quad n \geq 1. \tag{2.2}$$

This equation may be taken as an infinite linear system. Its inversion would yield the Stieltjes coefficients in terms of polygammic constants.

We now introduce Dirichlet L-functions $L_{\pm k}(s)$ (e.g., [**16**, Chapter 16]), that are known to be expressible as linear combinations of Hurwitz zeta functions. We let χ_k be a real Dirichlet character modulo k, where the corresponding L function is written with subscript $\pm k$ according to $\chi_k(k-1) = \pm 1$. We have

$$L_{\pm k}(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} = \frac{1}{k^s} \sum_{m=1}^{k} \chi_k(m) \zeta\left(s, \frac{m}{k}\right), \quad \operatorname{Re} s > 1. \tag{2.3}$$

This equation holds for at least $\operatorname{Re} s > 1$. If χ_k is a nonprincipal character, as we typically assume in the following, then convergence obtains for $\operatorname{Re} s > 0$.

The L functions, extendable to the whole complex plane, satisfy the functional equations [**31**]

$$L_{-k}(s) = \frac{1}{\pi}(2\pi)^s k^{-s+1/2} \cos\left(\frac{s\pi}{2}\right)\Gamma(1-s)L_{-k}(1-s), \tag{2.4}$$

and

$$L_{+k}(s) = \frac{1}{\pi}(2\pi)^s k^{-s+1/2} \sin\left(\frac{s\pi}{2}\right)\Gamma(1-s)L_{+k}(1-s). \tag{2.5}$$

Due to the relation

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin \pi s}, \tag{2.6}$$

these functional equations may also be written in the form

$$L_{-k}(1-s) = 2(2\pi)^{-s} k^{s-1/2} \sin\left(\frac{\pi s}{2}\right)\Gamma(s)L_{-k}(s), \tag{2.7}$$

and

$$L_{+k}(1-s) = 2(2\pi)^{-s} k^{s-1/2} \cos\left(\frac{\pi s}{2}\right)\Gamma(s)L_{+k}(s). \tag{2.8}$$

Integral representations are known for these L-functions. In particular, we have

Lemma 1 [**31**]. *For χ_k a nonprincipal Dirichlet character we have*

$$L_{\pm k}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{1-e^{-ku}} \left(\sum_{m=1}^{k} \chi_k(m) e^{-mu}\right) du, \quad \operatorname{Re} s > 0. \tag{2.9}$$

Proof. For $\operatorname{Re} s > 1$ and $\operatorname{Re} a > 0$, we have the integral representation (e.g., [**24**, page 89])

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a-1)t}}{e^t - 1}\, dt. \tag{2.10}$$

We use equation (2.3) to write

$$L_{\pm k}(s) = \frac{k^{-s}}{\Gamma(s)} \sum_{m=1}^{k} \chi_k(m) \int_0^\infty \frac{t^{s-1} e^{-(m/k-1)t}}{e^t - 1}\, dt. \tag{2.11}$$

We then put $u = t/k$, giving the Lemma for $\operatorname{Re} s > 1$. By analytic continuation for χ_k nonprincipal, it also holds for $\operatorname{Re} s > 0$.

3. Proofs of Propositions.

Proof of Proposition 1. We supply two different proofs. For the first, we apply the relation [**24**, pages 87, 92]

$$\zeta'(0,a) = \ln \Gamma(a) - \ln \sqrt{2\pi}. \tag{3.1}$$

From equation (1.2) we have

$$\zeta'(s,a) = -\frac{1}{(s-1)^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \gamma_{n+1}(a)(s-1)^n. \tag{3.2}$$

Then putting $s = 0$ and using equation (3.1) we have

$$\zeta'(0,a) - \zeta'(0,b) = \ln \left[\frac{\Gamma(a)}{\Gamma(b)}\right], \tag{3.3}$$

and the Proposition follows.

For the second proof we initially assume that $a > 0$ and $b > 0$. We apply the representation [**8**, Proposition 3a]

$$\begin{aligned} \gamma_k(a) &= \frac{1}{2a} \ln^k a - \frac{\ln^{k+1} a}{k+1} \\ &\quad + \frac{2}{a} \operatorname{Re} \int_0^\infty \frac{(y/a - i) \ln^k (a - iy)}{(1 + y^2/a^2)(e^{2\pi y} - 1)}\, dy, \\ &\qquad \operatorname{Re} a > 0, \end{aligned} \tag{3.4}$$

and have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{k!}\gamma_{k+1}(a) &= \frac{1}{2}\ln a + a - a\ln a - 1\\ &\quad + \frac{2}{a}\mathrm{Re}\int_0^\infty \frac{(y/a-i)(a-iy)\ln(a-iy)}{(1+y^2/a^2)(e^{2\pi y}-1)}\,dy\\ &= a + \left(\frac{1}{2}-1-a\right)\ln a\\ &\quad + 2\mathrm{Im}\int_0^\infty \frac{\ln(a-iy)}{(e^{2\pi y}-1)}\,dy,\end{aligned} \tag{3.5}$$

where we have used elementary exponential sums, and Im $\ln(a-iy) = -\tan^{-1}(y/a)$. As $\int_a^b \psi(s)\,ds = \ln[\Gamma(b)/\Gamma(a)]$, the Proposition follows from Binet's second expression for $\ln\Gamma(z)$ [**24**, pages 17, 91],

$$\begin{aligned}\ln\Gamma(z) &= \left(z-\frac{1}{2}\right)\ln z - z\\ &\quad + \frac{1}{2}\ln(2\pi) + 2\int_0^\infty \frac{\tan^{-1}(t/z)}{e^{2\pi t}-1}\,dt,\\ &\qquad \mathrm{Re}\, z > 0.\end{aligned} \tag{3.6}$$

By analytic continuation, the result (1.4) is extended to $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, b > 0$.

Remarks. The representation [**30**]

$$\begin{aligned}\gamma_n(a) &= \sum_{k=0}^{m}\frac{\ln^n(k+a)}{k+a} - \frac{\ln^{n+1}(m+a)}{n+1} - \frac{\ln^n(m+a)}{2(m+a)}\\ &\quad + \int_m^\infty P_1(x)f_n'(x)\,dx,\\ &\qquad 0 < a \le 1, \quad m, n = 0, 1, 2, \dots ,\end{aligned} \tag{3.7}$$

where $f_n(x) \equiv \ln^n(x+a)/(x+a)$, or its equivalent may also be used to prove Proposition 1, in view of [**10**, page 107]

$$\begin{aligned}\ln\Gamma(s+1) &= (s+1/2)\ln s - s\\ &\quad + \frac{1}{2}\ln 2\pi - \int_0^\infty \frac{P_1(x)}{(x+s)}\,dx, \quad \mathrm{Re}\, s > 0.\end{aligned} \tag{3.8}$$

The representation (3.4) may be used to develop many other summations, including for $|z| \leq 1$,

$$\sum_{k=0}^{\infty} \frac{z^k}{k!}\gamma_{k+1}(a) = \frac{1}{2}a^{z-1}\ln a + (a^z - 1)\frac{1}{z^2} - \frac{a^z}{z}\ln a + 2\mathrm{Im}\int_0^\infty \frac{(a-iy)^{z-1}}{(e^{2\pi y}-1)}\ln(a-iy)\,dy. \tag{3.9}$$

Since $\zeta'(-1) = 1/12 - \ln A$, where is A is Glaisher's constant, we find from equation (3.2) that

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}\gamma_{n+1} = \ln A - \frac{1}{3} < 0. \tag{3.10}$$

Proof of Proposition 2. For part (a), equations (1.5) and (1.6) follow from equation (2.3). For equation (1.7), we differentiate equation (2.3) and use the important relation for χ_k nonprincipal

$$\sum_{r=1}^{k-1} \chi_k(r) = \sum_{r=1}^{k} \chi_k(r) = 0. \tag{3.11}$$

Evaluation of the result at $s = 1$ gives equation (1.7). For equation (1.8) we have from Lemma 1

$$\frac{\partial}{\partial s}L_{\pm k}(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{u^{s-1}[\ln u - \psi(s)]}{1-e^{-ku}}\left(\sum_{m=1}^{k}\chi_k(m)e^{-mu}\right)du, \tag{3.12}$$

where we used $\Gamma'(s) = \Gamma(s)\psi(s)$. Evaluating at $s = 1$, with $\psi(1) = -\gamma$, gives equation (1.8).

Part (b) follows very similar steps. In obtaining equation (1.12), we use $\psi'(1) = \zeta(2) = \pi^2/6$.

Remark. A key feature of Proposition 2 connecting the values $L'_{\pm k}(1)$ with differences of Stieltjes constants is the nullification of polar singularities. An example of various relations of this Proposition is given by

$$\gamma_1\left(\frac{a}{2}\right) - \gamma_1\left(\frac{a+1}{2}\right) = \frac{\partial}{\partial s}\bigg|_{s=1} 2^s \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s} \tag{3.13a}$$

$$= \ln 2\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right] + 2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\ln(n+a)}{(n+a)}. \tag{3.13b}$$

When $a = 1/2$, the Dirichlet L-function appearing on the right side of equation (3.13a) is $L_{-4}(s)$.

Proof of Proposition 3. The first two equalities in Proposition 3 follow directly from Proposition 2. For the third, we express $L'_{-k}(1)$ in terms of known quantities of $L'_{-k}(0)$ from the functional equation (2.4) or (2.7). Given that $\zeta(0, a) = 1/2 - a$, we have from equation (2.3) that

$$L_{\pm k}(0) = \sum_{m=1}^{k} \chi_k(m)\left(\frac{1}{2} - \frac{m}{k}\right) = -\frac{1}{k}\sum_{m=1}^{k} m\chi_k(m), \tag{3.14}$$

where we used property (3.11). Parallel to equation (1.6) we obtain

$$\begin{aligned} \frac{\partial}{\partial s}L_{\pm k}(s)\bigg|_{s=0} &= k^{-1}\sum_{m=1}^{k}\chi_k(m)\left[\zeta'\left(0, \frac{m}{k}\right) - \ln k\ \zeta\left(0, \frac{m}{k}\right)\right] \\ &= -\ln k\ L_{\pm k}(0) + \sum_{m=1}^{k}\chi_k(m)\ln\Gamma\left(\frac{m}{k}\right), \end{aligned} \tag{3.15}$$

where we used both equations (3.1) and (3.11). From the functional equation (2.7) we find

$$-L'_{-k}(0) = \frac{k^{1/2}}{\pi}(\ln k - \ln 2\pi - \gamma)L_{-k}(1) + \frac{k^{1/2}}{\pi}L'_{-k}(1). \tag{3.16}$$

This equation can be solved for $L'_{-k}(1)$, where equation (3.15) gives $L'_{-k}(0)$ and by the functional equation (2.4) we have $L_{-k}(1) = (\pi/k^{1/2})L_{-k}(0)$. The result is

(3.17)
$$L'_{-k}(1) = -\frac{\pi}{k^{3/2}}(\ln 2\pi + \gamma)\sum_{m=1}^{k} m\chi_k(m) - \frac{\pi}{k^{1/2}}\ln\prod_{m=1}^{k}\Gamma^{\chi_k(m)}\left(\frac{m}{k}\right),$$

yielding, by the first line of equation (1.13), the conclusion of the Proposition.

Remark. We note that the digamma function, like the Gamma function itself, satisfies a number of identities that may be used to re-express equation (1.13). These include the reflection formula

$$\psi(z) - \psi(1-z) = -\pi\cot\pi z, \tag{3.18}$$

as well as the multiplication formula for integers m,

$$\psi(mz) = \ln m + \sum_{k=0}^{m-1}\psi\left(z + \frac{k}{m}\right). \tag{3.19}$$

In addition, we recall that by means of Gauss's formula ([**24**, page 19]) the value $\psi(p/q)$ for any rational argument can be written as a finite combination of elementary function values. As concerns differences of higher order Stieltjes coefficients, similar identities may be written for the trigamma and higher order polygamma functions.

Examples. As examples of Proposition 3, we may write the following, using equation (3.18).

(3.20)
$$\gamma_1\left(\frac{1}{3}\right) - \gamma_1\left(\frac{2}{3}\right) = -\frac{\pi}{\sqrt{3}}\left\{\ln 2\pi + \gamma - 3\ln\left[\frac{\Gamma(1/3)}{\Gamma(2/3)}\right] + \ln 3\right\},$$

(3.21)
$$\gamma_1\left(\frac{1}{4}\right) - \gamma_1\left(\frac{3}{4}\right) = -\pi\left\{\ln 8\pi + \gamma - 2\ln\left[\frac{\Gamma(1/4)}{\Gamma(3/4)}\right]\right\},$$

(3.22)
$$\gamma_1\left(\frac{1}{6}\right)-\gamma_1\left(\frac{5}{6}\right)=\pi\left\{2\sqrt{\frac{2}{3}}(\ln 2\pi+\gamma)-\sqrt{6}\ln\left[\frac{\Gamma(1/6)}{\Gamma(5/6)}\right]-\sqrt{3}\ln 6\right\},$$

(3.23)
$$\begin{aligned}&\gamma_1\left(\frac{1}{7}\right)+\gamma_1\left(\frac{2}{7}\right)-\gamma_1\left(\frac{3}{7}\right)+\gamma_1\left(\frac{4}{7}\right)-\gamma_1\left(\frac{5}{7}\right)-\gamma_1\left(\frac{6}{7}\right)\\&=-\sqrt{7}\pi\left\{(\ln 2\pi+\gamma)-\ln\left[\frac{\Gamma(1/7)\,\Gamma(2/7)\,\Gamma(4/7)}{\Gamma(3/7)\,\Gamma(5/7)\,\Gamma(6/7)}\right]+\ln 7\right\},\end{aligned}$$

and

(3.24)
$$\begin{aligned}&\gamma_1\left(\frac{1}{11}\right)-\gamma_1\left(\frac{2}{11}\right)+\gamma_1\left(\frac{3}{11}\right)+\gamma_1\left(\frac{4}{11}\right)+\gamma_1\left(\frac{5}{11}\right)\\&-\gamma_1\left(\frac{6}{11}\right)-\gamma_1\left(\frac{7}{11}\right)-\gamma_1\left(\frac{8}{11}\right)+\gamma_1\left(\frac{9}{11}\right)-\gamma_1\left(\frac{10}{11}\right)\\&=-\pi\sqrt{11}\left\{(\ln 2\pi+\gamma)-\ln\left[\frac{\Gamma\left(\frac{1}{11}\right)\Gamma\left(\frac{3}{11}\right)\Gamma\left(\frac{4}{11}\right)\Gamma\left(\frac{5}{11}\right)\Gamma\left(\frac{9}{11}\right)}{\Gamma\left(\frac{2}{11}\right)\Gamma\left(\frac{6}{11}\right)\Gamma\left(\frac{7}{11}\right)\Gamma\left(\frac{8}{11}\right)\Gamma\left(\frac{10}{11}\right)}\right]\right.\\&\qquad\left.+\ln 11\right\}.\end{aligned}$$

In such equations, we could just as well use the duplication formula

$$\frac{\Gamma(x)}{\Gamma(2x)}=\sqrt{\pi}\frac{2^{1-2x}}{\Gamma(x+1/2)} \tag{3.25}$$

to re-express the Gamma function ratios. More generally, we may use the multiplication formula

$$\Gamma(nx)=(2\pi)^{(1-n)/2}n^{nx-1/2}\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right). \tag{3.26}$$

Proposition 3 shows that the logarithmic sums

$$\gamma_\ell(a)-\gamma_\ell(b)=\sum_{n=0}^{\infty}\left[\frac{\ln^\ell(n+a)}{n+a}-\frac{\ln^\ell(n+b)}{n+b}\right] \tag{3.27}$$

for rational values of a and b are essentially logarithmic constants. The integrals corresponding to the examples (3.20)–(3.24) are easily written from equation (1.13) and we omit the details. Comparison can be made to tabulated integrals that are expressible in terms of logarithmic ratios of Gamma function values [**2, 12, 20, 27**]. These include entries on pages 532, 571–573 and 580–581 of a standard table [**12**].

Regarding examples (3.20), (3.23), and (3.24) we may recall that, for k an odd prime, there is exactly one nonprincipal Dirichlet character χ_k modulo k.

By [**5**, Proposition 3] or as a special case of Proposition 5.1 of [**6**], for integers $q \geq 2$ we have

(3.28)
$$\sum_{r=1}^{q-1} \gamma_k\left(\frac{r}{q}\right) = -\gamma_k + q(-1)^k \frac{\ln^{k+1} q}{(k+1)} + q\sum_{j=0}^{k} \binom{k}{j}(-1)^j(\ln^j q)\gamma_{k-j}.$$

Given Proposition 3, we may now combine various sums and differences of Stieltjes coefficients to find identities in terms of the values $\gamma_k \equiv \gamma_k(1)$. As a very particular instance, we have

Corollary 1. *The values $\gamma_1(1/3)$ and $\gamma_1(2/3)$ may be separately written in terms of γ_1.*

This statement follows from Proposition 3 at $k = 3$ (Example (3.20)) together with equation (3.28) at $q = 3$ and $k = 1$.

Proof of Proposition 4. When $\chi_k(k-1) = 1$, $L_{+k}(0) = 0$, $\sum_{m=1}^{k} m\chi_k(m) = 0$, and from (3.15)

$$L'_{+k}(0) = \sum_{m=1}^{k} \chi_k(m) \ln\Gamma\left(\frac{m}{k}\right). \tag{3.29}$$

Differentiating equation (2.3) twice we have

$$L''_{+k}(0) = \sum_{m=1}^{k} \chi_k(m)\left[-2\ln k\ \ln\Gamma\left(\frac{m}{k}\right) + \zeta''\left(0, \frac{m}{k}\right)\right]. \tag{3.30}$$

Differentiating the functional equation (2.5) we find

$$L'_{+k}(1) = \frac{1}{k^{1/2}}\left[2\left(\gamma + \ln\left(\frac{2\pi}{k}\right)\right)L'_{+k}(0) - L''_{+k}(0)\right]. \tag{3.31}$$

Then using equations (3.29) and (3.30) we determine

$$\begin{aligned} &L'_{+k}(1) \\ &= k^{-1/2}\left[2(\ln 2\pi + \gamma)\ln\prod_{m=1}^{k}\Gamma^{\chi_k(m)}\left(\frac{m}{k}\right) - \sum_{m=1}^{k}\chi_k(m)\zeta''\left(0, \frac{m}{k}\right)\right]. \end{aligned} \tag{3.32}$$

Substituting into the first line of equation (1.14), the Proposition is completed.

Examples. We have from Proposition 4 at $k = 5$ using equation (2.6),

$$\begin{aligned} \gamma_1\left(\frac{1}{5}\right) - \gamma_1\left(\frac{2}{5}\right) - \gamma_1\left(\frac{3}{5}\right) + \gamma_1\left(\frac{4}{5}\right) \\ = -\sqrt{5}\biggl\{2(\gamma + \ln 2\pi)\ln\frac{1}{2}(1+\sqrt{5}) - \zeta''\left(0, \frac{1}{5}\right) \\ + \zeta''\left(0, \frac{2}{5}\right) + \zeta''\left(0, \frac{3}{5}\right) - \zeta''\left(0, \frac{4}{5}\right) \\ + 2\ln 5\coth^{-1}\sqrt{5}\biggr\}. \end{aligned} \tag{3.33}$$

We have from Proposition 4 at $k = 10$ using equation (2.6),

$$\begin{aligned} \gamma_1\left(\frac{1}{10}\right) - \gamma_1\left(\frac{3}{10}\right) - \gamma_1\left(\frac{7}{10}\right) + \gamma_1\left(\frac{9}{10}\right) \\ = \sqrt{10}\biggl\{2(\gamma + \ln 2\pi)\ln\frac{1}{2}(3+\sqrt{5}) - \zeta''\left(0, \frac{1}{10}\right) \\ + \zeta''\left(0, \frac{3}{10}\right) + \zeta''\left(0, \frac{7}{10}\right) - \zeta''\left(0, \frac{9}{10}\right) \\ + 3\sqrt{2}\ln 10\coth^{-1}\sqrt{5}\biggr\}. \end{aligned} \tag{3.34}$$

From the well-known Hermite formula for $\zeta(s,a)$ (e.g., [**24**, page 91])

$$\zeta(s,a)=\frac{a^{-s}}{2}+\frac{a^{1-s}}{s-1}+2\int_0^\infty \frac{\sin(s\tan^{-1}y/a)}{(y^2+a^2)^{s/2}}\frac{dy}{(e^{2\pi y}-1)}, \tag{3.35}$$

we obtain

$$\begin{aligned}\zeta''(0,a)&=\frac{1}{2}\ln^2 a+2a\ln a-a\ln^2 a-2a\\&\quad-2\int_0^\infty \tan^{-1}\left(\frac{y}{a}\right)\frac{\ln(a^2+y^2)}{(e^{2\pi y}-1)}\,dy.\end{aligned} \tag{3.36}$$

Evaluation of the integral for rational values of a, or combinations of rational values of a, would be of interest in regard to Proposition 4. It appears that a contour integral evaluation may be possible, with the integrand having simple poles along the imaginary axis at $y=ji$ and residues $(i/2\pi)\tanh^{-1}(j/a)\ln(a^2-j^2)$ there. However, this is likely to give an infinite series representation of $\zeta''(0,a)$, whereas another closed form is desirable.

Proof of Proposition 5. From the representation (3.7) we may proceed as follows:

$$\begin{aligned}\int_m^\infty P_1(x)f_n'(x)\,dx&=\sum_{j=m}^\infty\int_j^{j+1}P_1(x)f_n'(x)\,dx\\&=\sum_{j=m}^\infty\int_j^{j+1}(x-j-1/2)f_n'(x)\,dx.\end{aligned} \tag{3.37}$$

By using the expression for f_n, we obtain

(3.38)
$$\begin{aligned}\int_m^\infty P_1(x)f_n'(x)\,dx&=\sum_{j=m}^\infty\int_{j+a}^{j+a+1}\left[\frac{1}{x}-(a+j+1/2)\frac{1}{x^2}\right]\\&\qquad\qquad\times\left(n\ln^{n-1}x-\ln^n x\right)dx\\&=\sum_{j=m}^\infty\Bigg\{-(a+j+1/2)\int_{j+a}^{j+a+1}\frac{[\ln^{n-1}x-\ln^n x]}{x^2}\,dx\end{aligned}$$

$$- \frac{1}{(n+1)}\left[\ln^{n+1}(j+a+1) - \ln^{n+1}(j+a)\right] \\ + \ln^{n}(j+a+1) - \ln^{n}(j+a)\Bigg\}.$$

The remaining integrals may be performed as [**12**]

$$\int_{j+a}^{j+a+1} \frac{\ln^n x}{x^2}\,dx = \Gamma[n+1, \ln(j+a)] - \Gamma[n+1, \ln(j+a+1)]. \tag{3.39}$$

The insertion of equations (3.38) and (3.39) into equation (3.7) provides the Proposition.

Remarks. Semi-infinite integrals over P_1 as we have just performed are of much interest in connection with applications of Euler-Maclaurin summation.

In conjunction with equation (1.15) we may note that for n a non-negative integer we have [**12**, page 941]

$$\Gamma(n+1, x) = n! e^{-x} \sum_{m=0}^{n} \frac{x^m}{m!}. \tag{3.40}$$

Similarly we may obtain other explicit summation representations of the Stieltjes constants by working with [**30**]

$$\begin{aligned} C_n(a) = (-1)^{n-1} n! \sum_{k=0}^{n+1} & \frac{s(n+1, n+1-k)}{k!} \\ & \times \int_1^\infty P_n(x-a)\frac{\ln^k x}{x^{n+1}}\,dx, \quad n \geq 1. \end{aligned} \tag{3.41}$$

We omit the details.

Proof of Proposition 6. By the use of [**3**]

$$\gamma_k(a) = \lim_{N\to\infty}\left[\sum_{j=0}^{N} \frac{\ln^k(j+a)}{j+a} - \frac{\ln^{k+1}(N+a)}{(k+1)}\right], \tag{3.42}$$

at $k=1$ we have

$$\begin{aligned}
\gamma_1(a) &= \lim_{N\to\infty}\left[\sum_{j=0}^N \frac{\ln(j+a)}{j+a} - \frac{\ln^2(N+a)}{2}\right]\\
&= \lim_{N\to\infty}\left[\sum_{j=0}^N \frac{\ln(j+a)}{j+a} - \int_{1-a}^N \frac{\ln(x+a)}{x+a}\,dx\right]\\
&= \sum_{j=0}^\infty \frac{\ln(j+a)}{j+a} - \int_{1-a}^1 \frac{\ln(x+a)}{x+a}\,dx\\
&\quad - \int_1^\infty \frac{\ln(x+a)}{x+a}\,dx\\
&= -\frac{1}{2}\ln^2(a+1) + \frac{\ln a}{a} + \sum_{j=1}^\infty \frac{\ln(j+a)}{j+a}\\
&\quad - \sum_{j=1}^\infty \int_j^{j+1} \frac{\ln(x+a)}{x+a}\,dx\\
&= -\frac{1}{2}\ln^2(a+1) + \frac{\ln a}{a}\\
&\quad + \sum_{j=1}^\infty \int_0^1 \left[\frac{\ln(j+a)}{j+a} - \frac{\ln(x+j+a)}{x+j+a}\right] dx.
\end{aligned} \tag{3.43}$$

We now apply the generating function for harmonic numbers

$$\sum_{n=1}^\infty (-1)^n H_n z^n = -\frac{\ln(1+z)}{1+z}, \quad |z|<1, \tag{3.44}$$

to expand the integrand in this equation. We have

$$\begin{aligned}
\frac{\ln y}{y} - \frac{\ln(x+y)}{x+y} &= -\frac{\ln(1+x/y)}{x+y} - \ln y\left(\frac{1}{x+y} - \frac{1}{y}\right)\\
&= -\frac{\ln(1+x/y)}{x+y} - \frac{\ln y}{y}\left[\frac{1}{(1+x/y)} - 1\right]\\
&= -\frac{1}{y}\frac{\ln(1+x/y)}{1+x/y} - \frac{\ln y}{y}\sum_{k=1}^\infty (-1)^k \frac{x^k}{y^k}\\
&= \sum_{k=1}^\infty (-1)^k \frac{(H_k - \ln y)}{y^{k+1}} x^k.
\end{aligned} \tag{3.45}$$

We obtain for the integral in equation (3.43)

$$\int_0^1 \sum_{k=1}^\infty (-1)^k \frac{[H_k - \ln(j+a)]}{(j+a)^{k+1}} x^k \, dx = \sum_{k=1}^\infty \frac{(-1)^k}{(k+1)} \frac{[H_k - \ln(j+a)]}{(j+a)^{k+1}}. \tag{3.46}$$

Summing this expression over $j = 1$ to ∞ gives

$$\begin{aligned} \sum_{k=1}^\infty \sum_{j=1}^\infty \frac{(-1)^k}{(k+1)} \frac{[H_k - \ln(j+a)]}{(j+a)^{k+1}} &= \sum_{k=1}^\infty \frac{(-1)^k}{(k+1)} [\zeta(k+1, a+1) H_k + \zeta'(k+1, a+1)] \\ &= \sum_{k=1}^\infty \frac{(-1)^k}{(k+1)} [(\zeta(k+1, a) - a^{-(k+1)}) H_k \\ &\quad + \zeta'(k+1, a) + a^{-(k+1)} \ln a], \end{aligned} \tag{3.47}$$

where we used the relation $\zeta(s, a+1) = \zeta(s, a) - a^{-s}$ and its derivative. Inserting equation (3.47) into equation (3.43) gives the Proposition.

We mention a second method for proving Proposition 6, by using the integral representation (2.10) for $\zeta(s, a)$. This method can also provide an integral representation of $\gamma_1(a)$. In particular, since

$$\zeta'(k+1, a) = \frac{1}{k!} \int_0^\infty \frac{t^k e^{-(a-1)t}}{e^t - 1} \ln t \, dt, \tag{3.48}$$

we have the term of equation (1.16)

$$\sum_{k=1}^\infty \frac{(-1)^k}{(k+1)} \zeta'(k+1, a) = \int_0^\infty \frac{e^{-(a-1)t}}{e^t - 1} \left[\frac{1}{t} - 1 - \frac{e^{-t}}{t}\right] \ln t \, dt. \tag{3.49}$$

This integral may be evaluated per logarithmic differentiation of

$$\int_0^\infty t^\beta \frac{e^{-(a-1)t}}{e^t - 1} \left[\frac{1}{t} - 1 - \frac{e^{-t}}{t}\right] dt = \Gamma(\beta)[a^\beta - \beta\zeta(\beta+1, a)], \quad \operatorname{Re}\beta > -1. \tag{3.50}$$

Then one may apply the operator $(\partial/\partial\beta)_{\beta=0}$, use the functional equation of the Hurwitz zeta function, and the relations $\zeta(0,a) = 1/2 - a$ and (3.3).

For the other summation term in equation (1.16), using (2.10), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)}\zeta(k+1,a)H_k &= -\int_0^\infty \frac{e^{-(a+1)t}}{(1-e^{-t})t}\left[\gamma(1+e^t) + 2e^{t/2}\cosh\left(\frac{t}{2}\right)\ln t\right. \\ &\qquad \left. - e^{t/2}\sqrt{\pi t}\,\frac{\partial I_\nu}{\partial \nu}\bigg|_{\nu=-1/2}\left(\frac{t}{2}\right)\right],\end{aligned} \tag{3.51}$$

where I_ν is the modified Bessel function of the first kind (e.g., [**12**, pages 958, 961]).

Note 1. By integrating the generating function relation (3.44) we obtain the term of Proposition 6

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)} a^{-(k+1)} H_k = -\frac{1}{2}\ln^2\left(\frac{a+1}{a}\right). \tag{3.52}$$

Corollary 2. *We have*

$$\begin{aligned}\gamma_1 &= -\frac{1}{2}\ln^2 2 + \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)}[(\zeta(k+1)-1)H_k + \zeta'(k+1)] \\ &= \sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)}[\zeta(k+1)H_k + \zeta'(k+1)].\end{aligned} \tag{3.53}$$

Corollary 3. *Proposition* 6 *permits the recovery of relation* (3.28) *at* $k = 1$,

$$\sum_{r=1}^{q-1}\gamma_1\left(\frac{r}{q}\right) = (q-1)\gamma_1 - q\left(\frac{1}{2}+\gamma\right)\ln q. \tag{3.54}$$

In verifying this statement we use

$$\sum_{r=1}^{q-1} \zeta\left(k+1, \frac{r}{q}\right) = (q^{k+1}-1)\zeta(k+1). \tag{3.55}$$

Proof of Proposition 7. We use [**7**, Corollary 3] for $\operatorname{Re} a > 1/2$,

$$\begin{aligned}\zeta'(s,a) = &-\frac{(a-1/2)^{1-s}}{s-1}\ln(a-1/2) - \frac{(a-1/2)^{1-s}}{(s-1)^2} \\ &-\sum_{k=1}^{\infty} \frac{(s)_{2k}}{4^k(2k+1)!} \\ &\times \{[\psi(s+2k)-\psi(s)]\zeta(s+2k,a)+\zeta'(s+2k,a)\},\end{aligned} \tag{3.56}$$

and apply the definition (1.2). For equation (1.18) we use the property

$$\frac{d}{ds}(s)_{pk} = (s)_{pk}[\psi(s+pk)-\psi(s)], \tag{3.57}$$

so that

$$\left.\frac{d}{ds}\right|_{s=1} (s)_{2k} = (2k)! H_{2k}. \tag{3.58}$$

More generally, we use the expansion

$$\left(a-\frac{1}{2}\right)^{1-s} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \ln^j\left(a-\frac{1}{2}\right)(s-1)^j, \tag{3.59}$$

and apply the product rule to find equation (1.20).

Proof of Proposition 8. We will use a generating function for the Stirling numbers $s(j,k)$,

$$\frac{1}{x}\ln^m(1+x) = m! \sum_{n=m-1}^{\infty} s(n+1,m)\frac{x^n}{(n+1)!}, \qquad |x|<1. \tag{3.60}$$

Proceeding similarly to the beginning of the proof of Proposition 6, we have generally

$$\begin{aligned}\gamma_k(a) = &-\frac{1}{k+1}\ln^k(a+1) + \frac{\ln^k a}{a}\\ &+\sum_{j=1}^\infty \int_0^1 \left[\frac{\ln^k(j+a)}{j+a} - \frac{\ln^k(x+j+a)}{x+j+a}\right] dx.\end{aligned} \tag{3.61}$$

Now we apply the expansion (3.60) to write for part (i)

$$\begin{aligned}&\frac{\ln^2 y}{y} - \frac{\ln^2(x+y)}{x+y}\\ &= -\sum_{k=1}^\infty \frac{x^k}{y^{k+1}}\left[(-1)^k \ln^2 y + \frac{2}{k!}s(k+1,2)\ln y + \frac{2}{k!}s(k+1,3)\right].\end{aligned} \tag{3.62}$$

We then substitute this equation into equation (3.61) at $k = 2$ and perform the integration. We then perform the summation over j, using

$$\sum_{j=1}^\infty \frac{\ln^p(j+a)}{(j+a)^{k+1}} = (-1)^p \zeta^{(p)}(k+1, a+1), \tag{3.63}$$

and part (i) follows. Similarly, for part (ii), in place of (3.62) we use

$$\begin{aligned}&\frac{\ln^n y}{y} - \frac{\ln^n(x+y)}{x+y}\\ &= \sum_{k=1}^\infty \frac{x^k}{y^{k+1}}\left[(-1)^k \ln^n y + \frac{1}{k!}\sum_{j=0}^{n-1}\frac{n!}{(n-j-1)!}s(k+1,j+2)\ln^{n-j-1} y\right].\end{aligned} \tag{3.64}$$

We substitute this equation into equation (3.61), perform the integration, perform the summation over j using (3.63), giving equation (1.22). The Proposition is complete.

Remarks. Proposition 6 corresponds to the well-known special case of the harmonic numbers wherein $s(n+1,2) = (-1)^{n+1}n!H_n$. The connection of Stirling numbers with sums of generalized harmonic numbers $H_n^{(r)}$ is well known. For example, we have $s(n+1,3) = (-1)^n n![H_n^2 - H_n^{(2)}]/2$.

As for $\gamma_1(a)$, an integral representation for $\gamma_n(a)$ may be developed using equation (2.10). We omit such consideration.

If desired, Proposition 8 may be used to write an expression for the differences $\gamma_n(a) - \gamma_n(b)$. We have been informed that Smith has also obtained equation (1.22), and numerically studied its rate of convergence [**23**].

Proof of Proposition 9. Although this result is supposedly proved in [**22**], we have not been able to obtain it, and so present our own proofs. These methods of proof may themselves be of independent interest.

Proof 1. We first establish

Lemma 2. *We have*

$$\int_x^\infty \frac{\{y\}}{y^2}\,dy = H_{[x]} - \gamma - \ln x + 1 - \frac{[x]}{x}. \tag{3.65}$$

We have

$$\begin{aligned}\int_x^\infty \frac{\{y\}}{y^2}\,dy &= \int_{[x]+\{x\}}^\infty \frac{\{y\}}{y^2}\,dy\\ &= \int_{[x]}^\infty \frac{\{y\}}{y^2}\,dy - \int_{[x]}^x \frac{\{y\}}{y^2}\,dy\\ &= \sum_{j=[x]}^\infty \int_j^{j+1} \frac{(y-[y])}{y^2}\,dy - \int_{[x]}^x \frac{(y-[x])}{y^2}\,dy\\ &= \sum_{j=[x]}^\infty \left[\ln\left(\frac{j+1}{j}\right) - j\left(\frac{1}{j} - \frac{1}{j+1}\right)\right]\\ &\quad - \ln\left(\frac{x}{[x]}\right) + [x]\left(\frac{1}{[x]} - \frac{1}{x}\right).\end{aligned} \tag{3.66}$$

The sum here is given by (cf. the Appendix)

$$\begin{aligned}\sum_{j=[x]}^\infty \left[\ln\left(\frac{j+1}{j}\right) - \frac{1}{j+1}\right] &= \sum_{j=1}^\infty \left[\ln\left(\frac{j+1}{j}\right) - \frac{1}{j+1}\right]\\ &\quad - \sum_{j=1}^{[x]-1} \left[\ln\left(\frac{j+1}{j}\right) - \frac{1}{j+1}\right]\end{aligned} \tag{3.67}$$

$$= 1 - \gamma - \sum_{j=2}^{[x]} \left[\ln j - \ln(j-1) - \frac{1}{j} \right]$$
$$= 1 - \gamma - \ln[x] + H_{[x]} - 1$$
$$= H_{[x]} - \ln[x] - \gamma.$$

Inserting this expression into equation (3.66) we obtain the Lemma.

Using the Lemma, we have

(3.68)
$$\begin{aligned} I_2 &= \int_1^\infty \left(H_{[x]} - \gamma - \ln x + 1 - \frac{[x]}{x} \right) \frac{dx}{x} \\ &= \lim_{M\to\infty} \left\{ \int_1^M (1-\gamma-\ln x)\frac{dx}{x} + \int_1^M \left(H_{[x]} - \frac{[x]}{x} \right) \frac{dx}{x} \right\} \\ &= \lim_{M\to\infty} \left[(1-\gamma)\ln M - \frac{1}{2}\ln^2 M + \sum_{j=1}^M \int_j^{j+1} \left(H_j - \frac{j}{x} \right) \frac{dx}{x} \right] \\ &= \lim_{M\to\infty} \left\{ (1-\gamma)\ln M - \frac{1}{2}\ln^2 M + \sum_{j=1}^M \left[H_j \ln\left(\frac{j+1}{j}\right) - \frac{1}{j+1} \right] \right\}. \end{aligned}$$

Now the term

(3.69)
$$\begin{aligned} \sum_{j=1}^M H_j \ln\left(\frac{j+1}{j}\right) &= \sum_{j=1}^M \sum_{k=1}^j \frac{1}{k} \ln\left(\frac{j+1}{j}\right) = \sum_{k=1}^M \frac{1}{k} \sum_{j=k}^M \ln\left(\frac{j+1}{j}\right) \\ &= \sum_{k=1}^M \frac{1}{k} [\ln(M+1) - \ln k] \\ &= H_M \ln(M+1) - \sum_{k=1}^M \frac{\ln k}{k}. \end{aligned}$$

Then by equation (3.68) we have

$$\begin{aligned} I_2 = \lim_{M\to\infty} \Bigg\{ &(1-\gamma)\ln M - \frac{1}{2}\ln^2 M \\ &+ H_M \ln(M+1) - \sum_{k=1}^M \frac{\ln k}{k} - H_{M+1} + 1 \Bigg\}. \end{aligned} \tag{3.70}$$

We recall the asymptotic form $H_M = \ln M + \gamma + O(1/M)$ as $M \to \infty$ and appeal to equation (3.42) at $a = k = 1$:

(3.71)

$$\begin{aligned} I_2 = \lim_{M\to\infty} \Bigg\{ \ln M - \gamma \ln M - \frac{1}{2}\ln^2 M \\ + \left[\gamma + \ln M + O\left(\frac{1}{M}\right)\right]\left[\ln M + O\left(\frac{1}{M}\right)\right] \\ - \sum_{k=1}^{M} \frac{\ln k}{k} - \left[\gamma + \ln M + O\left(\frac{1}{M}\right)\right] + 1 \Bigg\} \\ = \lim_{M\to\infty} \left\{ \frac{1}{2}\ln^2 M - \sum_{k=1}^{M} \frac{\ln k}{k} - \gamma + 1 + O\left(\frac{\ln M}{M}\right)\right\} \\ = 1 - \gamma - \gamma_1. \end{aligned}$$

The alternative form of I_2 in equation (1.24) results from the change of variable $(u, v) = (1/xy, x)$, with inverse transformation $(x, y) = (v, 1/uv)$, and Jacobian

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 0 & 1 \\ -\frac{1}{u^2 v} & -\frac{1}{uv^2} \end{vmatrix} = \frac{1}{u^2 v}. \tag{3.72}$$

Then we obtain

$$I_2 = \int_0^1 \int_{1/v}^{\infty} \frac{\{u\}}{u^2 v}\, du\, dv = \int_1^{\infty} \int_t^{\infty} \frac{\{z\}}{z^2 t}\, dz\, dt. \tag{3.73}$$

For the second equality we have used the simple transformation $(v, u) = (1/t, z)$, with corresponding Jacobian

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -\frac{1}{t^2}. \tag{3.74}$$

We may obtain a second shorter proof of Proposition 9 by using a known result, namely the integral representation [**17**]

(3.75)

$$\gamma_k = -\int_1^{\infty} \frac{1}{t} \ln^k t\, dP_1(t) = \int_1^{\infty} \frac{\ln^{k-1} t}{t^2}(k - \ln t) P_1(t)\, dt - \delta_{k0}/2,$$

where δ_{jk} is the Kronecker symbol. We interchange the order of integration in the second form of I_2 on the right side of equation (1.24), giving

$$
\begin{aligned}
I_2 &= \int_1^\infty \int_1^y \frac{\{y\}}{xy^2}\,dx\,dy \\
&= \int_1^\infty \frac{\{y\}}{y^2} \ln y\,dy \qquad (3.76)\\
&= -\int_0^1 \left\{\frac{1}{v}\right\} \ln v\,dv. \qquad (3.77)
\end{aligned}
$$

Therefore, from equation (3.76) and equation (3.75) at $k = 1$ we obtain

$$
\begin{aligned}
I_2 &= \int_1^\infty \frac{[P_1(w) + 1/2]}{w^2} \ln w\,dw \\
&= -\gamma - 1 + \int_1^\infty \frac{P_1(w)}{w^2}\,dw + \frac{1}{2}\int_1^\infty \frac{\ln w}{w^2}\,dw \qquad (3.78)\\
&= -\gamma_1 - \gamma + \frac{1}{2} + \frac{1}{2} = -\gamma_1 - \gamma + 1.
\end{aligned}
$$

Remark. Since $0 < \{x\} \leq 1$ for $x > 0$, we easily have from equations (A.1) and (3.73) or (3.76) the inequalities $0 < I_1 \leq 1$ and $0 < I_2 \leq 1$.

From Proposition 10 follows

Corollary 4. *We have*

$$
\lim_{n\to\infty} I_n = -\zeta(0) = \frac{1}{2}. \qquad (3.79)
$$

Proof of Proposition 10. For this we use the change of variable $(x_1, x_2, \ldots, x_n) = (u_n, u_{n-1}, \ldots, u_2, 1/u_1u_2\cdots u_n)$, with the Jacobian

$$
J = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ -\frac{1}{u_1^2u_2\cdots u_n} & 0 & \cdots & 0 & 0 \end{vmatrix} = \frac{\varepsilon_n}{u_1^2u_2\cdots u_n}, \qquad (3.80)
$$

where $\varepsilon_n = \pm 1$. Specifically, if n is of the form $4m$ or $4m+1$, $\varepsilon_n = -1$, and $\varepsilon_n = 1$ otherwise. Then

$$I_n = \varepsilon_n \int_1^\infty \int_{1/u_2}^1 \cdots \int_{1/u_{n-1}}^1 \frac{\{u_1\}}{u_1^2} \frac{\prod_{i=n}^1 du_i}{\prod_{i=2}^n u_i}. \tag{3.81}$$

We then multiply integrate to find

$$I_n = \frac{1}{(n-1)!} \int_1^\infty \frac{\{u_1\}}{u_1^2} \ln^{n-1} u_1 \, du_1. \tag{3.82}$$

We now note from equation (3.75)

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\gamma_k}{k!} &= -\frac{1}{(n-1)!} \int_1^\infty \Gamma(n, \ln t)\, dP_1(t) \\ &= -\frac{1}{(n-1)!} \int_1^\infty \frac{\ln^{n-1} t}{t^2} P_1(t)\, dt + \frac{1}{2}, \end{aligned} \tag{3.83}$$

where we applied [**12**, page 941], integrated by parts, and used $\Gamma(n, 0) = (n-1)!$ for $n \geq 1$ an integer. We then have the Proposition, as

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\gamma_k}{k!} &= -\frac{1}{(n-1)!} \int_1^\infty \frac{\ln^{n-1} t}{t^2} \left(\{t\} - \frac{1}{2} \right) dt + \frac{1}{2} \\ &= -\frac{1}{(n-1)!} \int_1^\infty \{t\} \frac{\ln^{n-1} t}{t^2}\, dt + 1. \end{aligned} \tag{3.84}$$

Remarks. We have found that the subjects of Propositions 9 and 10 have recently been of interest elsewhere [**11**]. It would be interesting to have a probabilistic argument for Corollary 4.

Summary. We have obtained new explicit analytic results for the Stieltjes coefficients including series representations and summatory relations. Other integral representations based upon the properties of Dirichlet L-functions provide the difference of Stieltjes coefficients at rational arguments, and these give inequalities. Our results have implications for other coefficients of analytic number theory and other fundamental mathematical constants.

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APPENDIX

A.1. Integral expression for $1-\gamma$. We show here that

Proposition A1. *We have*

$$I_1 \equiv \int_0^1 \left\{\frac{1}{x}\right\} dx = \int_1^\infty \frac{\{x\}}{x^2}\, dx = 1-\gamma. \tag{A.1}$$

Proof. We recall that $\{x\} = P_1(x) + 1/2 = x - [x]$, giving

$$\begin{aligned} I_1 &= \int_1^\infty \frac{\{x\}}{x^2}\, dx \\ &= \sum_{j=1}^\infty \int_j^{j+1} \left(\frac{1}{x} - \frac{j}{x^2}\right) dx \\ &= \sum_{j=1}^\infty \left[\ln\left(\frac{j+1}{j}\right) - \frac{1}{j+1}\right] \\ &= \lim_{N\to\infty} \sum_{j=1}^N \left[\ln(j+1) - \ln j - \frac{1}{j+1}\right] \\ &= \lim_{N\to\infty} [\ln N - H_N + 1] = 1-\gamma, \end{aligned} \tag{A.2}$$

where we used the telescoping nature of the sum in the next-to-last line.

As a variation on this proof, we may note that a sum above is a case of the summation [**13**, 44.9.1, page 290]

$$\sum_{k=1}^\infty \left[\ln\left(\frac{k+x}{k}\right) - \frac{x}{k}\right] = -\gamma x + \ln\Gamma(x+1). \tag{A.3}$$

We then obtain

(A.4)

$$\sum_{j=1}^{\infty}\left[\ln\left(\frac{j+1}{j}\right)-\frac{1}{j+1}\right]=\sum_{j=1}^{\infty}\left[\ln\left(\frac{j+1}{j}\right)-\frac{1}{j}+\frac{1}{j}-\frac{1}{j+1}\right]$$
$$=-\gamma+\ln\Gamma(2)+\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right)$$
$$=1-\gamma.$$

Similarly, this sum may be found at $x=0$ in [**13**, 44.9.4, page 290] or at $x=1$ in [**13**, 44.9.5, page 290].

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30. N.-Y. Zhang and K.S. Williams, *Some results on the generalized Stieltjes constants*, Analysis **14** (1994), 147–162. In addition to a typographical error present in the proof of Theorem 5, Remark 5, and equation (1.11) of this reference, where 2^{-n} should read 2^{-n-1}, the first term on the right side of equation (6.2) should read $(\log^n a)/a$, and in both equations (6.5) and (6.11), P_n should appear in place of P_1. On page 148, in equation (1.4), $f_n(x)$ should be replaced by $f_n'(x)$. On page 157, (7.2) should be replaced with (7.1) in the second line of text from the bottom, and on page 158 (7.2) should be replaced with (1.9) in the third line of text.

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