ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 5, 2011

CARLESON MEASURES AND A CLASS OF GENERALIZED INTEGRATION OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. In this paper, we consider a linear operator

$$I_{h,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))\, h(\zeta)\, d\zeta$$

induced by holomorphic maps h and φ of the open unit disk $\mathbf{D}$, where $\varphi(\mathbf{D}) \subset \mathbf{D}$ and n is a non-negative integer. A complete characterization of when $I_{h,\varphi}^{(n)}$ is bounded on the Bergman space $\mathcal{A}^2$ is established by using Luecking's result for Carleson measures. We also compute upper and lower bounds for the essential norm of this operator on the Bergman space.

1. Introduction. Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$. Throughout this paper, we denote by $H(\mathbf{D})$ the space of holomorphic functions on $\mathbf{D}$. Let $dA(z) = (1/\pi)\, dx\, dy$, where $z = x + iy$, denote the normalized Lebesgue measure on $\mathbf{D}$. Recall that the Bergman space $\mathcal{A}^2$ is a Hilbert space of holomorphic functions on $\mathbf{D}$ with the norm

$$(1.1) \qquad ||f||_{\mathcal{A}^2} = \left(\int_{\mathbf{D}} |f(z)|^2 dA(z) \right)^{1/2} < \infty.$$

Also, if $f \in \mathcal{A}^2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is its Taylor series in $\mathbf{D}$, then $||f||_{\mathcal{A}^2}$ may also be defined as

$$(1.2) \qquad ||f||_{\mathcal{A}^2} = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

2010 AMS *Mathematics subject classification.* Primary 47B35, 46E10, Secondary 30H05.
Keywords and phrases. Generalized integration operator, Riemann-Stieltjes operator, composition operator, Bergman space.
Received by the editors on March 13, 2008, and in revised form on January 26, 2009.
DOI:10.1216/RMJ-2011-41-5-1711

Moreover, it is well known that

$$(1.3)\qquad ||f||_{\mathcal{A}^2} \approx |f(0)| + \left(\int_{\mathbf{D}} |f'(z)|^2 (1-|z|^2)^2 dA(z) \right)^{1/2},$$

where the notation $A \approx B$ means that there is a positive constant C such that $B/C \leq A \leq CB$. See [**5, 14**] for more details on the Bergman space.

For $z, w \in \mathbf{D}$, let $\beta_z(w) = (z-w)/(1-\overline{z}w)$ be the Möbius transformation of $\mathbf{D}$ which interchanges 0 and z. Then the nth derivative of β_z, $\beta_z^{(n)}(w) = n!(|z|^2-1)(\overline{z})^{n-1}/(1-\overline{z}w)^{n+1}$. Also $K_z(w) = 1/(1-\overline{z}w)^2$ is the Bergman kernel and $k_z(w) = (1-|z|^2)/(1-\overline{z}w)^2 = (1-|z|^2)K_z(w) = -\beta_z'(w)$ is the normalized kernel function in $\mathcal{A}^2$. Moreover, $k_z^{(n)}(w) = (n+1)!(1-|z|^2)(\overline{z})^n/(1-\overline{z}w)^{n+2} = -\beta_z^{(n+1)}(w)$.

Let g, h and φ be holomorphic maps on $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$. For a non-negative integer n, we define a linear operator

$$I_{h,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))\, h(\zeta)\, d\zeta, \quad f \in H(\mathbf{D}).$$

We call it the generalized integration operator, since it induces many known operators. When $\varphi(z) = z$, we drop φ and simply write $I_h^{(n)}$ for $I_{h,\varphi}^{(n)}$. If $n = 0$ and $h(z) = g'(z)$, then we get the operator $T_{g,\varphi}$ induced by g and φ as

$$\begin{aligned} T_{g,\varphi} f(z) &= \int_0^z f(\varphi(\zeta))\, dg(\zeta) = \int_0^z f(\varphi(\zeta)) g'(\zeta)\, d\zeta \\ &= \int_0^1 f(\varphi(tz))\, z\, g'(tz)\, dt. \end{aligned}$$

The operator $T_{g,\varphi}$ can be viewed as a generalization of the Riemann-Stieltjes operator T_g induced by g, defined by

$$T_g f(z) = \int_0^z f(\zeta)\, dg(\zeta) = \int_0^1 f(tz) z g'(tz)\, dt, \quad z \in \mathbf{D}.$$

Pommerenke [**7**] initiated the study of Riemann-Stieltjes operator on H^2, where he showed that T_g is bounded on H^2 if and only if g is in

$BMOA$. This was extended to other Hardy spaces H^p, $1 \le p < \infty$, in [**1, 2**], where compactness of T_g on H^p and Schatten class membership of T_g on H^2 was also completely characterized in terms of the symbol g. Recently, several authors have studied these operators on different spaces of analytic functions. For example, one can refer to ([**3, 8, 9, 11, 12**]) and the related references therein for the study of these operators on different spaces of analytic functions. If $n = 1$ and $h(z) = \varphi'(z)g(z)$, then we get the operator $J_{g,\varphi}$ induced by g and φ, defined as

$$J_{g,\varphi}f(z) = \int_0^z f'(\varphi(\zeta))\varphi'(\zeta)g(\zeta)\, d\zeta, \quad z \in \mathbf{D}.$$

The operator $J_{g,\varphi}$ is the generalization of the operator J_g, recently defined by Yoneda in [**13**] as

$$J_g f(z) = \int_0^z f'(\zeta)g(\zeta)\, d\zeta, \quad z \in \mathbf{D}.$$

Also, if $n = 1$ and $h(z) = \varphi'(z)$, then $I_{h,\varphi}^{(n)}$ reduces to the difference of the composition operator and the point evaluation map, defined as $C_\varphi f = f \circ \varphi - f(\varphi(0))$, $f \in H(\mathbf{D})$. These operators have gained increasing attention during the last three decades, mainly due to the fact that they provide ways and means to link classical function theory to functional analysis and operator theory. For general background on composition operators, we refer to [**4, 10**] and the references therein.

The main goal of this work is to estimate the essential norm of the operator $I_{h,\varphi}^{(n)}$ on the Bergman space $\mathcal{A}^2$. Recall that the essential norm $||T||_e$ of a bounded linear operator on a Banach space X is given by

$$||T||_e = \inf\{||T-K|| : K \text{ is compact on } X\},$$

that is, its distance in the operator norm from the space of compact operators on X. The essential norm provides a measure of non-compactness of T. Clearly T is compact if and only if $||T||_e = 0$.

2. Boundedness of $I_{h,\varphi}^{(n)}$. In what follows, we make extensive use of Carleson measure techniques, so we give a short introduction to Carleson sets and Carleson measures first. For $b \in \partial\mathbf{D}$ and $0 < \delta < 1$, let $S(b, \delta)$ denote the Carleson set:

$$S(b, \delta) = \{z \in \mathbf{D} : |z - b| < \delta\}.$$

We need a special case of the Luecking's result [**6**] for $p = q = 2$ and $\alpha = 0$ in which he characterized positive measures μ with the property

$$||f^{(n)}||_{L^2(\mu)} \leq C||f||_{\mathcal{A}^2}.$$

Theorem 2.1. *Let $n \in \mathbf{N} \cup \{0\}$. Then for a positive Borel measure μ on $\mathbf{D}$ the following are equivalent*:

(1) *There is a constant $C_1 > 0$ such that, for $b \in \partial\mathbf{D}$ and $0 < \delta < 1$,*

$$\mu(S(b,\delta)) \leq C_1 \delta^{2(1+n)}.$$

(2) *There is a constant $C_2 > 0$ such that, for every $f \in \mathcal{A}^2$,*

$$\int_{\mathbf{D}} |f^{(n)}(w)|^2 d\mu(w) \leq C_2||f||_{\mathcal{A}^2}.$$

(3) *There is a constant $C_3 > 0$ such that, for every $z \in \mathbf{D}$,*

$$\int_{\mathbf{D}} |\beta_z^{(1+n)}(w)|^2 d\mu(w) \leq C_3.$$

A positive Borel measure μ which satisfies the above equivalent conditions is called a $2(n+1)$-Carleson measure for the Bergman space $\mathcal{A}^2$. If we define

$$||\mu|| = \sup_{\delta > 0} \sup_{b \in \partial\mathbf{D}} \frac{\mu(S(b,\delta))}{\delta^{2(n+1)}},$$

then $||\mu||$ and the constants in Theorem 2.1 are comparable.

A positive Borel measure μ on $\mathbf{D}$ is called a vanishing $2(n+1)$-Carleson measure if

$$\lim_{\delta \to 0} \sup_{b \in \partial\mathbf{D}} \frac{\mu(S(b,\delta))}{\delta^{2(n+1)}} = 0.$$

The following result completely characterizes the bounded generalized integration operator $I_{h,\varphi}^{(n)}$ on the Bergman space.

Theorem 2.2. *Let h and φ be holomorphic maps of $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$. Then the following are equivalent*:

(1) *$I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$.*

(2) *The pull-back measure $\mu_{h,\varphi} = \nu_h \circ \varphi^{-1}$ of ν_h induced by φ is a $2(n+1)$-Carleson measure. Here $d\nu_h(z) = (1-|z|^2)^2|h(z)|^2 dA(z)$.*

(3) $\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} [(1-|z|^2)^2(1-|w|^2)^2]/|1-\overline{z}\varphi(w)|^{2(n+2)}|h(w)|^2 dA(w) < \infty$.

Proof. Since $I_{h,\varphi}^{(n)} f(0) = 0$ for all $n \geq 0$, so by (1.3) we have

$$\begin{aligned} ||I_{h,\varphi}^{(n)} f||_{\mathcal{A}^2}^2 &\approx \int_{\mathbf{D}} |(I_{h,\varphi}^{(n)} f)'(z)|^2 (1-|z|^2)^2 dA(z) \\ &= \int_{\mathbf{D}} |h(z)|^2 |f^{(n)}(\varphi(z))|^2 (1-|z|^2)^2 dA(z). \end{aligned}$$

Thus, by definition, $I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$ if and only if there is a constant $C > 0$ such that for any $f \in \mathcal{A}^2$,

$$\int_{\mathbf{D}} |f^{(n)}(\varphi(z))|^2 |h(z)|^2 (1-|z|^2)^2 dA(z) \leq C||f||_{\mathcal{A}^2}^2.$$

Let $d\nu_h(z) = (1-|z|^2)^2|h(z)|^2 dA(z)$ and $\mu_{h,\varphi} = \nu_h \circ \varphi^{-1}$ be the pull-back measure of ν_h induced by φ. If we change the variable $w = \varphi(z)$, then we get

$$\begin{aligned} &\int_{\mathbf{D}} |f^{(n)}(w)|^2 d\mu_{h,\varphi}(w) \\ &\qquad = \int_{\mathbf{D}} |f^{(n)}(\varphi(z))|^2 d\nu_h(z) \\ &\qquad = \int_{\mathbf{D}} |f^{(n)}(\varphi(z))|^2 |h(z)|^2 (1-|z|^2)^2 dA(z). \end{aligned}$$

Thus, by Theorem 2.1, (1) is equivalent to

$$\int_{\mathbf{D}} |f^{(n)}(w)|^2 d\mu_{h,\varphi}(w) \leq C||f||_{\mathcal{A}^2}^2.$$

Hence (1) and (2) are equivalent. Again, by Theorem 2.1, the condition that $\mu_{h,\varphi}$ is a $2(n+1)$-Carleson measure is equivalent to

$$\int_{\mathbf{D}} |\beta_z^{(n+1)}(w)|^2 d\mu_{h,\varphi}(w) \leq C.$$

Changing the variable, we get

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^2|z|^{2n}(1-|w|^2)^2}{|1-\overline{z}\varphi(w)|^{2(n+2)}}|h(w)|^2 dA(w) < \infty. \tag{2.1}$$

Thus, (1), (2) and (2.1) are equivalent. Clearly, (3) $\Rightarrow$ (2.1). Suppose that (2.1) holds. Then, for any $0 < r_0 < 1$, we have

$$\sup_{r_0<|z|<1} \int_{\mathbf{D}} \frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\overline{z}\varphi(w)|^{2(n+2)}}|h(w)|^2 dA(w) < \infty. \tag{2.2}$$

By (1), $I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$. Thus by taking $f(z) = z^n/n!$ in $\mathcal{A}^2$, we get

$$\int_{\mathbf{D}} (1-|w|^2)^2|h(w)|^2 dA(w) \leq C$$

and so

$$\begin{aligned} \sup_{0\leq|z|\leq r_0} \int_{\mathbf{D}} & \frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\overline{z}\varphi(w)|^{2(n+2)}}|h(w)|^2 dA(w) \\ & \leq \frac{1}{(1-r_0)^{2(n+2)}} \int_{\mathbf{D}} (1-|w|^2)^2|h(w)|^2 dA(w). \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we have (2.1) $\Rightarrow$ (3). □

Corollary 2.3. *Let h and φ be holomorphic maps of $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$. If $I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$, then*

$$\sup_{z\in\mathbf{D}} \frac{(1-|z|^2)^2}{(1-|\varphi(z)|^2)^{n+1}}|h(z)| < \infty.$$

Proof. Suppose that $I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$. By Theorem 2.2, we have

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\overline{z}\varphi(w)|^{2(n+2)}}|h(w)|^2 dA(w) < \infty.$$

In particular,

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}\frac{(1-|\varphi(a)|^2)^2(1-|w|^2)^2}{|1-\overline{\varphi(a)}\varphi(w)|^{2(n+2)}}|h(w)|^2dA(w)<\infty. \tag{2.4}$$

For a fixed $a\in\mathbf{D}$, let $\Omega(a)=\{z\in\mathbf{D}:|z-a|\le(1-|a|^2)/2\}$. Then $\Omega(a)\subset\mathbf{D}$. By subharmonicity of the function $|h(w)|^2/|1-\overline{\varphi(a)}\varphi(w)|^{2(n+2)}$, we get

$$\begin{aligned}\frac{|h(a)|^2}{(1-|\varphi(a)|^2)^{2(n+1)}}&\le\frac{C}{(1-|a|^2)^2}\\&\quad\times\int_{\Omega(a)}\frac{(1-|\varphi(a)|^2)^2}{|1-\overline{\varphi(a)}\varphi(w)|^{2(n+2)}}|h(w)|^2dA(w)\\&\le\frac{CC'}{(1-|a|^2)^4}\\&\quad\times\int_{\Omega(a)}\frac{|h(w)|^2(1-|\varphi(a)|^2)^2(1-|w|^2)^2}{|1-\overline{\varphi(a)}\varphi(w)|^{2(n+2)}}dA(w).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{(1-|a|^2)^4|h(a)|^2}{(1-|\varphi(a)|^2)^{2(n+1)}}\\&\qquad\le CC'\int_{\Omega(a)}\frac{(1-|\varphi(a)|^2)^2(1-|w|^2)^2}{|1-\overline{\varphi(a)}\varphi(w)|^{2(n+2)}}|h(a)|^2dA(w).\end{aligned} \tag{2.5}$$

The result follows by (2.4) and (2.5). □

It seems that the result for the boundedness of the operator $I_h^{(n)}$ on the Bergman space $\mathcal{A}^2$ has not appeared in the literature. Therefore, we single it out as a corollary. In its formulation, we write $\mathcal{A}_\infty^{-1}$ for the space of holomorphic functions f on $\mathbf{D}$ for which

$$\sup_{z\in\mathbf{D}}(1-|z|^2)|f(z)|<\infty.$$

Corollary 2.4. *Let h be a holomorphic map of $\mathbf{D}$. Then $I_h^{(n)}$ is bounded on $\mathcal{A}^2$ if and only if $h\in X$, where*

$$X=\begin{cases}\mathcal{A}_\infty^{-1} & \textit{if } n=0\\ H^\infty & \textit{if } n=1\\ \{0\} & \textit{if } n\ge 2.\end{cases}$$

Proof. First suppose that $I_h^{(n)}$ is bounded on $\mathcal{A}^2$. Then, by taking $\varphi(z) = z$ in Corollary 2.3, we have

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^{1-n}|h(z)| < \infty. \tag{2.6}$$

Thus, for $n = 0$, $h \in \mathcal{A}_\infty^{-1}$ and for $n = 1$, $h \in H^\infty$. Again if $n \geq 2$, then (2.6) implies that there is a positive constant C such that $|h(z)| \leq C(1-|z|^2)^{n-1}$ for all $z \in \mathbf{D}$. It follows that $|h(z)| \to 0$ as $|z| \to 1^-$, so by the maximum modulus theorem, we have $h \equiv 0$. Conversely, suppose that $h \in X$. Now, if $n \geq 2$, then the result is obvious. If $n = 0$, then

$$\begin{aligned}\int_{\mathbf{D}} \frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\overline{z}w|^4}|h(w)|^2 dA(w) & \\ &\leq ||h||^2_{\mathcal{A}_\infty^{-1}} \int_{\mathbf{D}} \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} dA(w) \\ &= ||h||^2_{\mathcal{A}_\infty^{-1}},\end{aligned}$$

and so by Theorem 2.2, $I_h^{(n)}$ is bounded on $\mathcal{A}^2$. If $n = 1$, then once again the proof follows by taking $\varphi(z) = z$ in Corollary 2.3 and Theorem 2.2. We omit the details. □

3. Essential norm. For holomorphic maps h and φ of $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$, define $\Lambda_h^\varphi(a)$ as

$$\Lambda_h^\varphi(a) = \int_{\mathbf{D}} \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}\varphi(z)|^{2(n+2)}}|h(z)|^2 dA(z).$$

Theorem 3.1. *Let h and φ be holomorphic maps of $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$. Let $I_{h,\varphi}^{(n)}$ be bounded on $\mathcal{A}^2$. Then there are positive constants C_1 and C_2 such that*

$$C_1 \limsup_{|a|\to 1} \Lambda_h^\varphi(a) \leq ||I_{h,\varphi}^{(n)}||_e^2 \leq C_2 \limsup_{|a|\to 1} \Lambda_h^\varphi(a).$$

In order to prove Theorem 3.1, we need several lemmas.

Lemma 3.2. *Let $1/2 < r < 1$, $n \in \mathbf{N} \cup \{0\}$ and $D(0,r) = \{z \in \mathbf{D} : |z| < r\}$. Let*

$$M_r^* = \sup_{|a| \geq r} \int_{\mathbf{D}} |\beta_a^{(n+1)}(z)|^2 d\mu(z).$$

Then, if μ is a $2(n+1)$-Carleson measure for the Bergman space $\mathcal{A}^2$, so is $\widetilde{\mu}_r = \mu|_{\mathbf{D}\setminus D(0,r)}$. Moreover, $||\widetilde{\mu}_r|| \leq NM_r^*$, where N is a constant depending upon n only.

Proof. Let

$$M_r = \sup_{\delta \leq 1-r} \sup_{b \in \partial\mathbf{D}} \frac{\mu(S(b,\delta))}{\delta^{2(1+n)}}.$$

For $b \in \partial\mathbf{D}$ and $0 < \delta < 1$, take any $S(b,\delta)$. Suppose that $\delta = \eta(1-r)$ for some constant η. If $0 < \eta \leq 1$, then $S(b,\delta) \subset \mathbf{D} \setminus D(0,r)$, and so

$$\widetilde{\mu}_r(S(b,\delta)) = \mu(S(b,\delta)) \leq M_r \delta^{2(1+n)}.$$

If $\eta > 1$, then $[\eta] + 1 > \eta$ and $[\eta] + 1 < 2\eta$, where $[\eta]$ is the greatest integer that is less than or equal to η. Let $m = [\eta] + 1$. In this case it is possible to cover $S(b,\delta)$ by $S(b_1,\delta_1)$, $S(b_2,\delta_2), \ldots, S(b_m,\delta_m)$ such that $\delta_k = (1-r)$, $k = 1, 2, \ldots, m$. Thus

$$\begin{aligned}
\widetilde{\mu}_r(S(b,\delta)) &= \mu(S(b,\delta) \cap (\mathbf{D} \setminus D(0,r))) \\
&\leq \sum_{k=1}^{m} \mu(S(b_k,\delta_k)) \\
&\leq M_r \sum_{k=1}^{m} \delta_k^{2(1+n)} \\
&\leq M_r \left(\sum_{k=1}^{m} \delta_k \right)^{2(1+n)} \\
&= M_r (m(1-r))^{2(1+n)} \\
&= M_r ([\eta]+1)^{2(1+n)} (1-r)^{2(1+n)} \\
&\leq M_r (2\eta)^{2(1+n)} (1-r)^{2(1+n)} \\
&= M_r 2^{2(1+n)} \delta^{2(1+n)}.
\end{aligned}$$

This implies that $||\widetilde{\mu}_r|| \leq 2^{2(1+n)} M_r$. Thus, to complete the proof, we just need to prove that $M_r \leq NM_r^*$, where $N > 0$ is a constant depending upon n only. Take $\delta \le 1-r$. Let $a=(1-\delta)e^{i\theta}$. Then $|a| = 1-\delta \ge r$. Again

$$\begin{aligned}
\frac{(1-|a|^2)|a|^n}{|1-\overline{a}z|^{2+n}} &\ge \Re\bigg(\frac{(1-|a|^2)|a|^n}{(1-\overline{a}z)^{2+n}}\bigg) \\
&= \frac{(1-|a|^2)|a|^n}{(1-|a|)^{2+n}}\Re\bigg(\frac{1-|a|}{1-\overline{a}z}\bigg)^{2+n} \\
&= \frac{(1-|a|^2)|a|^n}{(1-|a|)^{2+n}}\Re\bigg(1+\frac{|a|(1-z\overline{\zeta})}{(1-|a|)}\bigg)^{-(2+n)} \\
&> \frac{1}{(2)^{(2+n)/2}}\frac{(1-|a|^2)|a|^n}{(1-|a|)^{2+n}} \\
&\ge \frac{1}{2^{(3n+2)/2}\delta^{1+n}}, \qquad \bigg(\zeta = \frac{a}{|a|}\bigg)
\end{aligned}$$

if $|1-z\overline{\zeta}|/(1-|a|) < \gamma_0$ for some fixed $0 < \gamma_0 < 1/4$, that is, if $z \in S(b, \gamma_0\delta)$. Thus

$$\begin{aligned}
\frac{\mu(S(b,\delta))}{\delta^{2(1+n)}} &\le 2^{3n+2}\int_{S(b,\gamma_0\delta)} |\beta_a^{(n+1)}(z)|^2 d\mu(z) \\
&\le 2^{3n+2}\int_{\mathbf{D}} |\beta_a^{(n+1)}(z)|^2 d\mu(z) \\
&\le 2^{3n+2} M_r^*.
\end{aligned}$$

Taking the supremum over all δ with $\delta \le 1-r$, we get $M_r \le 2^{3n+2}CM_r^*$. □

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be holomorphic on $\mathbf{D}$, $Q_m f(z) = \sum_{k=0}^{m-1} a_k z^k$ and $R_m f(z) = I - Q_m$, where I is the identity map. Then R_m is the orthogonal projection of $\mathcal{A}^2$ onto $z^m\mathcal{A}^2$ and $R_m f(z) = \sum_{k=m}^{\infty} a_k z^k$.

We need the following lemma which generalizes Proposition 3.15 of [**4**]. Though it can be easily extended to weighted Hardy spaces considered by Cowen and MacCluer [**4**, page 133], we state it for the Bergman space $\mathcal{A}^2$ only.

Lemma 3.3. *For each r, $0 < r < 1$ and $f \in \mathcal{A}^2$, we have*

$$|(R_m f)^n(z)| \leq ||f||_{\mathcal{A}^2} \left(\sum_{k=\max\{m,n\}}^{\infty} \left(\frac{k!}{(k-n)!} \right)^2 r^{2(k-n)}(k+1) \right)^{1/2}$$

for $|z| \leq r$.

Proof. Let n be a non-negative integer. Then for each $z \in \mathbf{D}$, the evaluation of nth derivative of functions in $\mathcal{A}^2$ at z is a bounded linear functional and $f^{(n)}(z) = \langle f, K_z^{(n)} \rangle$, where

$$\begin{aligned} K_z^{(n)}(w) &= \sum_{k=n}^{\infty} \left(\frac{k!}{(k-n)!} \right)^2 (\overline{z})^{(k-n)} (\overline{w})^k (k+1). \\ ||K_z^{(n)}||_{\mathcal{A}^2}^2 &= \left\| \sum_{k=n}^{\infty} \left(\frac{k!}{(k-n)!} \right)^2 (\overline{z})^{(k-n)} (\overline{w})^k (k+1) \right\|_{\mathcal{A}^2} \\ &= \sum_{k=n}^{\infty} \left| \left(\frac{k!}{(k-n)!} \right)^2 (\overline{z})^{(k-n)} (k+1) \right|^2 \frac{1}{(k+1)} \\ &= \sum_{k=n}^{\infty} \left(\frac{k!}{(k-n)!} \right)^2 |z|^{2(k-n)} (k+1). \end{aligned}$$

Thus

$$\begin{aligned} |(R_m f)^n(z)| &= \left| \left\langle R_m f, K_z^{(n)} \right\rangle \right| \\ &= \left| \left\langle f, R_m K_z^{(n)} \right\rangle \right| \\ &\leq ||f||_{\mathcal{A}^2} ||K_z^{(n)}||_{\mathcal{A}^2} \\ &\leq ||f||_{\mathcal{A}^2} \left(\sum_{k=\max\{m,n\}}^{\infty} \left(\frac{k!}{(k-n)!} \right)^2 r^{2(k-n)}(k+1) \right)^{1/2}. \end{aligned}$$

Finally, we need the following lemma of Cowen and MacCluer [**4**, page 134] to estimate the essential norm of $I_{h,\varphi}^{(n)}$.

Lemma 3.4. *Let T be a bounded operator on $\mathcal{A}^2$. Then*

$$||T||_e = \lim_{m \to \infty} ||TR_m||.$$

Proof of Theorem 3.1. *Upper bound.* By Lemma 3.4,

$$||I_{h,\varphi}^{(n)}||_e^2 = \lim_{m\to\infty} ||I_{h,\varphi}^{(n)} R_m||_e^2 = \lim_{m\to\infty} \sup_{||f||_{\mathcal{A}^2}\le 1} ||(I_{h,\varphi}^{(n)} R_m)f||_{\mathcal{A}^2}^2.$$

Thus by (1.3),

$$\begin{aligned}||(I_{h,\varphi}^{(n)} R_m)f||_{\mathcal{A}^2}^2 &\approx \int_{\mathbf{D}} |h(z)|^2 |(R_m f)^{(n)}(\varphi(z))|^2 (1-|z|^2)^2 dA(z)\\ &= \int_{\mathbf{D}} |(R_m f)^{(n)}(z)|^2 d\mu_{h,\varphi}(z)\\ &= \int_{\mathbf{D}\setminus D(0,r)} |(R_m f)^{(n)}(z)|^2 d\mu_{h,\varphi}(z)\\ &\quad + \int_{D(0,r)} |(R_m f)^{(n)}(z)|^2 d\mu_{h,\varphi}(z)\\ &= I_1 + I_2.\end{aligned}$$

Since $I_{h,\varphi}^{(n)}$ is bounded on $\mathcal{A}^2$, $d\mu_{h,\varphi}$ is a $2(n+1)$-Carleson measure for the Bergman space. So

$$\begin{aligned}I_2 &\le \sup_{|z|\le r} |(R_m f)^{(n)}(z)|^2 \int_{D(0,r)} d\mu_{h,\varphi}(z)\\ &\le ||f||_{\mathcal{A}^2}^2 \bigg(\sum_{k=\max\{m,n\}}^{\infty} \bigg(\frac{k!}{(k-n)!}\bigg)^2 r^{2(k-n)}(k+1)\bigg).\end{aligned}$$

Thus for fixed r as $m\to\infty$, we have $\sup_{||f||\le 1} I_2 \to 0$. On the other hand, if we denote by $\widetilde{\mu_{h,\varphi_r}} = \mu_{h,\varphi}|_{\mathbf{D}\setminus D(0,r)}$, then by Lemma 3.2 we have

$$I_1 \le \int_{\mathbf{D}\setminus D(0,r)} |(R_n f)^{(n)}(z)|^2 d\mu_{h,\varphi}(z) \le \lim_{n\to\infty} KNM_r^* = KNM_r^*.$$

Therefore,

$$\begin{aligned}||I_{h,\varphi}^{(n)} R_m||_e^2 &\le KN \lim_{n\to\infty} M_r^*\\ &= KN \limsup_{|a|\to 1} \int_{\mathbf{D}} |\beta_a^{(n+1)}(z)|^2 d\mu_{h,\varphi}(z)\\ &= KN \limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}\varphi(z)|^{2(n+2)}} |h(z)|^2 dA(z),\end{aligned}$$

which gives the desired upper bound.

Lower bound. Consider the normalized kernel function $k_a(z) = 1-|a|^2/(1-\overline{a}z)^2$. Then $||k_a||_{\mathcal{A}^2} = 1$ and $k_a \to 0$ uniformly on compact subsets of $\mathbf{D}$ as $|a| \to 1$. Fix a compact operator K on $\mathcal{A}^2$. Then $||Kk_a||_{\mathcal{A}^2} \to 0$ as $|a| \to 1$. Therefore,

$$\begin{aligned}||I_{h,\varphi}^{(n)} - K|| &\geq \limsup_{|a|\to 1} ||(I_{h,\varphi}^{(n)} - K)k_a||_{\mathcal{A}^2} \\ &\geq \limsup_{|a|\to 1} (||I_{h,\varphi}^{(n)} k_a||_{\mathcal{A}^2} - ||Kk_a||_{\mathcal{A}^2}) \\ &\geq C \limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}\varphi(z)|^{2(n+2)}} |h(z)|^2 dA(z).\end{aligned}$$

Thus,

$$\begin{aligned}||I_{h,\varphi}^{(n)}||_e^2 &\geq ||I_{h,\varphi}^{(n)} - K|| \\ &\geq C \limsup_{|a|\to 1} \int_{\mathbf{D}} \frac{(1-|a|^2)^2(1-|z|^2)^2}{|1-\overline{a}\varphi(z)|^{2(n+2)}} |h(z)|^2 dA(z). \qquad \square\end{aligned}$$

Routine calculations yield the following corollary.

Corollary 3.5. *Let h and φ be holomorphic maps of $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$. Then the following are equivalent*:

(1) $I_{h,\varphi}^{(n)}$ *is compact on $\mathcal{A}^2$.*

(2) *The pull-back measure $\mu_{h,\varphi} = \nu_h \circ \varphi^{-1}$ of ν_h induced by φ is a vanishing $2(n+1)$-Carleson measure. Here $d\nu_h(z) = (1-|z|^2)^2|h(z)|^2 dA(z)$.*

(3) $\lim_{|z|\to 1} \int_{\mathbf{D}} \frac{(1-|z|^2)^2(1-|w|^2)^2}{|1-\overline{z}\varphi(w)|^{2(n+2)}} |h(w)|^2 dA(w) = 0.$

Acknowledgments. We are thankful to the referee for several valuable comments and suggestions, which improved the manuscript a lot.

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