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COASSOCIATED PRIMES OF LOCAL HOMOLOGY AND LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ an ideal of R and A an Artinian R-module. Let t be a positive integer such that the local homology module $H_i^{\mathfrak{a}}(A)$ is Artinian for all $i < t$. Then $\operatorname{Tor}_j^R(R/\mathfrak{a}, H_t^{\mathfrak{a}}(A))$ is Artinian for $j = 0, 1$. In particular, the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_t^{\mathfrak{a}}(A))$ is finite, where $\operatorname{V}(\mathfrak{a})$ denotes the set of all prime ideals of R containing $\mathfrak{a}$. Moreover, we show that whenever $\operatorname{Cosupp}(H_i^{\mathfrak{a}}(A))$ is finite for all $i < t$, then the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_t^{\mathfrak{a}}(A))$ is finite. Also, for a finitely generated module M, we show that $R/\mathfrak{a} \otimes_R H_{\mathfrak{a}}^t(M)$ is Artinian whenever the local cohomology module $H_{\mathfrak{a}}^i(M)$ is Artinian for all $i > t$.

In particular, the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_{\mathfrak{a}}^t(M))$ is finite.

1. Introduction. Throughout this paper we assume that R is a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ is an ideal of R. We use M and A to denote a finitely generated and an Artinian R-module respectively. For each non-negative integer i, the ith local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $H_{\mathfrak{a}}^i(M)$. We refer the reader to [**2**] for the definition of local cohomology and its basic properties.

In [**8**], Huneke asked whether the number of associated prime ideals of a local cohomology module $H_{\mathfrak{a}}^i(M)$ is always finite. In [**20**], Singh has given an example of Noetherian non-local ring R and an ideal $\mathfrak{a}$ such that $H_{\mathfrak{a}}^3(R)$ has infinitely many associated primes. More recently, in [**9**], Katzman constructed a hypersurface S and an ideal $\mathfrak{a}$ such that $H_{\mathfrak{a}}^2(S)$ has infinitely many associated primes (see also [**21**]). However, it is known that this conjecture is true in many situations. For example, Brodmann and Lashgari [**1**] showed that the first non finitely generated local cohomology module $H_{\mathfrak{a}}^i(M)$ with respect to an ideal $\mathfrak{a}$ has only finitely many associated primes. Also, Khashyarmanesh

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and Salarian [**11**] showed that if $\operatorname{Supp}(H^i_{\mathfrak{a}}(M))$ is finite for all $i < t$, then $\operatorname{Ass}(H^t_{\mathfrak{a}}(M))$ is finite. For some other work on this question, we refer the reader to [**6, 12, 13, 15**]. There have been some attempts to dualize the theory of local cohomology. First, Matlis in [**16, 17**] studied the left derived functors $L^{\mathfrak{a}}_{\bullet}(-)$ of the $\mathfrak{a}$-adic completion functor $\Lambda_{\mathfrak{a}}(-) = \varprojlim_n (R/\mathfrak{a}^n \otimes_R -)$, where the ideal $\mathfrak{a}$ was generated by a regular sequence in a local Noetherian ring R and proved some duality between this functor and the local cohomology functor by using a duality which is today called the Matlis dual functor. Next, Simon in [**19**] suggested investigating the module $L^{\mathfrak{a}}_i(N)$. Later, Greenlees and May in [**7**] use the homotopy colimit, or telescope, of the cochain of Koszul complexes to define so-called *local homology groups* of a module N (over a commutative ring R) by

$$H^{\mathfrak{a}}_{\bullet}(N) = H_{\bullet}(\operatorname{Hom}(\operatorname{Tel} \mathrm{K}^{\bullet}(\underline{x}^n), N)),$$

where $\underline{x}$ is a finitely generated system of $\mathfrak{a}$. Then they showed, under some condition on $\underline{x}$ which are automatically satisfied when R is Noetherian, that $H^{\mathfrak{a}}_{\bullet}(N) \cong L^{\mathfrak{a}}_{\bullet}(N)$. Note that a strong connection between local cohomology and local homology was shown in [**5, 7**]. Recently in [**4**], Cuong and Nam defined the ith local homology module $H^{\mathfrak{a}}_i(N)$ of an R-module N with respect to the ideal $\mathfrak{a}$ by

$$H^{\mathfrak{a}}_i(N) = \varprojlim_n \operatorname{Tor}^R_i(R/\mathfrak{a}^n, N).$$

(This definition slightly differs from that of Greenlees and May. However, both are the same for Artinian modules.) Cuong and Nam also proved that in [**4**] many basic properties of local homology modules and that $H^{\mathfrak{a}}_i(N) \cong L^{\mathfrak{a}}_i(N)$ when N is Artinian. Hence we can say that there exists, on the Noetherian local ring, a theory for the left derived functors $L^{\mathfrak{a}}_{\bullet}(-)$ of the $\mathfrak{a}$-adic completion (as the local homology functors) on the category of Artinian modules parallel to the theory of local cohomology functors on the category of Noetherian modules. However, while the local cohomology functors $H^{\bullet}_{\mathfrak{a}}(-)$ are still defined as the right derived functors of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-) = \varinjlim_n \operatorname{Hom}_R(R/\mathfrak{a}^n, -)$ for non finitely generated modules, the above definition of local homology module may not coincide with $L^{\mathfrak{a}}_i(N)$ in this case. To this end, we will use the definition of Cuong and Nam.

The concept of *coassociated prime* ideals, which is dual of associated prime ideals, was studied by Macdonald [**14**], Chambless [**3**], Zöschinger [**24**] and Yassemi [**23**]. However, these concepts are equivalent (see [**23**]). In [**23**], the concept of coassociated prime ideals is introduced in terms of *cocyclic* modules: an R-module L is cocyclic if it is a submodule of the injective envelope $E(R/\mathfrak{m})$ of $R/\mathfrak{m}$ for some maximal ideals $\mathfrak{m}$ of R. Then the coassociated prime ideals (respectively the *cosupport*) of an R-module N, denoted by $\operatorname{Coass}(N)$ (respectively $\operatorname{Cosupp}(N)$), to be the set of prime ideals $\mathfrak{p}$ of R such that there exists a cocyclic homomorphic image L of N such that $\operatorname{Ann}(L) = p$ (respectively $\operatorname{Ann}(L) \subseteq \mathfrak{p}$).

The associated prime ideals of local cohomology modules are studied in great detail; not so much is known about the coassociated prime ideals of local homology and local cohomology modules. The main objective of this paper is to show the following theorem.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a local ring, and let t be a positive integer.*

(i) *If $H_i^{\mathfrak{a}}(A)$ is Artinian for all $i < t$, then $\operatorname{Tor}_j^R(R/\mathfrak{a}, H_t^{\mathfrak{a}}(A))$ is Artinian for $j = 0, 1$. In particular, the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_t^{\mathfrak{a}}(A))$ is finite.*

(ii) *If $\operatorname{Cosupp}(H_i^{\mathfrak{a}}(A))$ is finite for all $i < t$, then the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_t^{\mathfrak{a}}(A))$ is finite.*

(iii) *If $H_{\mathfrak{a}}^i(M)$ is Artinian for all $i > t$, then $R/\mathfrak{a} \otimes_R H_{\mathfrak{a}}^t(M)$ is Artinian. In particular, $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_{\mathfrak{a}}^t(M))$ is finite.*

2. The results.

The following theorem is dual of [**1**, Theorem 2.2].

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a local ring, and let t be a non-negative integer such that $H_i^{\mathfrak{a}}(A)$ is Artinian for all $i < t$. Then $\operatorname{Tor}_j^R(R/\mathfrak{a}, H_t^{\mathfrak{a}}(A))$ is Artinian for $j = 0, 1$. In particular, the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_t^{\mathfrak{a}}(A))$ is finite.*

Proof. By using [**18**, Theorem 11.39], [**4**, Lemma 4.3] and [**19**, Corollary 2.4], there is a Grothendieck spectral sequence

$$E^2_{p,q} := \operatorname{Tor}_p^R(R/\mathfrak{a}, H_q^{\mathfrak{a}}(A)) \underset{p}{\Longrightarrow} \operatorname{Tor}_{p+q}^R(R/\mathfrak{a}, A).$$

Since $E^i_{p,q}$ is a subquotient of $E^2_{p,q}$ for all $i \geq 2$, our assumptions give us that $E^i_{p,q}$ is Artinian for all $i \geq 2$, $p \geq 0$, and $q < t$. For all $i \geq 2$ and $p = 0, 1$, we consider the exact sequence

$$0 \longrightarrow \operatorname{im} d^i_{p+i,t-i+1} \longrightarrow \ker d^i_{p,t} \longrightarrow E^{i+1}_{p,t} \longrightarrow 0, \ (\sharp)$$

where $\operatorname{im} d^i_{p+i,t-i+1} = \operatorname{im}(E^i_{p+i,t-i+1} \to E^i_{p,t})$ and $\ker d^i_{p,t} = \ker(E^i_{p,t} \to E^i_{p-i,t+i-1})$. Now, $\operatorname{im} d^i_{p+i,t-i+1}$ is Artinian for all $i \geq 2$, $E^\infty_{p,t}$ is isomorphic to a subquotient of $\operatorname{Tor}^R_{p+t}(R/\mathfrak{a}, A)$ and so is Artinian for all p and $E^i_{p,t} = \ker d^i_{p,t}$ for all $i \geq 2$ and $p = 0, 1$. Since $E^i_{p,t} \cong E^\infty_{p,t}$ for i sufficiently large, we have that $E^i_{p,t}$ is Artinian for all p and all large i. Fix i and suppose $E^{i+1}_{p,t}$ is Artinian for $p = 0, 1$. By using $(\sharp)$ with conjunction $E^i_{p,t} = \ker d^i_{p,t}$ for $p = 0, 1$, we see that $E^i_{p,t}$ is Artinian for $p = 0, 1$. Continuing in this fashion, we see that $E^i_{p,t}$ is Artinian for all $i \geq 2$ and $p = 0, 1$. In particular, $\operatorname{Tor}^R_j(R/\mathfrak{a}, H^\mathfrak{a}_t(A))$ is Artinian for $j = 0, 1$ and so, by [**23**, Theorem 1.21] the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H^\mathfrak{a}_t(A))$ is finite. □

A sequence of elements $x_1, \dots, x_n$ in R is said to be an A-coregular sequence (cf. [**22**]) if $0 :_A (x_1, \dots, x_n) \neq 0$ and $0 :_A (x_1, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_A (x_1, \dots, x_{i-1})$ is surjective for $i = 1, \dots, n$. We denote by $\operatorname{width}(\mathfrak{a}, A)$ the supremum of the lengths of all maximal A-coregular sequences in the ideal $\mathfrak{a}$.

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a local ring and $0 :_A \mathfrak{a} \neq 0$ such that $t = width(\mathfrak{a}, A)$. Then $\operatorname{Tor}^R_j(R/\mathfrak{a}, H^\mathfrak{a}_t(A))$ is Artinian and the set $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H^\mathfrak{a}_t(A))$ is finite.*

Proof. This follows from [**5**, Theorem 4.11] and Theorem 2.1. □

The following theorem is dual of [**11**, Theorem B].

Theorem 2.3. *Let t be a non-negative integer such that* Cosupp $(H^\mathfrak{a}_i(A))$ *is finite for all $i < t$. Then the set* $\operatorname{V}(\mathfrak{a}) \cap \operatorname{Coass}(H^\mathfrak{a}_t(A))$ *is finite.*

Proof. We proceed by induction on t. If $t = 0$, then $H^\mathfrak{a}_0(A)$ is Artinian and so there is nothing to prove. Suppose, inductively, that $t \geq 1$ and the result is true for all $i < t$. By [**4**, Corollary 4.5], we can replace A by $\cap_{n>0}\mathfrak{a}^n A$. The last module is just equal to $\mathfrak{a}^n A$ for n sufficiently large. Therefore we may assume that $\mathfrak{a}A = A$. Since A is Artinian, there is an element x in $\mathfrak{a}$ such that $xA = A$. From the exact sequence

$$0 \longrightarrow (0 :_A x) \longrightarrow A \xrightarrow{x} A \longrightarrow 0$$

and using [**4**, Corollary 4.2], we obtain the following long exact sequence

$$\cdots \longrightarrow H_t^{\mathfrak{a}}(A) \xrightarrow{x} H_t^{\mathfrak{a}}(A) \xrightarrow{g} H_{t-1}^{\mathfrak{a}}(0 :_A x) \xrightarrow{f} H_{t-1}^{\mathfrak{a}}(A) \xrightarrow{x} H_{t-1}^{\mathfrak{a}}(A) \longrightarrow \cdots .$$

By [**23**, Theorem 2.7] and our assumption, it can be seen that $\operatorname{Cosupp}(H_i^{\mathfrak{a}}(0 :_A x))$ is finite for all $i < t-1$. Now, by using inductive hypothesis, we have $\mathrm{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_{t-1}^{\mathfrak{a}}(0 :_A x))$ is finite. Furthermore, it is easy to see that

$$\operatorname{Coass}(\operatorname{im} g) \subseteq \operatorname{Coass}(H_{t-1}^{\mathfrak{a}}(0 :_A x)) \cup \operatorname{Cosupp}(H_{t-1}^{\mathfrak{a}}(A)).$$

It therefore follows that $\mathrm{V}(\mathfrak{a}) \cap \operatorname{Coass}(\operatorname{im} g)$ is finite and so, by using the isomorphism, $\frac{H_t^{\mathfrak{a}}(A)}{xH_t^{\mathfrak{a}}(A)} \cong \operatorname{im} g$ the result follows. $\square$

The following theorem is a generalization of [**10**, Corollary 2.3].

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a local ring, and let t be a positive integer such that $H_{\mathfrak{a}}^i(M)$ is Artinian for all $i > t$. Then $R/\mathfrak{a} \otimes H_{\mathfrak{a}}^t(M)$ is Artinian. In particular,* $\mathrm{V}(\mathfrak{a}) \cap \operatorname{Coass}(H_{\mathfrak{a}}^t(M))$ *is finite.*

Proof. We proceed by induction on $n = \dim M$. If $n = 0$, there is nothing to prove. Assume, inductively, that $n \geq 1$ and the result is settled for any finitely generated R-module with Krull dimension less than n. Since $H_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M))$ for all $i \geq 1$, we can assume that M is an $\mathfrak{a}$-torsion free R-module. Therefore $\mathfrak{a}$ contains an element x which is a non-zero divisor on M. Now, the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces the long exact sequence

$$\cdots \longrightarrow H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M) \xrightarrow{f} H^t_{\mathfrak{a}}(M/xM) \xrightarrow{g} H^{t+1}_{\mathfrak{a}}(M) \longrightarrow \cdots .$$

It can be seen that $H^i_{\mathfrak{a}}(M/xM)$ is Artinian for all $i > t$. Since $\dim M/xM < n$, by inductive hypothesis, $R/\mathfrak{a} \otimes H^t_{\mathfrak{a}}(M/xM)$ is Artinian. Furthermore, by hypothesis, $H^{t+1}_{\mathfrak{a}}(M)$ is Artinian and so is $\operatorname{im} g$. By the above long exact sequence, we have two exact sequences

$$0 \longrightarrow \operatorname{im} f \longrightarrow H^t_{\mathfrak{a}}(M/xM) \longrightarrow \operatorname{im} g \longrightarrow 0$$

and

$$H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M) \longrightarrow \operatorname{im} f \longrightarrow 0,$$

and applying the right exact functor $R/\mathfrak{a} \otimes_R -$ on them, we deduce the exact sequence

$$\begin{aligned}\operatorname{Tor}^R_1(R/\mathfrak{a}, \operatorname{im} g) \longrightarrow R/\mathfrak{a} \otimes_R \operatorname{im} f \longrightarrow R/\mathfrak{a} \otimes_R H^t_{\mathfrak{a}}(M/xM)\\ \longrightarrow R/\mathfrak{a} \otimes_R \operatorname{im} g \longrightarrow 0\end{aligned}$$

and an isomorphism $R/\mathfrak{a} \otimes_R H^t_{\mathfrak{a}}(M) \cong R/\mathfrak{a} \otimes_R \operatorname{im} f$. It therefore follows $R/\mathfrak{a} \otimes H^t_{\mathfrak{a}}(M)$ is Artinian and, by [**23**, Theorem 1.21], $\mathrm{V}(\mathfrak{a}) \cap \operatorname{Coass}(H^t_{\mathfrak{a}}(M))$ is finite, as required. □

Corollary 2.5. *Let M be a finitely generated R-module of dimension n. Then $R/\mathfrak{a} \otimes_R H^{n-1}_{\mathfrak{a}}(M)$ is Artinian. In particular,* $\operatorname{Coass}(H^{n-1}_{\mathfrak{a}}(M)) \cap \mathrm{V}(\mathfrak{a})$ *is finite.*

Proof. This follows from [**2**, Exercise 7.1.7] and Theorem 2.4. □

Corollary 2.6. *Let M be a finitely generated R-module of finite Krull dimension. Then $R/\mathfrak{a} \otimes_R H^t_{\mathfrak{a}}(M)$ is Artinian. In particular,* $\mathrm{V}(\mathfrak{a}) \cap \operatorname{Coass}(H^t_{\mathfrak{a}}(M))$ *is finite, where $t = ara(\mathfrak{a})$.*

Proof. This follows from [**2**, Corollary 3.3.3] and Theorem 2.4. □

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REFERENCES

1. M.P. Brodmann and F.A. Lashgari, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (2000), 2851–2853.

2. M.P. Brodmann and R.Y. Sharp, *Local cohomology–An algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.

3. L. Chambless, *Coprimary decompositions, N-dimension and divisibility*: *Application to Artinian modules*, Comm. Algebra **9** (1981), 1131–1146.

4. N.T. Cuong and T.T. Nam, *The I-adic completion and local homology for Artinian modules*, Math. Proc. Camb. Phil. Soc. **131** (2001), 61–72.

5. ———, *A local homology theory for lineary compact modules*, J. Algebra **319** (2008), 4712–4737.

6. K. Divaani-Aazar and A. Mafi, *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **133** (2005), 655–660.

7. J.P.C. Greenlees and J.P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), 438–453.

8. C. Huneke, *Problems on local cohomology*, *Free resolutions in commutative algebra and algebraic geometry*, Res. Notes Math. **2** (1992), 93–108.

9. M. Katzman, *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra **252** (2002), 161–166.

10. K. Khashyarmanesh, *On the coassociated primes of generalized local cohomology modules*, Indian J. Pure Appl. Math. **37** (2006), 247–253.

11. K. Khashyarmanesh and Sh. Salarian, *On the associated primes of local cohomology modules*, Comm. Algebra **27** (1999), 6191–6198.

12. G. Lyubeznik, *Finiteness properties of local cohomology modules* (*An application of D-modules to commutative algebra*), Inv. Math. **113** (1993), 41–55.

13. ———, *Finiteness properties of local cohomology modules: a characteristic-free*, J. Pure Appl. Algebra **151** (2000), 43–50.

14. I.G. Macdonald, *Secondary representation of modules over a commutative ring*, Sympos. Math. **11** (1973), 23–43.

15. A. Mafi and H. Zakeri, *A note on local cohomology*, Algebra Colloquium **15** (2008), 97–100.

16. E. Matlis, *The Koszul complex and duality*, Comm. Algebra **1** (1974), 87–144.

17. ———, *The higher properties of R-sequences*, J. Algebra **50** (1978), 77–112.

18. J. Rotman, *Introduction to homological algebra*, Academic Press, New York, 1979.

19. A.M. Simon, *Some homological properties of complete modules*, Math. Proc. Camb. Phil. Soc. **108** (1990), 231–246.

20. A.K. Singh, *p-torsion elements in local cohomology modules*, Math. Res. Lett. **7** (2000), 165–176.

21. A.K. Singh and I. Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, International Mathematics Research Notices **33** (2004), 1703–1733.

22. Z. Tang, *Local homology theory for Artinian Modules*, Comm. Algebra **22** (1994), 1675–1684.

23. S. Yassemi, *Coassociated primes*, Comm. Algebra **23** (1995), 1473–1498.

24. H. Zöschinger, *Linear-kompakte moduln ber Noetherschen ringen*, Arch. Math. (Basel) **41** (1983), 121–130.

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