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BOUNDARY PRESERVATION BY ANALYTIC MAPS BETWEEN BORDERED RIEMANN SURFACES

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ABSTRACT. This note shows that an analytic map from one compact bordered Riemann surface onto another must map the boundary of the domain surface onto the boundary of the image. To establish this result, we use a certain rigidity property of analytic arcs. In light of this discussion, we note that the Riemann-Hurwitz formula for an analytic map between bordered Riemann surfaces hold when the condition of boundary preservation is replaced by the condition that the map in question be surjective.

1. Introduction. Let X_1 and X_2 be compact bordered Riemann surfaces. Suppose f is an analytic map mapping X_1 onto X_2. By the open mapping theorem, any point in the interior $\overset{\circ}{X}_1$ of X_1 must be mapped into the interior $\overset{\circ}{X}_2$ of X_2. In this note, we show that f must map the boundary ∂X_1 of X_1 onto the boundary ∂X_2 of X_2.

Applying this result to the special case in which the two bordered Riemann surfaces are subsets of the Riemann sphere Σ, we conclude that a meromorphic function from one compact subset of Σ with analytic boundary onto another must map the boundary of the domain onto the boundary of the image. This, in particular, allows a complete description of the family of analytic functions mapping the closed unit disc *onto* itself (*without* the assumption that the maps preserve the unit circle).

We also note that, in the Riemann-Hurwitz formula for an analytic map between bordered Riemann surfaces, the hypothesis that the map in question preserve boundary can be replaced by the condition that the map be surjective.

The proof of our main result will draw upon a certain rigidity property of analytic arcs, which we first establish in Section 2. We then prove

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the main theorem in Section 3 and conclude with a discussion of some consequences of this result.

Throughout this note, we let I denote the unit interval $[0,1]$, S the unit circle, D the open unit disc, H the open upper half plane, and Σ the Riemann sphere $\mathbf{C} \cup \{\infty\}$.

2. Rigidity of analytic arcs. We make some observations regarding analytic arcs which will be essential for the proof of our main theorem.

Let X be a Riemann surface and $\gamma : J \to X$ (J being a real interval) be an analytic arc. For convenience, we denote by $[\gamma]$ the image set $\gamma[J]$, i.e., the trajectory of γ. The analytic arc $\gamma : J \to X$ is said to be *regular* if $\gamma'(t) \neq 0$ for all $t \in J$.

It is known that the trajectory $[\gamma]$ of a *regular* analytic arc γ is a rigid object, in the sense that any arbitrarily small portion of it determines the unique maximal regular analytic arc containing $[\gamma]$ as a subarc; see [**3**]. (This rigidity is not the same as uniqueness of analytic continuation, as a given curve may have two completely different parametrizations that are not continuations of each other in any way. For example, both $\alpha(t) = t + i\sqrt{1-t^2}$ and $\beta(t) = ie^{i\pi t/2}$ for $t \in (-1,1)$ parametrize the upper semicircle $S \cap H$, but α and β are not related at all by analytic continuation.)

We show a variant of the general rigidity result under a somewhat different hypothesis. We begin with a lemma.

Lemma 2.1. *Let $\gamma : I \to \mathbf{C}$ be an analytic arc* (not necessarily regular)*. If $[\gamma] \cap \mathbf{R}$ is infinite, then $[\gamma] \subset \mathbf{R}$.*

Proof. Consider $\operatorname{Im} \gamma(t)$, which is *real analytic* on I. The infinitude of $[\gamma] \cap \mathbf{R}$ implies that $\operatorname{Im} \gamma$ vanishes infinitely often on I. By compactness of I, we conclude that $\operatorname{Im} \gamma \equiv 0$ on I, i.e., $\gamma(t) \in \mathbf{R}$ for $t \in I$. □

Remark 2.2. The following "naïve" generalization of Lemma 2.1 is false: "Given γ an analytic arc (not necessarily regular) and δ a *maximal regular* analytic arc on a Riemann surface X, if $[\gamma] \cap [\delta]$ is infinite, then $[\gamma] \subset [\delta]$." Consider the following two simple counter-examples:

1. For all $t \in \mathbf{R}$, let $\gamma(t) = t + i \sin t$, regarded as an arc on the Riemann sphere Σ. Let $C = \mathbf{R} \cup \{\infty\}$, which is a great circle on Σ and the trajectory of a maximal regular analytic arc. Clearly $[\gamma] \nsubseteq C$, despite the fact that $[\gamma] \cap C$ is infinite.

2. Let X be the standard torus $\mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$. For all $t \in \mathbf{R}$, let $\gamma(t) = t - it$ and $\delta(t) = t + \sqrt{2}it$ (both considered as curves on X). Again, $[\gamma] \cap [\delta]$ is infinite but $[\gamma] \nsubseteq [\delta]$.

In the first example, $[\gamma]$ is not compact but $[\delta]$ is, whereas in the second, $[\gamma]$ is compact but $[\delta]$ is not. However, if we require that both $[\gamma]$ and $[\delta]$ be compact, the conclusion that $[\gamma] \subset [\delta]$ can indeed be drawn.

Theorem 2.3. *Suppose that $\gamma : I \to X$ is an analytic arc* (not necessarily regular) *and that $\delta : \mathbf{R} \to X$ is a periodic regular analytic arc on X. If $[\gamma] \cap [\delta]$ is infinite, then $[\gamma] \subset [\delta]$.*

Proof. Let w be an accumulation point of $[\gamma] \cap [\delta]$ (which may be a self-intersection point of δ). There can only be finitely many elements of $\delta^{-1}(w)$ within any period of δ. By the choice of w, there exists a $t_0 \in \delta^{-1}(w)$ such that every open interval J containing t_0 satisfies the condition that $[\gamma] \cap [\delta|_J]$ is infinite. By regularity of δ, we can find an open interval J_0 containing t_0 such that δ extends to a conformal map (also named δ) on a neighborhood $U \subset \mathbf{C}$ of J_0. Then $V = \delta[U]$ is an open neighborhood of w and there is an analytic inverse $(\delta|_U)^{-1} : V \to U$. For some interval $K_0 \subset I$, the arc $(\delta|_U)^{-1} \circ \gamma|_{K_0}$ satisfies the hypothesis of Lemma 2.1, and therefore $[\gamma|_{K_0}] \subset [\delta|_{J_0}] \subset [\delta]$. Compactness of I and $[\delta]$ then allows this set inclusion to propagate throughout $[\gamma]$. □

Corollary 2.4. *Suppose that g is a function analytic on a neighborhood of the unit circle S and that $g[S] \cap S$ is infinite. Then $g[S] \subset S$.*

By giving S the standard parametrization $t \mapsto e^{2\pi it}$ for $t \in I$, $g[S]$ can be realized as the trajectory of an analytic arc, which may not be regular as g may have critical points on S. Corollary 2.4 then follows immediately from Theorem 2.3. However, note that, under the hypothesis of Corollary 2.4, it is not necessarily true that $g[S] = S$, e.g., consider $g(z) = \exp[i(z + (1/z))]$.

3. Boundary preservation. We now establish the main result of this article.

Theorem 3.1. *Suppose that X_1 and X_2 are compact bordered Riemann surfaces and that f is an analytic map mapping X_1 onto X_2. Then $f[\partial X_1] = \partial X_2$.*

Before giving the proof, we recall certain facts about bordered Riemann surfaces.

Remark 3.2. 1. Analyticity of f on ∂X_1 is understood to mean that f is analytic on some ambient Riemann surface Y_1 (open or compact without boundary) containing X_1 as a compact subset. The choice of Y_1 causes no ambiguity.

2. Every compact bordered Riemann surface X can be obtained from some compact Riemann surface without boundary by excising a finite number of coordinate discs; see [**2**, page 429]. Thus, each boundary circle C of X can be given a global analytic parametrization by the unit circle S, i.e., C does not have to be broken down into local coordinate patches for its analytic parametrization. (This is not essential, but provides convenience for the following argument.)

Proof of Theorem 3.1. By the open mapping theorem, $\partial X_2 \subset f[\partial X_1]$. In particular, $f[\partial X_1] \cap \partial X_2$ is an infinite set.

Then, for some boundary circles C_1 and C_2 of X_1 and X_2, respectively, $f[C_1] \cap C_2$ is an infinite set. By the preceding remark, we may use some global analytic parametrizations $\mu_k : S \to C_k$ ($k \in \{1, 2\}$) to construct $g = \mu_2^{-1} \circ f \circ \mu_1$. Then, $g[S] \cap S$ is infinite. By Corollary 2.4, $g[S] \subset S$, i.e., $f[C_1] \subset C_2$.

Since there are only finitely many boundary circles, applying this argument a number of times yields the desired inclusion $f[\partial X_1] \subset \partial X_2$. □

In the context of classical function theory, we obtain the following result as a direct consequence of Theorem 3.1.

Theorem 3.3. *Suppose that X_1 and X_2 are compact subsets of Σ each bounded by finitely many disjoint regular analytic Jordan curves. If f is meromorphic on X_1 with $f[X_1] = X_2$, then $f[\partial X_1] = \partial X_2$.*

It is well known that any nonconstant analytic function g mapping the closed unit disc $\overline{D}$ into itself with $g[S] \subset S$ is a finite Blaschke product, i.e.,

$$g(z) = c \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \quad \text{for some } n \in \mathbf{N},\ \alpha_k \in D \text{ and } c \in S.$$

Theorem 3.3 shows that the condition $g[S] \subset S$ can be replaced by the condition $g[\overline{D}] = \overline{D}$.

Corollary 3.4. *An analytic function mapping the closed unit disc* onto *itself is a finite Blaschke product.*

Remark 3.5. To see the significance of the analyticity condition on ∂X_1 in Theorem 3.3, consider for example $f(z) = z^2$ and X_1 the closure of the Jordan domain with $\partial X_1 = (H \cap S) \cup \mathcal{C}$ where $\mathcal{C} \subset \overline{D}$ is the trajectory of any simple arc in the lower semidisk joining -1 and 1. The curve $\mathcal{C}$ can be so chosen that ∂X_1 is the trajectory of a C^∞ arc. Clearly, $f[X_1] = \overline{D}$. However, unless $\mathcal{C}$ *is* the lower semicircle, $f[\mathcal{C}] \not\subset S = \partial\overline{D}$. Of course, analyticity of the boundary curves is *not* an absolute necessity for boundary preservation. For certain special cases in which ∂X_1 and ∂X_2 are piecewise regular analytic, boundary preservation does hold; see [**1**] for the case when X_1 and X_2 are polygons in $\mathbf{C}$.

Finally, we remark upon the relation between valence and boundary preservation of analytic maps.

Remark 3.6. Suppose that X_k $(k \in \{1, 2\})$ are compact bordered Riemann surfaces of genera g_k with n_k boundary components and that $f : X_1 \to X_2$ is a surjective analytic map. Theorem 3.1 shows that f maps ∂X_1 onto ∂X_2. By analyticity of ∂X_k, it is easy to see that f has no critical point on ∂X_1. Thus, the map f is a ramified covering of

X_2 by X_1, and, in this situation, the Riemann-Hurwitz formula relates the topology of the two surfaces to the valence of f. Let N denote the valence of f and ν the total order of the branch points (i.e., the number of critical points of f counted with multiplicity). The Riemann-Hurwitz formula in this context states that

$$2g_1 - 2 + n_1 = N(2g_2 - 2 + n_2) + \nu.$$

In the classical case when $X_k \subset \Sigma$, $g_k = 0$ and the formula takes the following simple form:

$$n_1 - 2 = N(n_2 - 2) + \nu.$$

(See [**4**] for the general Riemann-Hurwitz formula and [**5**] for the classical case.) Due to Theorem 3.1, these formulae remain valid without the additional hypothesis that the map f preserve boundary.

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REFERENCES

1. J. Bak and P. Ding, *Shape distortion by analytic functions*, Amer. Math. Monthly **116** (2009), 143–150.

2. T.W. Gamelin, *Complex analysis*, Springer, New York, 2000.

3. C.D. Minda, *Regular analytic arcs and curves*, Colloq. Math. **38** (1977), 73–82.

4. L. Sario and H. Noshiro, *Value distribution theory*, Van Nostrand, Princeton, New Jersey, 1966.

5. N. Steinmetz, *The formula of Riemann-Hurwitz and iteration of rational functions*, Complex Variables Theory Appl. **22** (1993), 203–206.

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