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OSCILLATION OF nTH ORDER SUPERLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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We dedicate this paper to the memory of Lloyd K. Jackson

ABSTRACT. Consider the following nth order superlinear dynamic equation

$$x^{\Delta^n}(t) + p(t)x^{\alpha}(\sigma(t)) = 0, \quad \alpha > 1,$$

where $p \in C_{rd}(\mathbf{T}, \mathbf{R}^+)$, and $\mathbf{T}$ is an isolated time scale, α is a ratio of odd positive integers. We obtain an analog of the Kiguradze-Ličko-Švec-type oscillation theorem for this dynamic equation. As an application, we obtain

(i) when n is even, every solution $x(k)$ of the difference equation

$$\Delta^n x(k) + p(k)x^{\alpha}(k+1) = 0,$$

where $p(k) \geq 0$ and $\alpha > 1$ is oscillatory if and only if

$$\sum_{k=1}^{\infty}(k+1)^{n-1}p(k) = \infty.$$

(ii) when n is odd, every solution $x(k)$ of this difference equation is either oscillatory or $\lim_{k\to\infty} x(k) = 0$ if and only if the above sum is infinite.

1. Introduction. Consider the following nth order superlinear dynamic equation on a time scale

$$x^{\Delta^n}(t) + p(t)x^{\alpha}(\sigma(t)) = 0, \quad \alpha > 1, \tag{1.1}$$

where $p \in C_{rd}(\mathbf{T}, \mathbf{R}^+)$, $\mathbf{T}$ is a time scale, and α is a ratio of odd positive integers.

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When $\mathbf{T} = \mathbf{R}$, the dynamic equation (1.1) is the nth order superlinear differential equation

$$x^{(n)}(t) + p(t)x^{\alpha}(t) = 0, \quad \alpha > 1. \tag{1.2}$$

For $n = 2$, when $p(t)$ is nonnegative, Atkinson [**3**] proved that

$$\int^{\infty} tp(t)\, dt = \infty, \tag{1.3}$$

is a necessary and sufficient condition for the oscillation of (1.2).

For $n = 2$, when $p(t)$ is allowed to take on negative values, Kiguradze [**10**] proved that (1.3) is sufficient for all solutions of the differential equation (1.2) to be oscillatory. These results have been further extended by Wong [**15**].

When $\mathbf{T} = \mathbf{N}_0$, the dynamic equation (1.1) is the nth order superlinear difference equation

$$\Delta^n x(k) + p(k)x^{\alpha}(k+1) = 0, \quad \alpha > 1. \tag{1.4}$$

For $n = 2$, when $p(k)$ is nonnegative, Hooker and Patula [**8**] and Mingarelli [**13**], respectively, proved that

$$\sum^{\infty} kp(k) = \infty \tag{1.5}$$

is a necessary and sufficient condition for the oscillation of all solutions of the difference equation (1.4).

For $n = 2$, when $p(k)$ is allowed to take on negative values, Jia, Erbe and Peterson [**5**] proved that (1.5) is sufficient for all solutions of the difference equation (1.4) to be oscillatory.

In 1963, Ličko and Švec (see [**12**] or [**2**]) established the following interesting necessary and sufficient condition for the oscillation of (1.2).

Theorem 1.1. *Suppose that* $p(t) \geq 0$. *Then* (i) *when* n *is even, every solution* $x(t)$ *of the differential equation* (1.2) *is oscillatory if and only if* $\int^{\infty} s^{n-1} p(s)\, ds = \infty$;

(ii) *when n is odd, every solution $x(t)$ of the differential equation* (1.2) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$ *if and only if* $\int^\infty s^{n-1}p(s)\,ds = \infty$.

The sufficiency part of Theorem 1.1 for n even was given for the first time by Kiguradze [**11**].

In this paper, we consider the oscillation of the nth order superlinear dynamic equation (1.1) on an *isolated* time scale where $p \in C_{rd}(\mathbf{T}, \mathbf{R}^+)$. We obtain a Kiguradze-Ličko-Švec-type oscillation theorem for equation (1.1). As an application, we get that

(i) when n is even, every solution $x(k)$ of the difference equation

$$\Delta^n x(k) + p(k)x^\alpha(k+1) = 0, \tag{1.6}$$

where $p(k) \geq 0$ and $\alpha > 1$, is oscillatory if and only if

$$\sum_{k=1}^{\infty}(k+1)^{n-1}p(k) = \infty, \tag{1.7}$$

(ii) when n is odd, every solution $x(k)$ of the difference equation (1.6) is either oscillatory or $\lim_{k\to\infty} x(k) = 0$ if and only if (1.7) holds.

For completeness, (see [**6, 7**] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbf{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbf{R}$) with $\sup \mathbf{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbf{T} : s < t\},$$

where $\sup \varnothing = \inf \mathbf{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbf{T}$ we say t is left-dense. The graininess function μ for a time scale $\mathbf{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbf{T} \to \mathbf{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbf{T} \to \mathbf{R}$ is differentiable at $t \in \mathbf{T}$ provided

$$x^\Delta(t) := \lim_{s\to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t)=t$ (here by $s\to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t)>t$

$$x^{\Delta}(t):=\frac{x(\sigma(t))-x(t)}{\mu(t)}.$$

Note that if $\mathbf{T}=\mathbf{R}$, then the delta derivative is just the standard derivative, and when $\mathbf{T}=\mathbf{Z}$ the delta derivative is just the forward difference operator. Our results here extend the theorems mentioned above to isolated time scales (i.e., all points are both left-scattered and right-scattered), and include, for example, the time scale $q^{\mathbf{N}_0}:=\{1,q,q^2,\dots\}$ which is very important in quantum theory [**14**].

2. Lemmas. The following lemma for the solution of a dynamic inequality on an unbounded above time scale can be regarded as a simple extension of [**1**, Corollary 1.7.14, pages 31–32] (see Ryder and Wend [**14**] for its continuous version). The proof is the same as in [**1**, Corollary 1.7.14], so we omit it.

Lemma 2.1. *Suppose that* $\mathbf{T}=[t_0,\infty)_{\mathbf{T}}$ *is a time scale interval which is unbounded above. Let* $x(t)$ *be defined on* $\mathbf{T}$ *with* $x(t)>0$, $x^{\Delta^n}(t)\le 0$ *and not identically zero, for large* $t\in\mathbf{T}$*. Then, exactly one of the following is true*:

(I) $\lim_{t\to\infty}x^{\Delta^i}(t)=0$, $1\le i\le n-1$.

(II) *there is an odd integer* j, $1\le j\le n-1$ *such that* $\lim_{t\to\infty}x^{\Delta^{n-i}}(t)=0$ *for* $1\le i\le j-1$, $\lim_{t\to\infty}x^{\Delta^{n-j}}x(t)\ge 0$ (*finite*), $\lim_{t\to\infty}x^{\Delta^{n-j-1}}(t)>0$ *and* $\lim_{t\to\infty}x^{\Delta^i}(t)=\infty$, $0\le i\le n-j-2$.

In addition, in Case (I) *we know that* $(-1)^{i+n-1}x^{\Delta^i}(t)>0$, *for* $1\le i\le n-1$, $t\in\mathbf{T}$ *and in Case* (II), $(-1)^{i+j}x^{\Delta^{n-i}}(t)>0$, *for* $1\le i\le j$, $t\in\mathbf{T}$.

Lemma 2.2. *Suppose that*

$$\mathbf{T}=\{t_0,t_1,t_2,\dots,t_k,\dots\},$$

where $0<t_0<t_1<\cdots<t_k<\cdots$, $\lim_{k\to\infty}t_k=\infty$*. Then for any*

$m \geq 2$, there exists $\varepsilon_{m-1} > 0$ such that

(2.1)
$$\int_{t_{k_0}}^{\sigma(\tau_{m-1})} \int_{t_{k_0}}^{\sigma(\tau_{m-2})} \cdots \int_{t_{k_0}}^{\sigma(\tau_2)} \int_{t_{k_0}}^{\sigma(\tau_1)} \Delta\tau_0 \Delta\tau_1 \Delta\tau_2 \cdots \Delta\tau_{m-2}$$
$$= \int_{t_{k_0}}^{\sigma(\tau_{m-1})} \int_{t_{k_0}}^{\sigma(\tau_{m-2})} \cdots \int_{t_{k_0}}^{\sigma(\tau_2)} [\sigma(\tau_1) - t_{k_0}] \Delta\tau_1 \Delta\tau_2 \cdots \Delta\tau_{m-2}$$
$$\geq \varepsilon_{m-1}[\sigma(\tau_{m-1})]^{m-1},$$

for $\tau_{m-1} > t_{k_0}$.

Proof. We prove the result by induction. When $m = 2$, we have

$$\int_{t_{k_0}}^{\sigma(\tau_1)} \Delta\tau_0 = \sigma(\tau_1) - t_{k_0} \geq \varepsilon_1 \sigma(\tau_1),$$

for $\tau_1 > t_{k_0}$, where $\varepsilon_1 = 1 - \frac{t_{k_0}}{t_{k_0+2}} > 0$.

Suppose that when $m = k$, (2.1) holds. Then when $m = k + 1$, supposing $\tau_k = t_l \in \mathbf{T}$, $l \geq k_0$, we have

$$\int_{t_{k_0}}^{\sigma(\tau_k)} \int_{t_{k_0}}^{\sigma(\tau_{k-1})} \cdots \int_{t_{k_0}}^{\sigma(\tau_2)} [\sigma(\tau_1) - t_{k_0}] \Delta\tau_1 \Delta\tau_2 \cdots \Delta\tau_{k-1}$$
$$\geq \varepsilon_{k-1} \int_{t_{k_0}}^{\sigma(\tau_k)} [\sigma(\tau_{k-1})]^{k-1} \Delta\tau_{k-1}$$
$$= \varepsilon_{k-1}[t_{k_0+1}^{k-1}(t_{k_0+1} - t_{k_0}) + t_{k_0+2}^{k-1}(t_{k_0+2} - t_{k_0+1}) + \cdots + t_{l+1}^{k-1}(t_{l+1} - t_l)]$$
$$\geq \varepsilon_{k-1} \int_{t_{k_0}}^{t_{l+1}} s^{k-1}\, ds$$
$$\geq \varepsilon_k [\sigma(\tau_k)]^k,$$

for $\tau_k > t_{k_0}$, where $\varepsilon_k = (\varepsilon_{k-1})/k(1 - (t_{k_0}/t_{k_0+2})^k)$, which shows that (2.1) holds for $m = k + 1$. □

Lemma 2.3. *Suppose that*

$$\mathbf{T} = \{t_0, t_1, t_2, \dots, t_k, \dots\},$$

where $\lim_{k\to\infty} t_k = \infty$ *and* $\alpha > 1$. *Assume that* $x(t) > 0$ *and* $x^{\Delta}(t) \geq 0$ *for* $t \geq t_{k_0}$. *Then*

$$\int_{t_{k_0}}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\Delta s \leq \frac{x^{1-\alpha}(t_{k_0})}{\alpha - 1}. \tag{2.2}$$

Proof. Let $t = t_m$. We have

$$\begin{aligned}\int_{t_{k_0}}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\Delta s &= \sum_{i=k_0}^{m-1} \int_{t_i}^{t_{i+1}} \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,\Delta s \\ &= \sum_{i=k_0}^{m-1} \frac{x^{\Delta}(t_i)\mu(t_i)}{x^{\alpha}(t_{i+1})} = \sum_{i=k_0}^{m-1} \frac{x(t_{i+1}) - x(t_i)}{x^{\alpha}(t_{i+1})} \\ &\leq \int_{x(t_{k_0})}^{x(t_m)} \frac{1}{u^{\alpha}}\,du \leq \frac{x^{1-\alpha}(t_{k_0})}{\alpha - 1}. \qquad \square\end{aligned}$$

The following lemma is from [**7**, Theorem 5.37 (i)].

Lemma 2.4 (Leibniz formula). *If* $f(t,s)$, $f^{\Delta_t}(t,s)$ *are rd-continuous, then*

$$\left[\int_a^t f(t,s)\Delta s\right]^{\Delta_t} = f(\sigma(t),t) + \int_a^t f^{\Delta_t}(t,s)\Delta s.$$

3. Main theorem. Assume that $\mathbf{T} = \{t_k\}_{k=0}^{\infty}$ where $0 < t_0 < t_1 < \cdots < t_k \cdots$, with $t_k \to \infty$.

Definition 3.1. We say that $\mathbf{T}$ satisfies condition (E) if there exists $L_1 > 1$ such that

$$\frac{t_k - t_0}{t_{k-1} - t_0} \leq L_1, \quad \text{for all } k > 1.$$

Clearly, if $\mathbf{T} = h\mathbf{N}_0$, $h > 0$, $\mathbf{T} = q^{\mathbf{N}_0}$, $q > 1$, or $\mathbf{T}$ is the set of harmonic numbers [**6**, Example 1.45] then $\mathbf{T}$ satisfies condition (E) but it is easy to show that $\mathbf{T} = \{2^{2^k}, k \in \mathbf{N}_0\}$, does not satisfy condition (E).

Remark 3.2. Condition (E) is a requirement on the asymptotic behavior of the graininess function and can be reformulated [**4**] as follows: The time scale $\mathbf{T}$ satisfies condition (E) if and only if for each fixed $i_0 > 1$, there exists $L_{i_0} > 1$ such that

$$\frac{t_k - t_{i_0}}{t_{k-1} - t_{i_0}} \leq L_{i_0}, \quad \text{for all } k > i_0 + 1.$$

We may now prove our main result.

Theorem 3.3. *Suppose that* $\mathbf{T}$ *satisfies condition* (E) *and consider the integral condition*:

$$\int_{t_0}^{\infty} [\sigma(t)]^{n-1} p(t) \Delta t = \infty. \tag{3.1}$$

(i) *Assume* n *is odd. Then every solution* $x(t)$ *of* (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$ *if and only if* (3.1) *holds.*

(ii) *Assume* n *is even. Then every solution* $x(t)$ *of* (1.1) *is oscillatory if and only if* (3.1) *holds.*

Proof. Suppose that (3.1) holds and assume that $x(t)$ is a nonoscillatory solution of (1.1). We may assume that $x(t) > 0$ for $t > T$. The case $x(t) < 0$ can be treated similarly. In view of (1.1),

$$x^{\Delta^n}(t) = -p(t)x^{\alpha}(\sigma(t)) \leq 0, \quad t \in [T, \infty)_{\mathbf{T}}. \tag{3.2}$$

Throughout this proof we will use the sign conditions on the derivatives of $x(t)$ given in Lemma 2.1. Since $x^{\Delta^n}(t) \leq 0$ and $x^{\Delta^{n-1}}(t) > 0$, $x^{\Delta^{n-1}}(t)$ decreases to a nonnegative limit as t increases to ∞. Integrating (3.2) from t to ∞, we get

$$x^{\Delta^{n-1}}(t) \geq \int_t^{\infty} p(\tau)x^{\alpha}(\sigma(\tau))\Delta\tau. \tag{3.3}$$

Suppose that case (I) of Lemma 2.1 holds. Integrate (3.3) from v to u with $T \le v \le u$. We have

$$\begin{aligned} x^{\Delta^{n-2}}(u) - x^{\Delta^{n-2}}(v) &\ge \left[(t-v)\int_t^\infty p(\tau)x^\alpha(\sigma(\tau))\Delta\tau\right]_{t=v}^{u} \\ &\quad + \int_v^u (\sigma(t)-v)p(t)x^\alpha(\sigma(t))\Delta t \\ &\ge \int_v^u (\sigma(t)-v)p(t)x^\alpha(\sigma(t))\Delta t. \end{aligned}$$

Letting u tend to ∞, we get

$$-x^{\Delta^{n-2}}(v) \ge \int_v^\infty (\sigma(t)-v)p(t)x^\alpha(\sigma(t))\Delta t.$$

Integrating from v_1 to u_1 with $T \le v_1 \le u_1$ and using the Leibniz formula (see Lemma 2.4), we obtain

$$\begin{aligned} &- x^{\Delta^{n-3}}(u_1) + x^{\Delta^{n-3}}(v_1) \\ &\qquad \ge \left[(v-v_1)\int_v^\infty (\sigma(t)-v)p(t)x^\alpha(\sigma(t))\Delta t\right]_{v=v_1}^{u_1} \\ &\qquad\quad + \int_{v_1}^{u_1}(\sigma(v)-v_1)\left[\int_v^\infty p(t)x^\alpha(\sigma(t))\Delta t\right]\Delta v \\ &\qquad \ge \int_{v_1}^{u_1}(\sigma(v)-v_1)\left[\int_v^\infty p(t)x^\alpha(\sigma(t))\Delta t\right]\Delta v \\ &\qquad = \left[\int_{v_1}^{v}(\sigma(u)-v_1)\Delta u\int_v^\infty p(t)x^\alpha(\sigma(t))\Delta t\right]_{v=v_1}^{u_1} \\ &\qquad\quad + \int_{v_1}^{u_1}\left[\int_{v_1}^{\sigma(v)}(\sigma(u)-v_1)\Delta u\right]p(v)x^\alpha(\sigma(v))\Delta v \\ &\qquad \ge \int_{v_1}^{u_1}\left[\int_{v_1}^{\sigma(v)}(\sigma(u)-v_1)\Delta u\right]p(v)x^\alpha(\sigma(v))\Delta v. \end{aligned}$$

Letting u_1 tend to ∞, we get that

$$x^{\Delta^{n-3}}(v_1) \ge \int_{v_1}^\infty \left[\int_{v_1}^{\sigma(v)}(\sigma(u)-v_1)\Delta u\right]p(v)x^\alpha(v)\Delta v.$$

Similarly, by integrating by parts we have that

$$-x^{\Delta^{n-4}}(t) \geq \int_t^\infty \left\{ \int_t^{\sigma(\tau_1)} \left[\int_t^{\sigma(\tau_2)} (\sigma(\tau) - t)\Delta\tau \right] \Delta\tau_2 \right\} \times p(\tau_1) x^\alpha(\sigma(\tau_1))\Delta\tau_1,$$

and

$$x^{\Delta^{n-5}}(t) \geq \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \int_t^{\sigma(\tau_3)} (\sigma(\tau) - t)\Delta\tau\Delta\tau_3\Delta\tau_2 \, p(\tau_1)x^\alpha(\sigma(\tau_1))\Delta\tau_1.$$

Repeating the above procedure we have

$$(-1)^n x^\Delta(t) \geq \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-3})} (\sigma(\tau) - t)\Delta\tau\Delta\tau_{n-3} \cdots \Delta\tau_3\Delta\tau_2 p(\tau_1)x^\alpha(\sigma(\tau_1))\Delta\tau_1. \tag{3.4}$$

For simplicity, we denote

$$g(\sigma(\tau_1), t) := \int_t^{\sigma(\tau_1)} \left[\int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-3})} (\sigma(\tau) - t)\Delta\tau\Delta\tau_{n-3} \cdots \Delta\tau_3 \right] \Delta\tau_2.$$

Let us now suppose n is even. Then from (3.4) we have

$$x^\Delta(t) \geq \int_t^\infty g(\sigma(\tau_1), t)p(\tau_1)x^\alpha(\sigma(\tau_1))\Delta\tau_1. \tag{3.5}$$

Note that (since x is increasing) $x(\sigma(\tau_1))/x(\sigma(u)) \geq 1$ for $\tau_1 \geq u$ and $g(\sigma(u), \sigma(u)) = 0$. By (3.5), integrating by parts from T_1 to t and using

Lemma 2.4, we get that

$$
\begin{aligned}
(3.6)\quad & \int_{T_1}^{t} \frac{x^{\Delta}(u)}{x^{\alpha}(\sigma(u))}\,\Delta u \\
& \geq \int_{T_1}^{t}\int_{u}^{\infty} g(\sigma(\tau_1),u)p(\tau_1)\frac{x^{\alpha}(\sigma(\tau_1))}{x^{\alpha}(\sigma(u))}\,\Delta\tau_1\,\Delta u \\
& \geq \int_{T_1}^{t}\int_{u}^{\infty} g(\sigma(\tau_1),u)p(\tau_1)\,\Delta\tau_1\,\Delta u \\
& = \left[(u-T_1)\int_{u}^{\infty} g(\sigma(\tau_1),u)p(\tau_1)\,\Delta\tau_1\right]_{u=T_1}^{t} \\
& \quad - \int_{T_1}^{t}(\sigma(u)-T_1)\Bigg[-g(\sigma(u),\sigma(u))p(u) \\
& \qquad\qquad + \int_{u}^{\infty} g^{\Delta_u}(\sigma(\tau_1),u)p(\tau_1)\Delta\tau_1\Bigg]\,\Delta u \\
& \geq -\int_{T_1}^{t}(\sigma(u)-T_1)\left[\int_{u}^{\infty} g^{\Delta_u}(\sigma(\tau_1),u)p(\tau_1)\Delta\tau_1\right]\Delta u.
\end{aligned}
$$

Note that by definition, we have

$$
\begin{aligned}
g^{\Delta_u}(\sigma(\tau_1),u) &= -\int_{u}^{\sigma(\tau_1)}\int_{u}^{\sigma(\tau_2)}\cdots\int_{u}^{\sigma(\tau_{n-4})}(\sigma(\tau_{n-3})-u)\Delta\tau_{n-3}\cdots\Delta\tau_3\Delta\tau_2 \\
&= -I_{n-3}(\tau_1,u),
\end{aligned}
$$

where

$$
I_k(\tau_1,u) := \int_{u}^{\sigma(\tau_1)}\int_{u}^{\sigma(\tau_2)}\cdots\int_{u}^{\sigma(\tau_{k-1})}(\sigma(\tau_k)-u)\Delta\tau_k\cdots\Delta\tau_3\Delta\tau_2.
$$

From (3.6), we get that

$$\begin{aligned}\int_{T_1}^{t} \frac{x^{\Delta}(u)}{x^{\alpha}(\sigma(u))}\Delta u & \\ &\geq \int_{T_1}^{t} (\sigma(u) - T_1) \\ &\quad \times \int_{u}^{\infty} I_{n-3}(\tau_1, u) p(\tau_1) \Delta\tau_1 \Delta u \\ &= \left\{ \int_{T_1}^{u} (\sigma(v_1) - T_1)\Delta v_1 \int_{u}^{\infty} I_{n-3}(\tau_1, u) p(\tau_1)\Delta\tau_1 \right\}_{u=T_1}^{t} \\ &\quad + \int_{T_1}^{t} \int_{T_1}^{\sigma(u)} (\sigma(v_1) - T_1)\Delta v_1 \int_{u}^{\infty} I_{n-4}(\tau_1, u) p(\tau_1)\Delta\tau_1 \Delta u \\ &\geq \int_{T_1}^{t} \int_{T_1}^{\sigma(u)} (\sigma(v_1) - T_1)\Delta v_1 \int_{u}^{\infty} I_{n-4}(\tau_1, u) p(\tau_1)\Delta\tau_1 \Delta u.\end{aligned}$$

By integrating by parts, repeating the above procedure and using (2.1) of Lemma 2.2, we have

(3.7)

$$\begin{aligned}\int_{T_1}^{t} \frac{x^{\Delta}(u)}{x^{\alpha}(\sigma(u))}\Delta u &\geq \int_{T_1}^{t} \left[\int_{T_1}^{\sigma(u)} \int_{T_1}^{\sigma(v_{n-2})} \cdots \int_{T_1}^{\sigma(v_2)} (\sigma(v_1) - T_1)\Delta v_1 \cdots \Delta v_{n-2} \right] p(u)\, \Delta u \\ &\geq \int_{\sigma(T_1)}^{t} \left[\int_{T_1}^{\sigma(u)} \int_{T_1}^{\sigma(v_{n-2})} \cdots \int_{T_1}^{\sigma(v_2)} (\sigma(v_1) - T_1)\Delta v_1 \cdots \Delta v_{n-2} \right] p(u)\Delta u \\ &\geq \varepsilon_{n-1} \int_{\sigma(T_1)}^{t} [\sigma(u)]^{n-1} p(u)\Delta u.\end{aligned}$$

Using Lemma 2.3, we have that

$$\int_{T_1}^{t} \frac{x^{\Delta}(u)}{x^{\alpha}(\sigma(u))}\, \Delta u \leq \frac{x^{1-\alpha}(T_1)}{\alpha - 1} < \infty. \tag{3.8}$$

Then, letting $t \to \infty$, from (3.1), (3.7) and (3.8), we obtain a contradiction.

Next let us suppose that n is odd. Then (3.4) reduces to

$$-x^{\Delta}(t) \geq \int_t^{\infty} g(\sigma(\tau_1), t)p(\tau_1)x^{\alpha}(\sigma(\tau_1))\,\Delta\tau_1, \tag{3.9}$$

and this implies that $x(t)$ is nonincreasing for $t \geq T$. Let $\lim_{t\to\infty} x(t) = L$. We shall prove that $L = 0$. Suppose $L > 0$. We take T so large that $x(t) \geq L/2$ for $t \geq T$. Integrating (3.9) by parts from T to t yields

$$\begin{aligned} x(T) - x(t) \geq & \left[(s-T)\int_s^{\infty} g(\sigma(\tau_1), s)p(\tau_1)x^{\alpha}(\sigma(\tau_1))\,\Delta\tau_1\right]_{s=T}^{t} \\ & - \int_T^t [\sigma(s) - T]\left[\int_s^{\infty} g^{\Delta_s}(\sigma(\tau_1), s)p(\tau_1)x^{\alpha}(\sigma(\tau_1))\Delta\tau_1\right]\Delta s \\ & \geq \int_T^t [\sigma(s) - T]\int_s^{\infty}\int_s^{\sigma(\tau_1)}\int_s^{\sigma(\tau_2)} \\ & \cdots \int_s^{\sigma(\tau_{n-4})} (\sigma(\tau_{n-3}) - s)\,\Delta\tau_{n-3}\cdots\Delta\tau_2 \\ & \qquad \cdot p(\tau_1)x^{\alpha}(\sigma(\tau_1))\Delta\tau_1\Delta s. \end{aligned}$$

Repeatedly integrating by parts from T to t and using (2.1) of Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned} x(T) & \geq x(T) - x(t) \\ & \geq \int_T^t \left[\int_T^{\sigma(\tau_{n-1})}\int_T^{\sigma(\tau_{n-2})} \cdots \int_T^{\sigma(\tau_2)} [\sigma(\tau_1) - T]\,\Delta\tau_1\cdots\Delta\tau_{n-2}\right] \\ & \quad \cdot p(\tau_{n-1})x^{\alpha}(\sigma(\tau_{n-1}))\,\Delta\tau_{n-1} \\ & \geq \varepsilon_{n-1}\left(\frac{L}{2}\right)^{\alpha}\int_{\sigma(T)}^t [\sigma(\tau_{n-1})]^{n-1}p(\tau_{n-1})\Delta\tau_{n-1}. \end{aligned} \tag{3.10}$$

Letting $t \to \infty$, by (3.1) we obtain a contradiction.

Suppose next case (II) of Lemma 2.1 holds. We observe that there exists a $T_2 \geq T$ such that $x^{\Delta^j}(t) > 0$ for $t \geq T_2$, $j = 0, 1, \dots, n-j-1$. Proceeding as in case (I), noticing that the j in Lemma 2.1 is odd, similar to relation (3.9) in case (I), we have

$$- x^{\Delta^{n-j+1}}(t) \geq \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{j-3})} [\sigma(\tau) - t]\, \Delta\tau \Delta_{j-3} \cdots \Delta\tau_2 p(\tau_1) x^\alpha(\sigma(\tau_1))\, \Delta\tau_1.$$

Integrating from t to u, $T_2 < t < u$, similar to (3.10), we have

$$\begin{aligned} x^{\Delta^{n-j}}(t) &\geq x^{\Delta^{n-j}}(t) - x^{\Delta^{n-j}}(u) \\ &\geq \int_t^u \int_t^{\sigma(\tau_{j-1})} \int_t^{\sigma(\tau_{j-2})} \\ &\quad \cdots \int_t^{\sigma(\tau_2)} [\sigma(\tau_1) - t]\, \Delta\tau_1 \cdots \Delta\tau_{j-2} \\ &\qquad \cdot p(\tau_{j-1}) x^\alpha(\sigma(\tau_{j-1}))\, \Delta\tau_{j-1}. \end{aligned}$$

Letting $u \to \infty$ and noticing $\lim_{u\to\infty} x^{\Delta^{n-j}}(u) \geq 0$, we obtain that

$$\begin{aligned} x^{\Delta^{n-j}}(t) \geq \int_t^\infty \int_t^{\sigma(\tau_{j-1})} \int_t^{\sigma(\tau_{j-2})} &\cdots \int_t^{\sigma(\tau_2)} [\sigma(\tau_1) - t]\, \Delta\tau_1 \cdots \Delta\tau_{j-2} \\ &\cdot p(\tau_{j-1}) x^\alpha(\sigma(\tau_{j-1}))\Delta\tau_{j-1}. \end{aligned}$$

Integrating from $T_3 > T_2$ to t and noticing that $x^{\Delta^{n-j-1}}(T_3) \geq 0$, we get that

$$
\begin{aligned}
x^{\Delta^{n-j-1}}(t) \ge x^{\Delta^{n-j-1}}(t) - x^{\Delta^{n-j-1}}(T_3) \\
&\ge \Bigg[(\tau - T_3)\int_\tau^\infty \int_\tau^{\sigma(\tau_{j-1})} \int_\tau^{\sigma(\tau_{j-2})} \\
&\cdots \int_\tau^{\sigma(\tau_2)} [\sigma(\tau_1) - \tau]\Delta\tau_1 \cdots \Delta\tau_{j-2} \\
&\qquad \cdot p(\tau_{j-1}) x^\alpha(\sigma(\tau_{j-1}))\Delta\tau_{j-1}\Bigg]_{\tau=T_3}^t \\
&\quad + \int_{T_3}^t [\sigma(\tau) - T_3] \int_\tau^\infty \int_\tau^{\sigma(\tau_{j-1})} \int_\tau^{\sigma(\tau_{j-2})} \\
&\cdots \int_\tau^{\sigma(\tau_3)} [\sigma(\tau_2) - \tau]\Delta\tau_2 \cdots \Delta\tau_{j-2} \\
&\qquad \cdot p(\tau_{j-1}) x^\alpha(\sigma(\tau_{j-1}))\Delta\tau_{j-1}\Delta\tau \\
&\ge \int_{T_3}^t [\sigma(\tau) - T_3] \int_\tau^\infty \int_\tau^{\sigma(\tau_{j-1})} \int_\tau^{\sigma(\tau_{j-2})} \\
&\cdots \int_\tau^{\sigma(\tau_3)} [\sigma(\tau_2) - \tau]\Delta\tau_2 \cdots \Delta\tau_{j-2} \\
&\qquad \cdot p(\tau_{j-1}) x^\alpha(\sigma(\tau_{j-1}))\Delta\tau_{j-1}\Delta\tau.
\end{aligned}
$$

Repeatedly integrating by parts, we get that

$$
\begin{aligned}
x^{\Delta^{n-j-1}}(t) &\ge \int_{T_3}^t \int_{T_3}^{\sigma(\tau_{j-1})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\Delta\tau_1 \\
&\qquad \cdots \Delta\tau_{j-2}\left[\int_{\tau_{j-1}}^\infty p(s) x^\alpha(\sigma(s))\Delta s\right]\Delta\tau_{j-1} \\
&= \Bigg[\int_{T_3}^{\tau_j} \int_{T_3}^{\sigma(\tau_{j-1})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\Delta\tau_1 \\
&\qquad \cdots \Delta\tau_{j-1} \cdot \int_{\tau_j}^\infty p(s) x^\alpha(\sigma(s))\Delta s\Bigg]_{\tau_j=T_3}^t \\
&\quad + \int_{T_3}^t \int_{T_3}^{\sigma(\tau_j)} \int_{T_3}^{\sigma(\tau_{j-1})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\Delta\tau_1 \\
&\qquad \cdots \Delta\tau_{j-1} p(\tau_j) x^\alpha(\sigma(\tau_j))\Delta\tau_j
\end{aligned}
$$

$$\geq \int_{T_3}^{t} \int_{T_3}^{\sigma(\tau_{j-1})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \Delta\tau_1 \cdots \Delta\tau_{j-1} \cdot \int_{t}^{\infty} p(s) x^{\alpha}(\sigma(s)) \Delta s.$$

Similarly, integrating from T_3 to t, we obtain that

$$x^{\Delta^{n-j-2}}(t) \geq x^{\Delta^{n-j-2}}(t) - x^{\Delta^{n-j-2}}(T_3) \geq \int_{T_3}^{t} \int_{T_3}^{\sigma(\tau_j)} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \Delta\tau_1 \cdots \Delta\tau_j \cdot \int_{t}^{\infty} p(s) x^{\alpha}(\sigma(s)) \Delta s.$$

Repeating the above procedure we get after $n-j-3$ steps

$$x^{\Delta}(t) \geq \int_{T_3}^{t} \int_{T_3}^{\sigma(\tau_{n-3})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \, \Delta\tau_1 \cdots \Delta\tau_{n-3} \cdot \int_{t}^{\infty} p(s) x^{\alpha}(\sigma(s)) \Delta s.$$

Note that $x(\sigma(s))/x(\sigma(\tau)) \geq 1$ for $s \geq \tau$. By (2.1) of Lemma 2.2, we get

(3.11)

$$\begin{aligned}
\int_{T_3}^{t} \frac{x^{\Delta}(\tau)}{x(\sigma(\tau))} \Delta\tau &\geq \int_{T_3}^{t} \int_{T_3}^{\tau} \int_{T_3}^{\sigma(\tau_{n-3})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \, \Delta\tau_1 \cdots \Delta\tau_{n-3} \cdot \left[\int_{\tau}^{\infty} p(s) \frac{x^{\alpha}(\sigma(s))}{x^{\alpha}(\sigma(\tau))} \Delta s \right] \Delta\tau \\
&\geq \int_{T_3}^{t} \int_{T_3}^{\tau} \int_{T_3}^{\sigma(\tau_{n-3})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \, \Delta\tau_1 \cdots \Delta\tau_{n-3} \left[\int_{\tau}^{\infty} p(s) \Delta s \right] \Delta\tau \\
&= \left[\int_{T_3}^{\tau} \int_{T_3}^{\sigma(\tau_{n-2})} \int_{T_3}^{\sigma(\tau_{n-3})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3] \, \Delta\tau_1 \right.
\end{aligned}$$

$$\cdots \Delta\tau_{n-2} \cdot \int_\tau^\infty p(s)\Delta s \Big]_{\tau=T_3}^t$$

$$+ \int_{T_3}^t \int_{T_3}^{\sigma(\tau)} \int_{T_3}^{\sigma(\tau_{n-2})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\,\Delta\tau_1 \cdots \Delta\tau_{n-2} p(\tau)\Delta\tau$$

$$\geq \int_{T_3}^t \int_{T_3}^{\sigma(\tau)} \int_{T_3}^{\sigma(\tau_{n-2})} \cdots \int_{T_3}^{\sigma(\tau_2)} [\sigma(\tau_1) - T_3]\,\Delta\tau_1 \cdots \Delta\tau_{n-2} p(\tau)\Delta\tau$$

$$\geq \varepsilon_{n-1} \int_{\sigma(T_3)}^t [\sigma(\tau)]^{n-1} p(\tau)\Delta\tau.$$

Using Lemma 2.3, we have that

$$\int_{T_3}^t \frac{x^\Delta(\tau)}{x^\alpha(\sigma(\tau))}\,\Delta\tau \leq \frac{x^{1-\alpha}(T_3)}{\alpha - 1} < \infty. \tag{3.12}$$

Letting $t \to \infty$, from (3.1), (3.11) and (3.12), we get a contradiction. This completes the proof of the sufficiency.

For the necessity, when n is odd, we shall show that if $\int_{t_0}^\infty [\sigma(t)]^{n-1} p(t)\,\Delta t < \infty$, then there exists a solution of (1.1) such that

$$x(\infty) = \frac{1}{2}, \quad \text{and} \quad x^{\Delta^j}(\infty) = 0, \quad j = 1, 2, \ldots, n-1. \tag{3.13}$$

Since $\mathbf{T}$ satisfies condition (E), there exists $K > 1$ such that $\sigma(t) \leq Kt$. So

$$\int_t^{\sigma(\tau_{n-3})} \int_t^{\sigma(\tau_{n-2})_{n-2}} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2}$$
$$\leq \int_t^{\sigma(\tau_{n-3})} [\sigma(\tau_{n-2}) - t]^2 \Delta\tau_{n-2}$$
$$\leq K^2(\sigma(\tau_{n-3}))^3.$$

In general, we have

$$\int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2} \cdots \Delta\tau_2 \tag{3.14}$$
$$\leq K^2 K^3 \cdots K^{n-2} (\sigma(\tau_1))^{n-1}$$
$$= K^{n^2-3n/2} (\sigma(\tau_1))^{n-1}.$$

From Lemma 2.4 and 3.14, it is easily verified that if the integral equation

$$x(t) = \frac{1}{2} + \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2}\cdots\Delta\tau_2 \cdot p(\tau_1)x^\alpha(\sigma(\tau_1))\Delta\tau_1 \tag{3.15}$$

has a solution $x(t)$ which is rd-continuous and uniformly bounded as $x \to \infty$, then it is also a solution of (1.1) with the supplementary conditions (3.13). The existence of a bounded continuous solution of (3.16) may be established by the Picard method of successive approximation. That is, we define a sequence of functions

$$x_m(t), \quad (m = 0, 1, 2, \dots),\ t \ge t_1,$$

by

$$x_0(t) \equiv 0$$

$$x_{m+1}(t) = \frac{1}{2} + \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2}\cdots\Delta\tau_2 \cdot p(\tau_1)x_m^\alpha(\sigma(\tau_1))\Delta\tau_1, \quad (m = 0, 1, \dots).$$

From (3.14) we can prove by induction that if t is so large that

$$\int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\,\Delta\tau\Delta\tau_{n-2} \cdots \Delta\tau_2 p(\tau_1)\,\Delta\tau_1 \le K^{n^2-3n/2} \int_t^\infty (\sigma(\tau_1))^{n-1}p(\tau_1)\,\Delta\tau_1 \le \frac{1}{2},$$

then $0 \le x_m(t) \le 1$. For simplicity, we set

$$G(\tau_1, t) = \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2}\cdots\Delta\tau_2.$$

We have
(3.16)

$$x_{m+2}(t) - x_{m+1}(t) = \int_t^\infty G(\tau_1, t)p(\tau_1)[x_{m+1}^\alpha(\sigma(\tau_1)) - x_m^\alpha(\sigma(\tau_1))]\Delta\tau_1.$$

By the mean value theorem,

$$|x_{m+1}^\alpha(\sigma(\tau_1)) - x_m^\alpha(\sigma(\tau_1))| = \alpha\xi^{\alpha-1}|x_{m+1}(\sigma(\tau_1)) - x_m(\sigma(\tau_1))|,$$

where $x_m(\sigma(\tau_1)) \le \xi \le x_{m+1}(\sigma(\tau_1))$ or $x_{m+1}(\sigma(\tau_1)) \le \xi \le x_m(\sigma(\tau_1))$.

So we get that

$$|x_{m+1}^\alpha(\sigma(\tau_1)) - x_m^\alpha(\sigma(\tau_1))| \le \alpha|x_{m+1}(\sigma(\tau_1)) - x_m(\sigma(\tau_1))|.$$

Therefore from (3.16), we have

$$|x_{m+2}(t) - x_{m+1}(t)| \le \alpha \max_{\tau_1 \ge t} |x_{m+1}(\tau_1) - x_m(\tau_1)| \int_t^\infty G(\tau_1, t)p(\tau_1)\Delta\tau_1.$$

From this we deduce the convergence of the sequence $x_m(t)$, $(m = 0, 1, \dots)$ for t so large that, by (3.14),

$$\alpha \int_t^\infty G(\tau_1, t)p(\tau_1)\Delta\tau_1 \le \alpha K^{n^2-3n/2} \int_t^\infty (\sigma(\tau_1))^{n-1}p(\tau_1)\Delta\tau_1 \le \beta < 1.$$

This proves the existence of a nonoscillatory solution of (1.1) for sufficiently large t, which establishes our result.

When n is even, we shall show that if $\int_{t_0}^\infty [\sigma(t)]^{n-1}p(t)\Delta t < \infty$, then there exists a solution of (1.1) such that

$$x(\infty) = 1, \qquad x^{\Delta^j}(\infty) = 0, \quad j = 1, 2, \cdots, n-1. \tag{3.17}$$

From Lemma 2.4 and (3.14), it is easily verified that if the integral equation

(3.18)

$$x(t) = 1 - \int_t^\infty \int_t^{\sigma(\tau_1)} \int_t^{\sigma(\tau_2)} \cdots \int_t^{\sigma(\tau_{n-2})} [\sigma(\tau_{n-2}) - \tau]\Delta\tau\Delta\tau_{n-2} \cdots \Delta\tau_2 p(\tau_1)x^\alpha(\sigma(\tau_1))\Delta\tau_1$$

has a solution $x(t)$ which is rd-continuous and uniformly bounded as $x \to \infty$, then it is also a solution of (1.1) with the supplementary conditions (3.17). The existence of a bounded continuous solution of (3.18) may be established by the Picard method of successive approximation.

The rest of the proof is the same as the case with n odd. This completes the proof of the theorem. □

Remark 3.4. We observe that in the sufficiency part of Theorem 3.3 we did not use the fact that $\mathbf{T}$ satisfies condition (E). Therefore, it is seen that for any time scale $\mathbf{T}$, (3.1) implies oscillation if n is even and if n is odd every solution is either oscillatory or approaches zero as n goes to ∞.

4. Examples. When $\mathbf{T} = \mathbf{N}_0$, equation (1.1) becomes

$$\Delta^n x(k) + p(k) x^\alpha (k+1) = 0. \tag{4.1}$$

By Theorem 3.3, we get the following example.

Example 4.1. The following hold: (i) When n is odd, every solution $x(t)$ of the difference equation (4.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$ if and only if

$$\sum_{k=1}^{\infty} (k+1)^{n-1} p(k) = \infty. \tag{4.2}$$

holds.

(ii) When n is even, every solution $x(t)$ of the difference equation (4.1) is oscillatory if and only if (4.2) holds.

As another simple illustration of Theorem 3.3 we get the following example:

Example 4.2. Let $\mathbf{T} = q^{\mathbf{N}_0}$, $q > 1$, and consider the dynamic equation

$$x^{\Delta^n}(t) + \frac{\beta}{t^n (\log_q t)^\gamma} \, x^\alpha (\sigma(t)) = 0, \quad \beta > 0. \tag{4.3}$$

Note that (4.3) is of the form (3.1) with

$$p(t) := \frac{\beta}{t^n (\log_q t)^\gamma}.$$

It is easy to see that

$$\int_q^\infty [\sigma(t)]^{n-1} p(t)\, \Delta t = \beta(q-1)\, q^{n-1} \sum_{j=1}^{\infty} \frac{1}{j^\gamma}$$

which diverges when $\gamma \le 1$ and converges when $\gamma > 1$. Hence from Theorem 3.3 we get that if $\gamma \le 1$ and if n is even, then all solutions of (4.3) are oscillatory, and if n is odd, then every solution $x(t)$ of (4.3) is either oscillatory or satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

On the other hand, if $\gamma > 1$, then for all positive integers n (4.3) has a nonoscillatory solution.

Many other interesting examples can be similarly given.

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