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PROJECTIVE OPERATOR SPACES, ALMOST PERIODICITY AND COMPLETELY COMPLEMENTED IDEALS IN THE FOURIER ALGEBRA

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ABSTRACT. We will show how projective operator spaces arise naturally as spaces of almost periodic functions. In particular, we will show that a locally compact group is compact if and only if its Fourier-Stieltjes algebra (or equivalently its Fourier algebra) is projective as an operator space. From this we see that if K is a compact subgroup of G, then the ideal $I(K)$ is completely complemented in $A(G)$.

1. Introduction. The notion of an operator space as a quantized analog of a Banach space has gained considerable attention of late. Four of the principal objects of abstract harmonic analysis; the group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$ and Fourier-Stieltjes algebra $B(G)$ of a locally compact group, are operator spaces by virtue of being preduals of von Neumann algebras. In commutative harmonic analysis, the fact that these algebras are preduals of von Neumann algebras is seldom if ever considered. Generally speaking, the same is true of the study of the algebras $L^1(G)$ and $M(G)$ even for noncommutative groups. However, for the Fourier and Fourier-Stieltjes algebras of noncommutative groups one can not stray far from the world of operator algebras. For this reason, it seems very natural to include the operator space structure as part of our working environment. In this paper, we will show how for certain types of problems concerning the Fourier and Fourier-Stieltjes algebras this additional structure is essential. We will also illustrate how the operator space structure was actually playing a fundamental role even in the study of commutative groups or in the study of $L^1(G)$ and $M(G)$ without our noticing.

To begin with, we note that the group algebra and the measure algebra are in a strong sense dual objects of the Fourier and Fourier

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algebras, respectively. This can be made more precise when G is an abelian group in that the classical Fourier-Stieltjes transform yields an isometric algebra isomorphism between $M(G)$ and $B(\widehat{G})$, where $\widehat{G}$ denotes the dual group of G, which when restricted to $L^1(G)$ yields $A(\widehat{G})$. In both cases the correspondence also preserves the operator space structure. However, when G is nonabelian the duality is more philosophical in nature. Still, one can very often translate theorems about $L^1(G)$ and $M(G)$ into corresponding conjectures about $A(G)$ and $B(G)$, respectively. While this principal is valid in a surprising number of circumstances, and it even "preserves" the main aspects of Pontryagan's duality, it fails miserably with respect to problems of cohomology. The reason for this stems from the fact that, while it can be seen that the operator space structures on $L^1(G)$ and $M(G)$ are essentially redundant, for nonabelian groups the operator space structure of the Fourier algebra $A(G)$ is known to carry important information about both $A(G)$ and the underlying group that cannot be obtained from studying its structure as a Banach algebra alone. The first striking illustration of this was an important cohomological result due to Ruan [**35**] which will be outlined below.

In [**22**] Johnson introduced the notion of amenability for a Banach algebra. (A Banach algebra $\mathcal{A}$ is amenable if every bounded derivation from $\mathcal{A}$ into an arbitrary dual Banach-bimodule $\mathcal{X}$ is inner. $\mathcal{A}$ is weakly amenable if every bounded derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner.) He also established a fundamental connection with the more classical notion of amenability for a locally compact group by showing that G is amenable if and only if the group algebra $L^1(G)$ is amenable as a Banach algebra. Since amenable Banach algebras always contain bounded approximate identities and since it is well known that $A(G)$ has a bounded approximate identity if and only if G is amenable, our duality principal would suggest that it would be reasonable to speculate that $A(G)$ would be amenable precisely when G is also amenable. However, Johnson [**23**] eventually provided examples of compact groups for which $A(G)$ is not even weakly amenable as a Banach algebra. Indeed, the author and Runde have recently shown that the only groups for which $A(G)$ is amenable are those groups with abelian subgroups of finite index [**17**]. The key to the difference in behavior between the two settings, $A(G)$ versus $L^1(G)$, seems to lie in the fact that while there is always a natural identification between $L^1(G)\widehat{\otimes}L^1(G)$ and the algebra

$L^1(G \times G)$, it is known that the natural mapping from $A(G)\widehat{\otimes}A(G)$ into $A(G \times G)$ is surjective if and only if G has an abelian subgroup of finite index [**31**]. This in turn can be shown to be a consequence of the fact that a group has an abelian subgroup of finite index precisely when the irreducible representations are of bounded finite degree [**31**]. In contrast, if we replace the Banach space projective tensor product with the operator space projective tensor product, we get a complete isometry between $A(G)\widehat{\otimes}_{op}A(G)$ and $A(G \times G)$. Ruan was able to exploit this identification to show that the proper notion of amenability for $A(G)$ is operator amenability in that the group G is amenable if and only if $A(G)$ is operator amenable (see [**35**] for the relevant definitions).

Carrying on with the theme, we note that Johnson has shown that the group algebra $L^1(G)$ is weakly amenable for every locally compact group [**24**]. That is, every bounded derivation from $L^1(G)$ into $L^\infty(G)$ is inner. Again, our principle of duality could lead us to speculate that the Fourier algebra might also share this property. In fact, as was indicated earlier, in stark contrast with the case of the group algebra it is known that there are compact groups for which the Fourier algebra fails to be weakly amenable. However, Spronk [**37**] has recently established what is surely the proper analog of Johnson's weak amenability theorem by making use of the operator structure to show that the Fourier algebra of every locally compact group is operator weakly amenable.

In [**26, 27, 28**], Khelemskii has given a detailed analysis of the homological properties of Banach and topological algebras. As our final illustration of our thesis that the operator structure contains important information, we will focus on one of Khelemskii's results, namely that $L^1(G)$ is *biprojective* in the category of Banach spaces if and only if the group G is compact. Biprojectivity is a rather strong property and as such we would expect to find that the class of groups for which $A(G)$ is biprojective to be rather restricted. Still, by appealing to the duality between compact and discrete groups, one might hope to show that $A(G)$ would be biprojective if and only if G is discrete. While Khelemskii's result shows that this is true for abelian groups, we should not be surprised to find that this is not true in general. For example, in [**38**], Steiniger showed that if $\mathbf{F}_2$ is the free group on two generators, then $A(\mathbf{F}_2)$ is not biprojective. Moreover, since biprojective algebras with approximate identities are known to be amenable, the above discussion leads us to conclude that the only amenable groups for which $A(G)$ is biprojective in the category of Banach algebras are discrete groups with abelian subgroups of finite index. It is therefore worth noting that as part of a pair of recent studies on the operator space analogs of the basic concepts from homology including biprojectivity, Wood [**39**] and Aristov [**6**] were independently able to establish the natural analog of Khelemskii's result quoted above by showing that $A(G)$ is operator biprojective if and only if G is discrete, lending further evidence to the theory that operator spaces provides the appropriate category for studying the homological or cohomological properties of $A(G)$.

In this paper, we will continue this theme that the operator structure for $A(G)$ carries vital structural information. However, rather than focusing on cohomological or homological properties we shall focus on a variant of Grothendieck's notion of projectivity for a Banach space [**19**]. Indeed, in [**8**], Blecher considered the operator space analog of projectivity for a Banach space. In particular, he was able to completely characterize those preduals of von Neumann algebras which are projective as operator spaces. In this note, we will show how projective operator spaces arise naturally as spaces of coefficient functions of finite dimensional representations of a locally compact group G. Along the way, we will show that the operator space version is again the proper formulation when considering projectivity for the Fourier and Fourier-Stieltjes algebras of a noncommutative locally compact group. In particular, we will show that $A(G)$ is projective as a Banach space if and only if G is compact and abelian while it is projective as an operator space if and only if the group is compact.

Finally, we will connect projectivity for operator spaces with some well-known geometric properties of Banach spaces and then apply what we know to the problem of identifying some completely complemented ideals in the Fourier algebra. of various locally compact groups.

2. Preliminaries. Let G be a locally compact group with a fixed left Haar measure μ_G. We will let $L^1(G)$ denote the group algebra of G and $M(G)$ denote the measure algebra of G. It is a standard fact that $M(G)$ decomposes into a direct sum $M(G) = M_d(G) \oplus M_a(G) \oplus M_s(G)$ where $M_d(G)$ denotes the discrete measures, $M_a(G)$ denotes the measures absolutely continuous with respect to μ_G and $M_s(G)$ denotes the continuous measures that are singular with respect to μ_G. It is also well known that $M_d(G) \cong l_1(G)$ and that $M_a(G) \cong L^1(G)$. Furthermore, if $\mu = \mu_d + \mu_a + \mu_s$, then $\|\mu\| = \|\mu_d\| + \|\mu_a\| + \|\mu_s\|$ [**21**].

Let π be a continuous unitary representation of G. By a coefficient function of π, we will mean a function of the form $u(x) = \langle \pi(x)\zeta, \eta\rangle$ where $\zeta, \eta \in \mathcal{H}_\pi$, the Hilbert space on which π acts.

We say that π is a character of G if $\mathcal{H}_\pi$ is one dimensional. We denote the set of all such characters by char (G).

We will let $B(G)$ denote the space of all such coefficient functions of G. $B(G)$ was introduced by Eymard in [**13**] where it was shown that $B(G)$ could be realized as the linear span of the set of all continuous positive definite functions on G. From this it was deduced that $B(G)$ is the Banach space dual of $C^*(G)$, the group C^*-algebra of G. Finally, with respect to the dual norm and pointwise operations $B(G)$ becomes a Banach algebra called the Fourier-Stieltjes algebra of G.

If π is a continuous unitary representation of G, we denote by $A_\pi(G)$ the closure in $B(G)$ of the linear span of the coefficient functions of π. The dual of $A_\pi(G)$ is the von Neumann algebra $VN_\pi(G)$ generated by $\{\pi(x) : x \in G\}$. $B_\pi(G)$ will denote the weak-* closure of $A_\pi(G)$ in $B(G)$. It can be shown that there exists a representation π_1 such that $A_{\pi_1}(G) = B_\pi(G)$. Moreover, $B(G) = A_\omega(G)$ where ω denotes the universal representation of G [**5**].

Let λ denote the left regular representation on G. Then $A_\lambda(G)$ is denoted simply by $A(G)$ and $VN_\lambda(G)$ is denoted simply by $VN(G)$. $A(G)$ is a closed ideal of $B(G)$ called the Fourier algebra of G. When G is abelian, the classical Fourier transform (or Fourier-Stieltjes transform) determines an isometric isomorphism between $A(G)$ and $L^1(\widehat{G})$ and between $B(G)$ and $M(\widehat{G})$. Here $\widehat{G}$ denotes the dual group of G [**13**]. Similarly $VN(G)$ can be identified with $L^\infty(\widehat{G})$.

Given a (bounded) continuous function $u(y)$ on G and $x \in G$, let $u_x(y) = u(xy)$. The function $u(y)$ is said to be almost periodic if $\{u_x : x \in G\}$ is relatively compact in $C(G)$, the Banach space of all bounded continuous functions on G. The space of all almost periodic functions on G is a C^*-subalgebra of $C(G)$ which we will denote by $AP(G)$. Let π_F denote the direct sum of all of the (nonequivalent) finite dimensional irreducible representations of G. Then it is well known that

$B(G) \cap AP(G) = A_{\pi_F}(G)$ (see [**7**]). If G is abelian, $B(G) \cap AP(G)$ is isometrically isomorphic with $\ell_1(\widehat{G})$.

Let π be a continuous unitary representation of G. Then there exists a central projection $P_\pi \in VN_\omega(G)$ such that $A_\pi(G) = P_\pi B(G)$. In particular, $A_\pi(G)$ is a complemented subspace of $B(G)$. Furthermore, $VN_\pi(G)$ is simply $P_\pi VN_\omega(G)$ [**5**].

Let I a closed ideal of $A(G)$. We define $Z(I) = \{x \in G \mid u(x) = 0$ for every $u \in I\}$. Clearly, $Z(I)$ is a closed subset of G. If $E \subseteq G$ is closed, then we define the closed ideal $I(E)$ of $A(G)$ by $I(E) = \{u \in A(G) \mid u(x) = 0$ for every $x \in E\}$. We let $\mathcal{R}(G)$ denote the ring of subsets of G generated by the set of left cosets of subgroups of G. We then define

$$\mathcal{R}_c(G) = \{E \subseteq G \mid E \in \mathcal{R}(G) \text{ and } E \text{ is closed in } G\}.$$

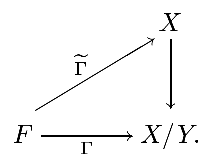
We say that a closed ideal I in $A(G)$ is complemented if there exists a bounded projection P from $A(G)$ onto I. I is completely complemented if the projection P can be chosen to be completely bounded.

A Banach space X is said to have the Schur property if any weakly convergent sequence is norm convergent. X has the Dunford-Pettis property (DPP) if whenever $\{x_n\}$ is weakly null in X and $\{x_n^*\}$ is weakly null in X^*, then $\{x_n^*(x_n)\}$ is null or equivalently if every weakly compact operator defined on X is completely continuous. Finally, X has the Radon-Nykodym property (RNP) if every vector measure of bounded variation on the Borel subsets of $[0,1]$ with values in X has a Bochner-integrable derivative with respect to its variation.

3. Projectivity in $L^1(G)$, $M(G)$, $A(G)$ and $B(G)$. We begin with a few easy facts about Banach space projectivity and the spaces $L^1(G)$, $M(G)$, $A(G)$ and $B(G)$. Recall the following definition:

3.1. Definition. A Banach space F is said to be projective if, given any Banach space X, a closed subspace Y and an $\varepsilon > 0$, every contractive map

$\Gamma : F \to X/Y$ lifts to a bounded map $\widetilde{\Gamma}$ with $\|\Gamma\| \leq 1 + \varepsilon$ such that the following diagram commutes



Grothendieck completely characterized projective Banach spaces by showing that F is projective if and only if F is isometrically isomorphic to $\ell_1(\Omega)$ for some set Ω [**19**]. It follows immediately from this that every projective Banach space must have the Schur property. It is clear from the definition that the Schur property is inherited by closed subspaces. As such, if $M(G)$ is projective, $L^1(G)$ has the Schur property. However, it is easy to see that this can only happen if the Haar measure is completely atomic, that is, if G is discrete. On the other hand, if G is discrete, then $M(G) = L^1(G) = \ell_1(\Omega)$ and as such it follows that $M(G)$ is projective. We have the following simple proposition:

3.2 Proposition. *Let G be a locally compact group. Then the following are equivalent.*

i) *$M(G)$ is projective.*

ii) *$L^1(G)$ is projective.*

iii) *G is discrete.*

A naive appeal to duality would suggest that $B(G)$, and hence $A(G)$, would be projective precisely when G is compact. For abelian groups, this is clearly the case by duality. However, what is less obvious is that commutativity turns out to be essential in this regard. Indeed, we have:

3.3 Proposition. *Let G be a locally compact group. Then the following are equivalent.*

i) *$B(G)$ is projective.*

ii) *$A(G)$ is projective.*

iii) *G is compact and commutative.*

Proof. If either $B(G)$ or $A(G)$ is projective, then $A(G)$ has the Schur property. However, it is known that $A(G)$ has the Schur property if and only if G is compact ([**7, 30**]). It then follows that $B(G) = A(G)$ and hence that i) and ii) are equivalent.

Assume that $A(G)$ is projective and hence that it has the Schur property. Then again, as above, G is compact. As such, we get that

$$A(G) = \ell_1 - \sum \oplus A_{\pi_\alpha},$$

where each π_α is a finite dimensional irreducible representation [**5**]. Since A_{π_α} is an ℓ_1-direct summand, there is a contractive projection from $A(G)$ onto A_{π_α}. From this it is routine to show that A_{π_α} is also projective. However, since π_α is irreducible, A_{π_α} is isometrically isomorphic as a Banach space with $TC(\mathcal{H}_{\pi_\alpha})$ [**5**], the space of trace class operators on the finite dimensional Hilbert space $\mathcal{H}_{\pi_\alpha}$ on which π_α is represented. Now, $TC(\mathcal{H}_{\pi_\alpha})$ is projective if and only if $\mathcal{H}_{\pi_\alpha}$ is one dimensional, that is, if and only if each π_α is a character. Finally, every irreducible representation of G is included in the decomposition. As such, G is a compact locally compact group with only one dimensional irreducible representations. This shows that G is also abelian. Hence i) implies ii).

That iii) implies i) or ii) follows from the dual version of the previous proposition. □

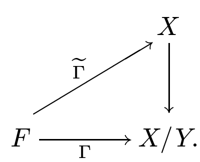
If we restrict ourselves to closed translation invariant subpaces of $B(G)$, then we can identify the projective part of $B(G)$. Indeed, the proof of the previous proposition shows us that such a space M is of the form:

$$M = \ell_1 - \sum_{\gamma \in \Omega} \oplus A_\gamma = \ell_1(\Omega)$$

where $\Omega \subseteq \mathit{char}\ (G)$. If G is abelian, then this result is as it should be since in this case $\Omega = \widehat{G}$ and $\ell_1(\widehat{G})$ is the "projective part" of $M(\widehat{G}) \cong B(G)$. However, we will see in the next section that when G is not abelian part of the story is missing. Indeed, in this case, the natural dual analog of $\ell_1(G)$ is not $\ell_1 - \sum_{\gamma \in \mathit{char}\,(G)} \oplus A_\gamma$, but is rather $B(G) \cap AP(G)$. The latter space carries all of the information encoded in the finite dimensional representations of G whereas the former space recognizes only the one dimensional representations and is therefore very much a "commutative" object.

4. Projective operator spaces in $B(G)$. We have seen in the previous section that Banach space projectivity for the Fourier and Fourier-Stieltjes algebras is an inherently commutative phenomenon. Our main focus in this section is to demonstrate how projective operator spaces arise naturally in noncommutative harmonic analysis. We begin by recalling the following definitions which were given by Blecher in [**8**]:

4.1. Definition. A uniformly closed operator space F is said to be projective if given any operator space X, a closed subspace Y and an $\varepsilon > 0$, every completely contractive map $\Gamma : F \to X/Y$ lifts to a completely bounded map $\widetilde{\Gamma}$ with $\|\widetilde{\Gamma}\|_{cb} \leq 1 + \varepsilon$ such that the following diagram commutes



We say that a dual operator space E is weak-* injective if whenever X is a dual operator space and Y is a weak-* closed subspace of X given any weak-* continuous completely bounded map $T : Y \to E$ and an $\varepsilon > 0$, there is a weak-* continuous completely bounded extension $\widetilde{T} : X \to E$ with $\|\widetilde{T}\|_{cb} \leq \|T\|_{cb} + \varepsilon$.

The next theorem can be viewed as an extension of [**8**, Theorem 3.12]. It shows that for preduals of von Neumann algebras projectivity can be determined by geometric means.

4.2. Theorem. *Let V be a von Neumann algebra. Then the following are equivalent*:

i) *V is a weak-* injective operator space*

ii) *V_* is a projective operator space*

iii) *V is a finite atomic von Neumann algebra*

iv) V_* *has the Schur property*

v) V_* *has the* RNP *and* DPP.

Proof. The equivalence of i) and ii) follows from the observation that an operator space E is projective if and only if its standard dual E^* is weak-* injective [**8**]. The equivalence of i) and iii) is [**8**, Theorem 3.12].

The equivalence of iii) and iv) is due to Hamana [**20**, Theorem 3].

Chu showed that the predual of a von Neumann algebra V has the RNP if and only if V is atomic [**10**]. Chu and Iochum [**11**] proved that if V is type I, finite, then V_* has the DPP and that if V_* has the DPP, then V must be finite. (Bunce [**9**] completed the characterization of preduals of von Neumann algebras with the DPP by showing that if V_* has the DPP, then V must be type I, finite.) Combining these results establishes the equivalence of i) and v) thereby completing the proof. □

We note that Blecher [**8**] showed that any projective operator space has the Schur property. This clearly implies that a projective operator space has the DPP. We will now show that any such space also has the RNP.

4.3. Proposition. *Let E be a projective operator space. Then E has both the* RNP *and the* DPP.

Proof. We have already established that E has the DPP.

If V_* is the predual of a finite atomic von Neumann algebra, then we have that V_* has the RNP by Theorem 3.1 (or [**10**]). It is well known that the RNP is inherited by closed subspaces [**12**]. This shows that any closed subspace of V_* will also have the RNP.

In [**8**, Theorem 3.12] Blecher showed that any projective operator space is linearly and topologically isomorphic to a complemented subspace of the predual of a finite atomic von Neumann algebra. Since the RNP is clearly preserved by Banach space isomorphisms, we conclude that E has the RNP. □

Our main goal in this section is to look for projective operator spaces that arise naturally in harmonic analysis. To motivate our search, we appeal to the abelian case. As we have seen, in this case $B(G) \cap AP(G) \approxeq \ell_1(\widehat{G})$ is a projective Banach space in the sense of Grothendieck. Moreover, since the space $\ell_1(\widehat{G})$ is the predual of an abelian von Neumann algebra its natural operator space structure is the MAX structure [**8**]. This is in a strong sense a trivial operator space structure in that any linear map from a MAX operator space into an operator space that is bounded is also completely bounded with the same bound. This means that when working with group algebras, or measure algebras, the operator space structure remains hidden. In particular, it means that projectivity in the category of operator spaces corresponds exactly to Grothendieck's notion for $\ell_1(\widehat{G})$. Moreover, since the identification between $B(G) \cap AP(G)$ and $\ell_1(\widehat{G})$ is a complete isometry, we conclude that if G is abelian, $B(G) \cap AP(G)$ is also a projective operator space.

For nonabelian groups, the situation with respect to the operator space structures on $A(G)$ and $B(G)$ is quite different. Indeed, the operator space structures on $A(G)$ and $B(G)$ are the respective MAX structures precisely when G is abelian [**15**]. This is the reason why the cohomology results translate from $L^1(G)$ to $A(G)$ for abelian groups but not for nonabelian groups. We have seen that with respect to our principal of duality the proper cohomology for group algebras really is the completely bounded cohomology. We can now show that this is also true for projectivity. In fact, we can use Theorem 3.1 to show how even for noncommutative groups, projective operator spaces once again arise as spaces of almost periodic functions.

4.4. Theorem. *Let π be a continuous unitary representation of G. Then the following are equivalent*:

i) *$A_\pi(G)$ is a projective operator space.*

ii) *$A_\pi(G) \subseteq B(G) \cap AP(G)$.*

iii) *π is the direct sum of finite dimensional irreducible representations.*

iv) *$A_\pi(G)$ has the* RNP *and the* DPP.

Proof. This follows immediately from Theorem 3.1 and from [**7**, Theorem 27]. □

4.5. Corollary. *Let M be a closed subspace of $B(G)$ that is projective as an operator space. Let M_1 be a closed translation invariant subspace of M. Then $M_1 \subseteq B(G) \cap AP(G)$.*

Proof. Since M_1 is closed and translation invariant, there exists a continuous unitary representation π of G such that $M_1 = A_\pi(G)$. Proposition 4.2 implies that M has both the RNP and the DPP. It is easy to see that this means that $M_1 = A_\pi(G)$ also has both the RNP and the DPP. Theorem 3.3 shows that $M_1 \subseteq B(G) \cap AP(G)$. □

We are now in a position to establish the proper analog of Proposition 4.1.

4.6. Theorem. *Let G be a locally compact group. Then the following are equivalent*:

i) *G is compact.*

ii) *$B(G)$ is a projective operator space.*

iii) *$A(G)$ is a projective operator space.*

Proof. If G is compact, then $B(G) = B(G) \cap AP(G)$ [**13**]. It follows from Theorem 4.3 that $B(G)$ is projective as an operator space.

If $B(G)$ is projective as an operator space, then since $B(G) = A_\omega(G)$, Theorem 4.3 implies that ω is the direct sum of finite dimensional irreducible representations. It follows that the left regular representation λ also has this property and hence that $A(G) = A_\lambda(G)$ is a projective operator space.

Finally, assume that $A(G)$ is a projective operator space. Then $A(G) \subseteq B(G) \cap AP(G)$. But $A(G) \subset C_0(G)$ [**13**]. If G is not compact, then it is easy to see that $C_0(G) \cap AP(G) = \{0\}$. Hence it must be that G is compact. □

Even if G is not compact it is certainly possible for $A(G)$ to contain closed subspaces that are projective as operator spaces. Simply take any one dimensional subspace for example. However, the previous theorem shows that if G is noncompact, then these spaces are from the perspective of harmonic analysis not quite natural in the sense that they are not translation invariant. In fact, we can show that if G is noncompact and if M is any subspace of $A(G)$ that is projective as an operator space, then for any nonzero $u \in M$ the set of translates of u that belong to M is necessarily very restricted.

4.7. Proposition. *Let M be a closed subspace of $B(G)$ which is projective as an operator space. Let $u \in A(G) \cap M$ be nonzero. Let $S = \{x \in G \mid u_x \in M\}$. Then S is compact.*

Proof. First observe that since the map $x \to u_x$ is continuous, S is closed.

Assume that S is not compact. Then certainly G is noncompact. By modifying an argument of Miao's [**32**, Lemma 3.1] in an obvious way, we get a sequence $\{x_n\} \subseteq S$ such that the sequence $\{u_{x_n}\}$ converges to 0 in the $\sigma(A(G), VN(G))$ topology. In particular, $\{u_{x_n}\}$ converges to 0 in the $\sigma(M, M^*)$ topology. Since M has the Schur property, $\{u_{x_n}\}$ converges to 0 in the norm topology. However, since $\|u\|_{B(G)} = \|u_{x_n}\|_{B(G)}$ for each $n \in \mathbf{N}$, this is only possible if $u = 0$. □

5. Completely complemented ideals in $A(G)$. The problem of characterizing complemented ideals in the group algebra of a locally compact abelian group, and hence the Fourier algebra of its dual, has a long history. In [**33**] Newman showed that if Π is the circle group and if $H^1 = \{f \in L^1(\Pi) \mid \widehat{f}(n) = 0 \text{ for every } n < 0\}$, then H^1 is not complemented in $L^1(\Pi)$. A year later, in [**36**], Walter Rudin showed that an ideal I in $A(G)$ is complemented if and only if $I = I(A)$ where

$$A = \bigcup_{i=1}^{n} (a_i \mathbf{Z} + b_i).$$

In [**34**], Rosenthal showed that a necessary condition for I to be complemented would be that $I = I(A)$ where $A \in \mathcal{R}_c(G)$. If G is compact, and hence $\widehat{G}$ is discrete, it is routine to show that the converse of Rosenthal's theorem holds. However, Rosenthal's condition is far from sufficient in general. For example, Alspach and Matheson [**2**] showed that in $L^1(\mathbf{R})$ a closed ideal I is complemented if and only if $I = I(A)$ where

$$A = \bigcup_{i=1}^{n} (a_i\mathbf{Z} + b_i)\backslash F$$

with the a_i's being pairwise rationally dependent and F is finite. Alspach, Matheson and Rosenblatt [**3**] (see also [**4**]) looked at arbitrary abelian locally compact groups. They succeeded in finding necessary and sufficient conditions for an ideal with a discrete hull to be complemented. They also developed a complicated inductive procedure that was then used successfully in [**1**] by Alspach to completely characterize the complemented ideals in $L^1(\mathbf{R}^2)$. Some further progress was made on the complementation problem for abelian groups by Kepert in his thesis [**25**] but there is still no characterization of the complemented ideals even in $L^1(\mathbf{R}^3)$.

In [**15**], we made the first attempt at studying the complemented ideal problem for the Fourier algebra of a nonabelian group. Our first goal was to try to establish the analog of Rosenthal's theorem by showing that if $I \subseteq A(G)$, then $I = I(A)$ where $A \in \mathcal{R}_c(G)$. We were unable to do this. In fact, the result turned out not to be true. As pointed out by Leinert, there are idempotent completely bounded multipliers of the Fourier algebra of the free group on two generators that are not in $B(G)$. These are characteristic functions of sets that are not in $\mathcal{R}_c(G)$. Multiplication by such functions leads to (completely) complemented ideals of $A(G)$ with hulls that are again not in $\mathcal{R}_c(G)$. In the end, we settled for trying to identify some nontrivial complemented ideals in some nonabelian groups.

It easy to show that for any closed subgroup H of an abelian locally compact group G, the ideal $I(H)$ is complemented [**3**]. Since such ideals are the building blocks of all known complemented ideals for abelian groups, one might hope that $I(H)$ would again generate a complemented ideal in any nonabelian group. While we showed that this was the case if the subgroup H was compact [**15**, Proposition 3.3 and Corollary 3.7] or central [**15**, Proposition 3.3 and Corollary 4.2], we were also able to show that even if H is a normal abelian subgroup of an amenable group, $I(H)$ need not be complemented in $A(G)$. While a substantial amount of work has been done since then by various authors on weakly complemented ideals, (ideals in which $I^{\perp}$ is complemented in $VN(G)$), no progress was made on the complemented ideal problem until very recently.

It is easy to see that Rosenthal's theorem can be obtained as a consequence of the Banach algebra amenability of the group algebra of an abelian group. After developing the necessary tools from completely bounded cohomology, Wood exploited Ruan's theorem on the operator amenability of $A(G)$ to show that if G is an amenable locally compact group and if I is a closed completely complemented ideal in $A(G)$, then $I = I(A)$ where $A \in \mathcal{R}_c(G)$ [**39**]. If we again note that the group algebra of an abelian group has a trivial operator space structure, we are led to the conclusion that Wood's result is the proper extension of Rosenthal's theorem to the nonabelian setting and that our focus [**15**] should really have been on completely complemented ideals. In the same paper, Wood was able to establish the converse of the above result and as such completely characterized the completely complemented ideals in the Fourier algebra of an amenable discrete group. However, for nondiscrete and nonabelian groups there were no examples of completely complemented ideals presented.

We will now show that if K is any compact subgroup of a locally compact group G, then $I(K)$ is completely complemented. To do so we begin with the following result that is a partial analog of [**15**, Proposition 3.3].

5.1. Lemma. *Let H be a closed subgroup of a locally compact group G. Assume that there exists a completely bounded linear map $\Gamma : A(H) \to A(G)$ such that $(\Gamma u)_{|_H} = u$ for every $u \in A(H)$. Then $I(H)$ is completely complemented in $A(G)$.*

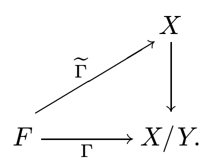
Proof. We first note that the restriction map is a completely contractive homomorphism from $A(G)$ onto $A(H)$. It follows that the map $Q : A(G) \to A(G)$ given by

$$Q(V) = v - (\Gamma v_{|_H})$$

ia a completely bounded map. Moreover, just as in [**15**, Proposition 3.3], Q is easily seen to be a projection from $A(G)$ onto $I(H)$. □

5.2. Theorem. *Let G be a locally compact group. If K is any compact subgroup of G, then there exists a completely bounded linear map $\Gamma : A(K) \to A(G)$ such that $(\Gamma u)_{|_H} = u$ for every $u \in A(K)$. In particular, $I(K)$ is completely complemented.*

Proof. By Theorem 4.5, $A(K)$ is operator projective. Also, the restriction map establishes a canonical completely isometric isomorphic between $A(K)$ and $A(G)/I(H)$ [**18**, Proposition 4.2]. Let $\varepsilon > 0$. If we let $F = A(K)$, $X = A(G)$, $Y = I(K)$ and $\Gamma : A(K) \to A(G)/I(H)$ be the inverse of the canonical isomorphism between $A(K)$ and $A(G)/I(H)$, then by operator projectivity, $\Gamma : F \to X/Y$ lifts to a completely bounded map $\widetilde{\Gamma}$ with $\|\widetilde{\Gamma}\|_{cb} \leq 1 + \varepsilon$ such that the following diagram commutes



However, for the diagram to commute, we must have that $(\widetilde{\Gamma} u)_{|_H} = u$ for every $u \in A(K)$. It now follows from Lemma 5.1 that $I(K)$ is completely complemented. □

We remarked earlier that if H is a closed subgroup of an abelian group G, then $I(H)$ is complemented in $A(G)$. Let $P : A(G) \to I(H)$ be a continuous projection. Since G is abelian, $A(G)$ has the MAX operator space structure. This means that P is also completely bounded and that $\|P\|_{cb} = \|P\|$. In particular, $I(H)$ is completely complemented in $A(G)$. We will now see that this holds whenever an abelian closed subgroup is such that $I(H)$ is complemented.

5.3. Proposition. *Let H be a closed abelian subgroup of G. Assume that $I(H)$ is complemented in $A(G)$. Then $I(H)$ is completely complemented in $A(G)$.*

Proof. Since $I(H)$ is complemented in $A(G)$, there exists a bounded linear map $\Gamma : A(H) \to A(G)$ such that $(\Gamma u)_{|_H} = u$ for every $u \in A(H)$ [**15**, Proposition 3.3]. Since H is abelian, $A(H)$ has the MAX operator space structure. Consequently, Γ is also completely bounded and $\|\Gamma\|_{cb} = \|\Gamma\|$. □

5.4. Corollary. *Let $G \in [SIN]$ be almost connected. Let H be a closed abelian subgroup of G. Then $I(H)$ is completely complemented in $A(G)$.*

Proof. It follows from [**16**, Corollary 4.15] that $I(H)$ is complemented in $A(G)$. The previous propositions show that $I(H)$ is completely complemented in $A(G)$. □

5.5. Corollary. *Let $G \in [IN]$ be almost connected. Let V be a closed vector subgroup of G. Then $I(V)$ is completely complemented in $A(G)$.*

Proof. It follows from [**15**, Proposition 4.16] that $I(V)$ is complemented in $A(G)$. Again, the previous proposition shows that $I(V)$ is completely complemented in $A(G)$. □

5.6. Definition. We will follow [**15**, Definition 3.1] and say that a closed subgroup H of G has the completely complemented ideal property $(CCIP)$ in G if $I(H)$ is completely complemented in $A(G)$. We let $CCIP(G) = \{H \mid H \text{ has the } CCIP \text{ in } G\}$.

Let G_1 and G_2 be locally compact groups. Let $\psi : A(G_1)\widehat{\otimes}_{op}A(G_2) \to A(G_1 \times G_2)$ be given by

$$\psi(u \otimes v)(s,t) = u(s)v(t).$$

Then ψ establishes a complete isometry between $A(G_1)\widehat{\otimes}_{op}A(G_2)$ and $A(G_1 \times G_2)$. The following lemma is similar to [**15**, Lemma 4.4].

5.7. Lemma. *Let $G = G_1 \times G_2$. Then for each $i = 1, 2$ there exists a linear complete isometry $\Gamma_i : A(G_i) \to A(G_1 \times G_2)$ such that $\Gamma_1(u)(g_1, e_2) = u(g_1)$ for each $u \in A(G_1), g_1 \in G_1$ and $\Gamma_2(v)(e_1, g_2) = v(g_2)$ for each $v \in A(G_2), g_2 \in G_2$.*

Proof. Choose $v_2 \in A(G_2)$ such that $v_2(e_2) = 1$ and $\|v_2\|_{A(G_2)} = 1$. Define $\Gamma_1(u) = \psi(u \otimes v_2)$. Since

$$\begin{aligned}\|(\Gamma_1)_n([u_{i,j}])\|_n &= \|[\psi(u_{i,j} \otimes v_2)]\|_n \\ &= \|\psi_n[u_{i,j} \otimes v_2]\|_n \\ &\leq \|\psi_n\|\|[u_{i,j}]\|_n\|v_2\|_{A(G_2)} \\ &= \|[u_{i,j}]\|_n,\end{aligned}$$

Γ_1 is a complete contraction. Moreover, $\Gamma_1(u)(g_1, e_2) = \psi(u \otimes v_2)(g_1, e_2) = u(g_1)v_2(e_2) = u(g_1)$ as desired. Finally, to see that Γ_1 is a complete isometry, we note that $A(G_1)$ is completely isometrically isomorphic to $A(G_1 \times \{e_2\})$ and the restriction map from $A(G_1 \times G_2)$ is also completely contractive.

The map Γ_2 is defined in a similar manner. □

5.8. Corollary. *Let $G = G_1 \times G_2$. Then $G_1 \times \{e_2\}, \{e_1\} \times G_2 \in CCIP(G_1 \times G_2)$.*

The next lemma follows exactly as in [**15**, Lemma 4.4] even though our map ψ is defined on the operator space projective tensor product $A(G_1)\widehat{\otimes}_{op}A(G_2)$.

5.9. Lemma. *Let $G = G_1 \times G_2$. Let $A = A_1 \times A_2$ where A_i is closed in G_i. Assume that A is a set of spectral synthesis in G. Then $I_G(A)$ is the closed linear span $\langle J\rangle^-$ of*

$$J = \{\psi(I_{G_1}(A_1) \otimes A(G_2)) \bigcup (A(G_1) \otimes I_{G_2}(A_2))\}.$$

The next theorem can be viewed as the completely bounded analog of [**15**, Proposition 4.5]. It is important to note that we have removed the assumption from [**15**, Proposition 4.5] that one of the groups have an abelian subgroup of finite index.

5.10. Theorem. *Let $G = G_1 \times G_2$. Let $A = A_1 \times A_2$ where A_i is closed in G_i. Assume that A is a set of spectral synthesis in G. Assume* *also that $I_{G_i}(A_i)$ is completely complemented in $A(G_i)$. Then $I(A)$ is completely complemented in $A(G_1 \times G_2)$.*

Proof. Since $\psi : A(G_1)\widehat{\otimes}_{op}A(G_2) \to A(G_1 \times G_2)$ is a completely isometric isomorphism so is $\psi^{-1} : A(G_1 \times G_2) \to A(G_1)\widehat{\otimes}_{op}A(G_2)$. Let P_i be a completely bounded projection from $A(G_i)$ onto $I(A_i)$. Define $P : A(G) \to A(G)$ by

$$P(u) = u - \psi\{((1 - P_1) \otimes (1 - P_2))(\psi^{-1}(u))\}.$$

Then P is completely bounded.

Let $u_i \in A(G_i)$, and let $(x_1, x_2) \in A = A_1 \times A_2$. If $u = \psi(u_1 \otimes u_2)$, then

$$\begin{aligned} P(u)(x_1, x_2) &= u_1(x_1)u_2(x_2) \\ &\quad - [1 - P_1]u_1(x_1)[1 - P_2]u_2(x_2) \\ &= u_1(x_1)u_2(x_2) - u_1(x_1)u_2(x_2) \\ &= 0. \end{aligned}$$

It follows that $P(u) \in I_G(A)$ for each $u \in A(G)$.

Let $u_1 \in I_{G_1}(A_1)$ and $u_2 \in A(G_2)$. Let $u = \psi(u_1 \otimes u_2)$. Then

$$\begin{aligned} &\psi\{((1 - P_1) \otimes (1 - P_2))(\psi^{-1}(u))\} \\ &\qquad = \psi\{(1 - P_1)(u_1) \otimes (1 - P_2)(u_2)\} \\ &\qquad = \psi\{0 \otimes (1 - P_2)(u_2)\} \\ &\qquad = 0. \end{aligned}$$

Thus $P(u) = u$. Similarly, we have that if $u_2 \in I_{G_2}(A_2)$, $u_1 \in A(G_1)$ and $u = \psi(u_1 \otimes u_2)$, then $P(u) = u$. It now follows immediately from the previous lemma that $P(u) = u$ for each $u \in I_G(A)$. □

5.11. Corollary. *Let $G = G_1 \times G_2$ where each G_i is amenable. Let $A = A_1 \times A_2$ where A_i is closed in G_i. Assume also that $I_{G_i}(A_i)$ is completely complemented in $A(G_i)$. Then $I(A)$ is completely complemented in $A(G_1 \times G_2)$.*

Proof. It follows from [**40**, Theorem 5] that if $I_{G_i}(A_i)$ is completely complemented in $A(G_i)$, then $A_i \in \mathcal{R}_c(G_i)$. This shows that $A \in$

$\mathcal{R}_c(G)$. However, since G is also amenable, A is a set of spectral synthesis [**16**, Lemma 2.2]. □

5.12. Corollary. *Let $G = G_1 \times K_2$ where each G_1 is abelian and K_2 is compact. Let $H = H_1 \times H_2$ where H_1 is a closed subgroup of G_1 and H_2 is a closed subgroup of K_2. Then $H \in CCIP(G)$.*

REFERENCES

1. D.E. Alspach, *A characterization of the complemented translation invariant subspaces of $L^1(\mathbf{R}^2)$*, J. London Math. Soc. **31** (1985), 115–124.

2. D.E. Alspach and A. Matheson, *Projections onto translation invariant subspaces of $L^1(\mathbf{R})$*, Trans. Amer. Math. Soc. **77** (1984), 815–823.

3. D.E. Alspach, A. Matheson and J. Rosenblatt, *Projections onto translation invariant subspaces of $L^1(G)$*, J. Functional Anal. **59** (1984), 254–292.

4. ———, *Separating sets by Fourier-Stieltjes transforms*, J. Functional Anal. **84** (1989), 297–311.

5. G. Arsac, *Sur l'espaces de Banach engendré par les coefficients d'une représentation unitaire*, Publ. Dép. Math. (Lyon) **13** (1976), 1–101.

6. O.Yu. Aristov, *Biprojective algebras and operator spaces*, J. Math. Sci. **111** (2002), 3339–3386.

7. A. Belanger and B. Forrest, *Geometric properties of coefficient function spaces determined by unitary representations of a locally compact group*, J. Math. Anal. **193** (1995), 390–405.

8. D.P. Blecher, *The standard dual of an operator space*, Pacific J. Math. **153** (1992), 15–30.

9. L.J. Bunce, *The Dunford-Pettis property in the predual of a von Neumann algebra*, Proc. Amer. Math. Soc. **116** (1992), 99–100.

10. C.H. Chu, *A note on scattered C^*-algebras and the Radon-Nikodym property*, J. London Math. Soc. **24** (1981), 533–536.

11. C.H. Chu and B. Iochum, *The Dunford-Pettis property in C^*-algebras*, Studia Math. **97** (1990), 59–64.

12. J. Diestel and J.J. Uhl, *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, RI, 1977.

13. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

14. B. Forrest, *Amenability and weak amenability of the Fourier algebra,* preprint.

15. ———, *Complemented ideals in the Fourier algebra and the Radon Nikodym property*, Trans. Amer. Math. Soc. **333** (1992), 689–700.

16. B. Forrest, E. Kaniuth, A.T. Lau and N. Spronk, *Ideals with bounded approximate identities in Fourier algebras*, J. Functional Anal. **203** (2003), 286–304.

17. B. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra*, Math. Z. **250** (2005), 731–744.

18. B.E. Forrest and P. Wood, *Cohomology and the operator space structure of the Fourier algebra and its second dual*, Indiana Univ. Math. J. **50** (2001), 1217–1240.

19. A. Grothendieck, *Une caractrisation vectorielle-metrique des espaces* L_1, Canad. J. Math. **7** (1955), 552–561.

20. M. Hamana, *On linear topological properties on some* C^**-algebras*, Tohôku Math. J. **29** (1977), 157–163.

21. E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer-Verlag, Berlin, 1973.

22. B. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972).

23. ———, *Nonamenability of the Fourier algebra for compact groups*, J. London Math. Soc. **50** (1994), 361–374.

24. ———, *Weak amenability of group algebras*, Bull. London Math. Soc. **50** (1994), 361–374.

25. A. Kepert, *Banach algebra amenability and group algebras*, Ph.D. thesis, Australian National University, 1992.

26. A.Ya. Khelemskii, *Flat Banach modules and amenable algebras*, Trans. Moscow Math. Soc., 1984; Amer. Math. Soc. Transl. (1985), 199–224.

27. ———, *The homology of Banach and toplogical algebras*, Mathematics and its Applications (Soviet Series) **41**, Kluwer Academic Publishers, 1986.

28. ———, *Banach and locally convex algebras*, Oxford Science Publications, 1992.

29. A.T. Lau, R.J. Loy and G.A. Willis, *Amenability of Banach and* C^**-algebras on locally compact groups*, Studia Math. **119** (1996), 161–178.

30. A.T. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier-Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. **377** (1993), 321–359.

31. V. Losert, *On tensor products of the Fourier algerba*, Arch. Math. **43** (1984), 370–372.

32. T. Miao, *Compactness of a locally compact group* G *and some geometric properties of* $A_p(G)$, Canad. J. Math. **48** (1996), 1273–1285.

33. D.J. Newman, *The nonexistence of projections from* L^1 *to* H^1, Proc. Amer. Math. Soc. **12** (1961), 98–99.

34. H.P. Rosenthal, *Projections onto translation invariant subspaces of* $L^p(G)$, Mem. Amer. Math. Soc. **63** (1966).

35. Z-J. Ruan, *The operator amenability of* $A(G)$, Amer. J. Math. **117** (1995), 1449–1476.

36. W. Rudin, *Projections on invariant subspaces*, Proc. Amer. Math. Soc. **13** (1962), 429–432.

37. N. Spronk, *Operator weak amenability of the Fourier algebra*, Proc. Amer. Math. Soc. **130** (2002), 3609–3617.

38. H. Steiniger, *Finite-dimensional extensions of Fourier algebras*, preprint.

39. Peter J. Wood, *The operator biprojectivity of the Fourier algerba*, Canad. J. Math. **54** (2002), 1100–1120.

40. ———, *Complemented ideals in the Fourier algebra of a locally compact group*, Proc. Amer. Math. Soc. **128** (2000), 445–451.

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