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A WEIGHTED PARTITION FUNCTION CONNECTED TO THE ROGERS-SZEGO POLYNOMIALS

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ABSTRACT. In this paper we present a weighted partition function which is connected to the Rogers-Szego polynomials. The function is also connected to the generating function for $C_4(n)$, the number of four-component multipartitions of n in which each part in the ith $(i = 1, 2, 3)$ component is larger than the number of parts in the next component with the 4th component's parts being larger than the number of parts in the 1st component.

1. Introduction. In [**1**] Andrews showed that the generating function for $C_4(n)$, the number of four-component multipartitions of n in which each part in the ith $(i = 1, 2, 3)$ component is larger than the number of parts in the next component with the 4th component's parts being larger than the number of parts in the 1st component, can be written as

$$\sum_{n=0}^{\infty} C_4(n)q^n = \sum_{i,j,k,m\geq 0} \frac{q^{i(j+1)+j(k+1)+k(m+1)+m(i+1)}}{(q)_i(q)_j(q)_k(q)_m} = \frac{1}{(q)_\infty} \sum_{i,j\geq 1} \frac{q^{ij-1}}{(q)_{i+j-1}}.$$

In this paper we will investigate the sum $\sum_{i,j\geq 1}(q^{(ij)})/(q)_{i+j-1}$. This is the second factor in Andrews' result multiplied by q. We will show that this generating function can be interpreted as a weighted partition function and that this function can be expressed in terms of the Rogers-Szego polynomials in two different ways. The proofs presented will be explained combinatorially.

2. How can we interpret our function as a weighted partition function? Using a Ferrers graph we will show how each term in our sum, $q^{ij}/(q)_{i+j-1}$, (for a fixed choice of i and j) can be used to generate

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ordinary partitions. Since for different choices of i and j we can get the same ordinary partition, the resulting interpretation will be that of a weighted partition generating function. The exponent on q in the numerator, ij, will be used to form i rows of j nodes. The partitions generated by $1/(q)_{i+j-1}$ are partitions where the parts are at most $i+j-1$ or, if we look at conjugates, these are partitions into at most $i+j-1$ parts. We will view these as the latter and will split these partitions into two pieces–the first piece will consist of the largest i parts in the partition (the portion of the large triangular region in Figure 1 that is above the line segment) and the second piece will consist of the remaining parts (the portion of the large triangular region below the line segment in Figure 1). Note that if the partition contains i or fewer parts then the first piece will be the complete partition and the second piece will be the empty partition (the portion below the line segment in Figure 1 will not exist). The number of parts in the second piece is at most $j-1$. To form our ordinary partition we will place the first piece to the right of our rectangle of $i \times j$ nodes and will place the conjugate of the second piece (a partition with parts $\leq j-1$) below our rectangle of $i \times j$ nodes. This process is illustrated in Figures 1 and 2.

Let me illustrate that this interpretation counts certain ordinary partitions more than once. The partition of $5 = 4 + 1$ is counted with $i = 1$, $j = 2$; $i = 1$, $j = 3$; and $i = 2$, $j = 1$ as shown in Figure 3.

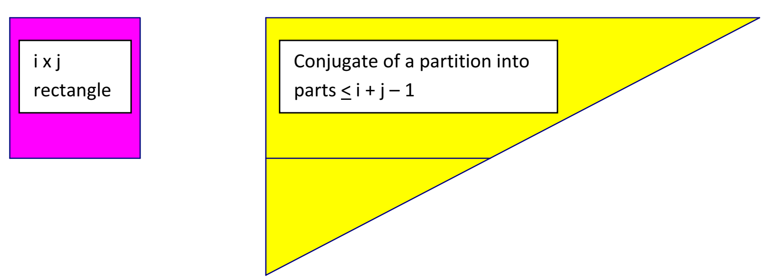

FIGURE 1.

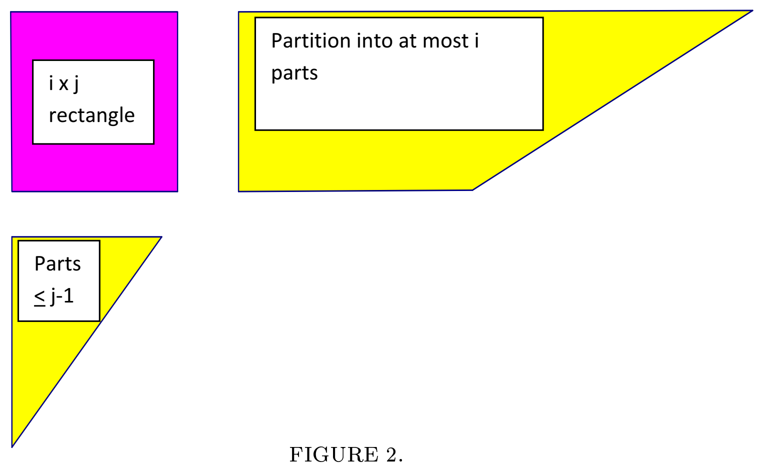

FIGURE 2.

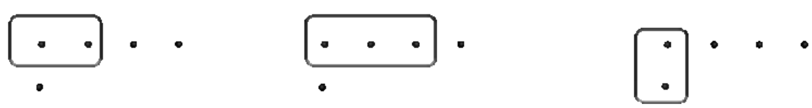

FIGURE 3.

The number of times that a particular partition is counted will be called its weight and we will describe how the weight is determined in the next section.

3. How do we determine the weight of an ordinary partition? All we have to do is look at the process of Section 2 in reverse. An ordinary partition, $\lambda = \lambda_r + \lambda_{r-1} + \cdots + \lambda_2 + \lambda_1$, where $\lambda_r \geq \lambda_{r-1} \geq \cdots \geq \lambda_2 \geq \lambda_1$, is counted for a particular i and j if its Ferrers graph has an $i \times j$ rectangle in the upper left hand corner and, when there are parts below the rectangle ($r > i$), there are at most $\lambda_{r-i+1} - j$ such parts and they are all less than j. For $i = r$ we see that j can vary from 1 to λ_1 and for $i < r$ we see that a rectangle of height i exists provided $\lambda_{r-i+1} - \lambda_{r-i} - (r-i) > 0$. The width of the rectangle, in this case, can vary from $\lambda_{r-i} + 1$ to $\lambda_{r-i+1} - (r-i)$. Thus we see that the weight of λ is given by $\lambda_1 + \sum_{i=1}^{r-1} v(i)$ where $v(i) = 0$ if $\lambda_{r-i+1} - \lambda_{r-i} - (r-i) \leq 0$ and equals $\lambda_{r-i+1} - \lambda_{r-i} - (r-i)$ otherwise.

In the next two sections we will look at two different ways to rewrite this generating function.

4. How can this generating function be expressed in a different form? We will rewrite our generating function based on the size of the part containing the bottom edge of the $i \times j$ rectangle. If the part containing the bottom edge of the rectangle is $m = 1$, then $j = 1$ as well and there are no parts below the rectangle. These partitions are generated by $q/(q)_\infty$. When the part containing the bottom edge of the rectangle is $m = 2$, $j = 1$ or 2 and again there are no parts below the rectangle. These partitions are generated by $q^2/(q^2)_\infty + q^2/(q^2)_\infty$. When $m = 3$, $j = 1$, 2, or 3, and we can have a single node below the rectangle for $j = 2$ and no nodes below the rectangle for $j = 1$ and $j = 3$. Thus, for $m = 3$, the partitions are generated by

$$\frac{q^3}{(q^3)_\infty} + \frac{q^3}{(q^3)_\infty} \cdot (q+1) + \frac{q^3}{(q^3)_\infty}.$$

In general, if the part containing the bottom edge of the rectangle is m, then $j = 1, 2, \ldots,$ or m and the generating function for these partitions is

$$\frac{q^m}{(q^m)_\infty} \sum_{j=1}^{m} \begin{bmatrix} m-1 \\ m-j \end{bmatrix}$$

since, for a given j, there can be at most $m - j$ parts below the rectangle with each part less than j. We obtain the following theorem.

Theorem 1.

$$\sum_{i,j \geq 1} \frac{q^{ij}}{(q)_{i+j-1}} = \sum_{m \geq 1} \frac{q^m}{(q^m)_\infty} \sum_{j=1}^{m} \begin{bmatrix} m-1 \\ m-j \end{bmatrix}.$$

This theorem is the result in the fourth line of Andrews' proof of Theorem 11 [**1**] (simply let $k + i = m - 1$). The inner sum is just the Rogers-Szego polynomial $H_k(t) = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} t^j$ [**2**] when $t = 1$ and $k = m - 1$. Thus our result can be expressed as

Theorem 2.

$$\sum_{i,j\geq 1} \frac{q^{ij}}{(q)_{i+j-1}} = \sum_{m\geq 1} \frac{q^m}{(q^m)_\infty} H_{m-1}(1).$$

We will now insert a parameter z to keep track of the part containing the bottom edge of the rectangle. Observing that any part $\geq i$ from $1/(q)_{i+j-1}$ adds a node to the part containing the bottom edge of the rectangle, we obtain

Theorem 3.

$$\sum_{i,j\geq 1} \frac{z^j q^{ij}}{(q)_{i-1}(zq^i)_j} = \sum_{m\geq 1} \frac{z^m q^m}{(q^m)_\infty} H_{m-1}(1).$$

5. Another way to express the generating function. We can also rewrite our generating function based on the size of the $i \times j$ rectangle plus the number of parts below the rectangle. When $j = 1$ there are no parts below the rectangle and the term in our generating function is

$$\sum_{i\geq 1} \frac{q^i}{(q)_i} = \frac{1}{(q)_\infty} - 1.$$

When $j = 2$ there are either no parts below the rectangle or the parts below the rectangle are all ones and the term in our generating function is

$$\sum_{a\geq 0} q^{1a} \sum_{i\geq 1} \frac{q^{2i+ai}}{(q)_i}.$$

In general, when $j = k+1$ there are either no parts below the rectangle or the parts below the rectangle are all less than or equal to k and the term in our generating function is

$$\sum_{a_1,a_2,\dots,a_k\geq 0} q^{1a_1+2a_2+\cdots+ka_k} \sum_{i\geq 1} \frac{q^{(k+1)i+(a_1+a_2+\cdots+a_k)i}}{(q)_i}.$$

The inner sum in this expression is equal to $1/(q^{a_1+a_2+\cdots+a_k+k+1})_\infty - 1$ by Corollary 2.2 in [**2**]. Thus we have the following theorem.

Theorem 4.

$$\sum_{i,j\geq 1} \frac{q^{ij}}{(q)_{i+j-1}} = \frac{1}{(q)_\infty} - 1 + \sum_{k\geq 1} \sum_{a_1,a_2,\dots,a_k\geq 0} q^{1a_1+2a_2+\cdots+ka_k} \left(\frac{1}{(q^{a_1+a_2+\cdots+a_k+k+1})_\infty} - 1\right).$$

If we insert a parameter z to keep track of the width of the rectangle plus the number of parts below the rectangle, the generating function on the righthand side of the equation in the previous theorem becomes

$$z\left(\frac{1}{(q)_\infty} - 1\right) + \sum_{k\geq 1} \sum_{a_1,a_2,\dots,a_k\geq 0} q^{1a_1+2a_2+\cdots+ka_k} z^{a_1+a_2+\cdots+a_k+k+1} \times \left(\frac{1}{(q^{a_1+a_2+\cdots+a_k+k+1})_\infty} - 1\right).$$

To insert the parameter z into the function on the lefthand side of the equation we simply note that any part greater than i from $1/(q)_{i+j-1}$ yields a part below our rectangle. So we have,

Theorem 5.

$$\sum_{i,j\geq 1} \frac{q^{ij}}{(q)_i(zq^{i+1})_{j-1}} = z\left(\frac{1}{(q)_\infty} - 1\right) + \sum_{k\geq 1} \sum_{a_1,a_2,\dots,a_k\geq 0} q^{1a_1+2a_2+\cdots+ka_k} z^{a_1+a_2+\cdots+a_k+k+1} \times \left(\frac{1}{(q^{a_1+a_2+\cdots+a_k+k+1})_\infty} - 1\right).$$

6. Is the generating function in Section 5 also related to the Rogers-Szego polynomials? Let's sort the terms in the generating function on the righthand side of the equation in Theorem 4 according to the appearance of $q^t((1/(q^{m+2})_\infty)-1)$, $m \geq 0$. Note that $q^t((1/(q^{m+2})_\infty)-1)$ appears when t has been partitioned into m parts ≤ 1, $m-1$ parts $\leq 2, \ldots, 0$ parts $\leq m+1$. The generating function for partitions into m parts ≤ 1, $m-1$ parts $\leq 2, \ldots, 0$ parts $\leq m+1$ is $H_m(q) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} q^j$. Thus our generating function can be expressed as follows.

Theorem 6.

$$\sum_{i,j\geq 1} \frac{q^{ij}}{(q)_{i+j-1}} = \frac{1}{(q)_\infty} - 1 + \sum_{m\geq 0} H_m(q)\left(\frac{1}{(q^{m+2})_\infty} - 1\right).$$

If we insert the parameter z to keep track of the width of the rectangle plus the number of parts below the rectangle, our generating function becomes

$$z\left(\frac{1}{(q)_\infty} - 1\right) + \sum_{m\geq 0} H_m(q) z^{m+2}\left(\frac{1}{(q^{m+2})_\infty} - 1\right).$$

7. Some other observations. It should be noted that we get similar results if we start with the function $\sum_{i,j\geq 1} q^{ij}/(q;q)_{i+j}$. In fact, we have the following theorem.

Theorem 7.

$$\sum_{i,j\geq 1} \frac{z^j q^{ij}}{(q)_{i-1}(zq^i)_{j+1}} = \sum_{m\geq 1} \frac{z^m q^m}{(q^m)_\infty}(H_m(1) - 1)$$

and

$$\sum_{i,j\geq 0} \frac{q^{ij} z^j}{(q)_i (zq^{i+1})_j} = z\left(\frac{1}{(q)_\infty} - 1\right) + \sum_{m\geq 0} H_m(q) z^{m+1}\left(\frac{1}{(q^{m+1})_\infty} - 1\right).$$

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