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SHARED SETS AND NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be three nonzero distinct finite complex numbers and the set $S = \{a_1, a_2, a_3\}$. If, for every $f \in \mathcal{F}$, $f(z) \in S \Rightarrow f'(z) \in S$, then $\mathcal{F}$ is normal in D.

1. Introduction and main results. Let D be a domain in C, and let $\mathcal{F}$ be a family of meromorphic functions defined in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in D, to a meromorphic function or ∞ (see Hayman [**3**], Schiff [**7**], Yang [**9**]).

Let f and g be two functions meromorphic on D in C, let $a \in C \cup \{\infty\}$, and let S be a set of complex numbers. If $g(z) \in S$ whenever $f(z) \in S$, then we write $f(z) \in S \Rightarrow g(z) \in S$. If $f(z) \in S \Rightarrow g(z) \in S$ and $g(z) \in S \Rightarrow f(z) \in S$, then we write $f(z) \in S \Rightarrow g(z) \in S$. If $f(z) \in S \Leftrightarrow g(z) \in S$, then we say that f and g share the set S in D. In particular, if $f(z) \in S \Leftrightarrow g(z) \in S$ and $S = \{a\}$, then we say that f and g share the value a in D.

Now let $\mathcal{F}$ be a family of meromorphic functions on D. Schwick proved in [**8**] that if there exist three distinct finite values $a_1, a_2, a_3 \in C$ such that f and f' share a_j, $j = 1, 2, 3$, for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D. The corresponding statement in which f and f' share two distinct finite values $a_1, a_2 \in C$ remains valid, as is shown by Pang and Zalcman [**5**].

On the other hand, Fang [**1**], Liu and Pang [**4**] extended Schwick's result in view of shared sets. Actually, they proved the following theorem.

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Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be three distinct finite complex numbers. If, for every $f \in \mathcal{F}$, f and f' share the set $S = \{a_1, a_2, a_3\}$, then $\mathcal{F}$ is normal in D.*

In this paper, we continue the investigations and prove the following results.

Theorem 1. *Let F be a family of meromorphic functions in a domain D, and let a_1, a_2 and a_3 be three nonzero distinct finite complex numbers and the set $S = \{a_1, a_2, a_3\}$. If, for every $f \in \mathcal{F}$, $f(z) \in S \Rightarrow f'(z) \in S$, then $\mathcal{F}$ is normal in D.*

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose poles are of multiplicity at least 3, let a_1, a_2 and a_3 be three distinct finite complex numbers and the set $S = \{a_1, a_2, a_3\}$, and let M be a positive number. If, for every $f \in \mathcal{F}$, $|f'(z)| \leq M$ whenever $f(z) \in S$, then $\mathcal{F}$ is normal in D.*

Example 1 [**2**]. Let $S = \{1, -1\}$. Set $\mathcal{F} = \{f_n(z) : n = 2, 3, 4, \dots\}$, where

$$f_n(z) = \frac{n+1}{2n} e^{nz} + \frac{n-1}{2n} e^{-nz}, \qquad D = \{z : |z| < 1\}.$$

Then, for any $f_n \in \mathcal{F}$, we have $n^2[f_n^2(z) - 1] = [f_n'(z)]^2 - 1$. Thus f_n and f_n' share the set $S = \{1, -1\}$, but $\mathcal{F}$ is not normal in D. This shows that the condition in Theorem 1 and Theorem 2 that the set S with three elements is the best possible.

Theorem 3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros are multiple. Let a_1 and a_2 be two nonzero distinct finite complex numbers and the set $S = \{a_1, a_2\}$. If, for every $f \in \mathcal{F}$, $f(z) \in S \Rightarrow f'(z) \in S$, then $\mathcal{F}$ is normal in D.*

2. A main lemma.

Lemma 1 ([**10**], cf. [**6**]). *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc. Then, if $\mathcal{F}$ is not normal, there exist*

(a) *a number* $0 < r < 1$,

(b) *points* z_n, $|z_n| < r$,

(c) *functions* $f_n \in \mathcal{F}$, *and*

(d) *positive numbers* $\rho_n \to 0$

such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \longrightarrow g(\xi)$$

locally uniformly with respect to the spherical metric, where g *is a nonconstant meromorphic function on* C.

3. Proofs of Theorems 1, 2, and 3.

3.1. Proof of Theorem 1. We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1 we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$, and $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \longrightarrow g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C.

We claim that

(i) $g(\xi) \neq a_1$ on C,

(ii) $g(\xi) \neq a_2$ on C, and

(iii) $g(\xi) \neq a_3$ on C.

Suppose now that $g(\xi_0) = a_1$. Clearly, $g(\xi) \not\equiv a_1$. Then by Hurwitz's theorem there exist ξ_n, $\xi_n \to \xi_0$, such that, for n sufficiently large,

$$a_1 = g(\xi_0) = g_n(\xi_n) = f_n(z_n + \rho_n \xi_n).$$

Thus there exists a positive number $M = \max\{|a_1|, |a_2|, |a_3|\}$ such that $|f_n'(z_n + \rho_n \xi_n)| \leq M$ because $f_n \in S \Rightarrow f_n' \in S$. It now follows that

$$g'(\xi_0) = \lim_{n \to \infty} g_n'(\xi_n) = \lim_{n \to \infty} \rho_n f_n'(z_n + \rho_n \xi_n) = 0.$$

This implies that ξ_0 is a multiple zero of $g - a_1$. Further, we can assume that ξ_0 is a zero of $g - a_1$ of multiplicity $k \geq 2$. Because $g^{(k)}(\xi_0) \neq 0$, there exists a positive number δ such that, for n sufficiently large,

$$g(\xi) \neq a_1, \qquad g'(\xi) \neq 0, \qquad g^{(k)}(\xi) \neq 0 \tag{1}$$

in $0 < |\xi - \xi_0| < \delta$. Note that ξ_0 is a zero of $g - a_1$ of multiplicity $k \geq 2$. Then by Rouché's theorem we know that for n sufficiently large $g_n(\xi) - a_1$ has k zeros $\xi_n^{(1)}, \xi_n^{(2)}, \ldots, \xi_n^{(k)}$ in $|\xi - \xi_0| < \delta/2$ and so $g_n(\xi_n^{(j)}) = f_n(z_n + \rho_n \xi_n^{(j)}) = a_1$ for $j = 1, 2, \ldots, k$. Since $f_n \in S \Rightarrow f_n' \in S$, it follows that there exists a subsequence of $\{f_n\}$, which we again denote by $\{f_n\}$, such that $g_n'(\xi_n^{(j)}) = \rho_n f_n'(z_n + \rho_n \xi_n^{(j)}) = \rho_n a_l \neq 0$ for $j = 1, 2, \ldots, k$ for some $l = 1, 2, 3$. Therefore, all k zeros of $g_n(\xi) - a_1$ are simple, so that $\xi_n^{(i)} \neq \xi_n^{(j)}$ as $i \neq j$ for $i, j = 1, 2, \ldots, k$. Now (1) and the fact that

$$\lim_{n \to \infty} g_n'(\xi_n^{(j)}) = \lim_{n \to \infty} \rho_n f_n'(z_n + \rho_n \xi_n^{(j)}) = 0$$

yield

$$\lim_{n \to \infty} \xi_n^{(j)} = \xi_0, \quad j = 1, 2, \ldots, k.$$

Since $g_n'(\xi) - \rho_n a_l$ has k zeros $\xi_n^{(1)}, \xi_n^{(2)}, \ldots, \xi_n^{(k)}$ in $|\xi - \xi_0| < \delta/2$ for some $l = 1, 2, 3$, ξ_0 is a zero of $g'(\xi)$ of multiplicity k. Hence $g^{(k)}(\xi_0) = 0$, contradicting (1). This proves (i). Likewise, we can prove (ii) and (iii).

Now by (i)–(iii) and Picard's theorem we see that g is reduced to a constant, which contradicts that g is nonconstant.

This completes the proof of Theorem 1. □

3.2. Proof of Theorem 2. We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1 we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$ and $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \longrightarrow g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose poles are of multiplicity at least 3.

We claim that

(i) all zeros of $g(\xi) - a_1$ are multiple,

(ii) all zeros of $g(\xi) - a_2$ are multiple, and

(iii) all zeros of $g(\xi) - a_3$ are multiple.

Suppose now that $g(\xi_0) = a_1$. Clearly, $g(\xi) \not\equiv a_1$. Then by Hurwitz's theorem there exist $\xi_n, \xi_n \to \xi_0$, such that, for n sufficiently large,

$$a_1 = g(\xi_0) = g_n(\xi_n) = f_n(z_n + \rho_n \xi_n).$$

Thus $|f_n'(z_n + \rho_n \xi_n)| \le M$ because $|f_n'(z)| \le M$ whenever $f_n(z) \in S$. It now follows that

$$g'(\xi_0) = \lim_{n\to\infty} g_n'(\xi_n) = \lim_{n\to\infty} \rho_n f_n'(z_n + \rho_n \xi_n) = 0.$$

This implies that ξ_0 is a multiple zero of $g - a_1$, completing the proof of (i). Likewise, we can prove (ii) and (iii).

Now let us use (i), (ii) and (iii) to derive a contradiction. By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} 2T(r,g) \le \overline{N}(r,g) &+ \overline{N}\left(r, \frac{1}{g-a_1}\right) + \overline{N}\left(r, \frac{1}{g-a_2}\right) \\ &+ \overline{N}\left(r, \frac{1}{g-a_3}\right) + S(r,g). \end{aligned}$$

From this, (i), (ii), (iii) and the fact that all poles of g have multiplicity at least 3, it follows that

$$\begin{aligned} 2T(r,g) \le \frac{1}{3}N(r,g) &+ \frac{1}{2}N\left(r, \frac{1}{g-a_1}\right) + \frac{1}{2}N\left(r, \frac{1}{g-a_2}\right) \\ &+ \frac{1}{2}N\left(r, \frac{1}{g-a_3}\right) + S(r,g), \end{aligned}$$

so that

$$2T(r,g) \le \frac{1}{3}T(r,g) + \frac{3}{2}T(r,g) + S(r,g),$$

i.e,

$$\frac{1}{6}T(r,g) \le S(r,g).$$

This implies that g is a constant, a contradiction.

This completes the proof of Theorem 2. □

3.3. Proof of Theorem 3. We may assume that $D = \Delta$, the unit disc. Suppose that $\mathcal{F}$ is not normal on Δ. Then by Lemma 1 we can find $f_n \in \mathcal{F}$, $z_n \in \Delta$ and $\rho_n \to 0^+$ such that

$$g_n(\xi) = f_n(z_n + \rho_n \xi) \longrightarrow g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros are multiple.

We claim that

(i) $g(\xi) \neq a_1$ on C, and

(ii) $g(\xi) \neq a_2$ on C.

Using reasoning similar to that in Theorem 1, we can prove (i) and (ii).

Now let us use (i) and (ii) to derive a contradiction. Note that a_1 and a_2 are two nonzero distinct finite complex numbers. By Nevanlinna's second fundamental theorem, we have

$$T(r,g) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-a_1}\right) + \overline{N}\left(r, \frac{1}{g-a_2}\right) + S(r,g).$$

From this, (i), (ii) and the fact that all zeros of g are multiple ,it follows that

$$T(r,g) \leq \frac{1}{2}N\left(r, \frac{1}{g}\right) + S(r,g) \leq \frac{1}{2}T(r,g) + S(r,g),$$

i.e,

$$\frac{1}{2}T(r,g) \leq S(r,g).$$

This implies that g is a constant, a contradiction.

This completes the proof of Theorem 3. □

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