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ASYMPTOTIC BEHAVIOR OF BOUNDED SOLUTIONS TO SOME SECOND ORDER EVOLUTION SYSTEMS

BEHZAD DJAFARI ROUHANI AND HADI KHATIBZADEH

ABSTRACT. By using previous results of Rouhani [**20–23**] for dissipative systems, we study the asymptotic behavior of solutions to the following system of second order evolution equation

$$\begin{cases} u''(t) - cu'(t) \in Au(t) & \text{a.e. } t \in (0, +\infty) \\ u(0) = u_0, \ \sup_{t \geq 0} |u(t)| < +\infty \end{cases}$$

where A is a maximal monotone operator in a real Hilbert space H and $c \geq 0$. We investigate weak and strong convergence theorems for solutions to this system. Our results extend and unify previous results by Mitidieri [**15**] and Morosanu [**17**] who studied the case $c = 0$ by assuming that A is maximal monotone and $A^{-1}(0) \neq \phi$, as well as previous results by Véron [**25**] who studied the case $c > 0$ by assuming A to be strongly monotone.

1. Introduction. Let H be a real Hilbert space with inner product $(.,.)$ and norm $|.|$. We denote weak convergence in H by w-lim and strong convergence by lim. $u'(t)$, respectively $u''(t)$, denotes $du/dt(t)$, respectively $d^2u/dt^2(t)$. A self-mapping T of a nonempty subset D of H is called nonexpansive, if: $|Tx - Ty| \leq |x - y|$ for all $x, y \in D$. Let A be a nonempty subset of $H \times H$ to which we shall refer as a (nonlinear) possibly multi-valued operator in H. A is called monotone, respectively strongly monotone, if $(y_2 - y_1, x_2 - x_1) \geq 0$, respectively $(y_2 - y_1, x_2 - x_1) \geq \alpha |x_1 - x_2|^2$ for some $\alpha > 0$, for all $[x_i, y_i] \in A$, $i = 1, 2$. A is maximal monotone if A is monotone and $R(I + A) = H$, where I is the identity operator on H.

Nonexpansive mappings as well as maximal monotone operators and semi-groups generated by them have been extensively studied. We

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refer the reader in particular to the beautiful books by Goebel and Kirk [**11**], Goebel and Reich [**12**], Brézis [**4**], Barbu [**3**] and Morosanu [**16**], as well as the recent articles by Falset et al. [**10**] and by Kaczor et al. [**13**] for a complete bibliography. The first mean ergodic theorem for nonexpansive mappings in Hilbert space was proved by Baillon [**1**]. It was extended to nonlinear contraction semi-groups associated with the solutions to dissipative evolution systems of the form

$$\begin{cases} u'(t) + Au(t) \ni 0 \quad \text{on } (0,+\infty) \\ u(0) = u_0 \end{cases} \tag{1}$$

where A is maximal monotone in H and $A^{-1}(0) \neq \phi$ by Baillon and Brézis [**2**]. When $A = \partial\varphi$, where φ is a proper convex lower semi-continuous function on H, Bruck [**7**] proved the weak convergence of solutions $u(t)$ to (1) as $t \to +\infty$ and the strong convergence of $u(t)$ when φ is even. Okochi [**18**] extended this result with a more general condition on φ.

The strong mean ergodic theorem for $u(t)$ was also proved by Brézis and Browder [**5**]. The asymptotic behavior of solutions to quasi-autonomous dissipative systems where A is assumed to be monotone, was studied in [**20–23**]. It was shown that conclusions about the asymptotic behavior of solutions can be drawn solely from the monotonicity assumption on A; since the maximal extension of a monotone operator in a Hilbert space requires the use of Zorn's lemma, this may be very useful from a practical and constructive point of view.

Existence, as well as asymptotic behavior of solutions to second order evolution systems of the form:

$$\begin{cases} u''(t) \in Au(t) & \text{a.e. on } \mathbf{R}^+ \\ u(0) = u_0, \ \sup_{t\geq 0} |u(t)| < +\infty \end{cases} \tag{2}$$

were studied by many authors, among others, by Barbu [**3**], Morosanu [**16, 17**] and the references therein, Mitidieri [**14, 15**] and Poffald and Reich [**19**]. For periodic forcing see Bruck [**8**]. Véron [**24**] showed that even for $A = \partial\varphi$, solutions to (2) may not converge strongly as $t \to +\infty$, although they always converge weakly.

In this paper, by using previous results of Djafari Rouhani [**20–23**] for dissipative systems, we extend those methods to study the asymptotic behavior of solutions to the second order evolution system:

$$(3)\qquad \begin{cases} u''(t) - cu'(t) \in Au(t) & \text{a.e. on } \mathbf{R}^+ \\ u(0) = u_0,\ \sup_{t\geq 0} |u(t)| < +\infty \end{cases}$$

where A is maximal monotone, $c \geq 0$, and $u_0 \in \overline{D(A)}$. We show among other things that the existence of a solution u to (3) implies that $A^{-1}(0) \neq \phi$, and $w - \lim_{t\to+\infty} u(t) = p$, where $p \in A^{-1}(0)$ and p is the asymptotic center (defined in Definition 2.3) of u. Moreover, we show the continuous dependence of p to the initial data u_0 and prove also strong convergence theorems for u. It was shown by Véron [**25, 26**] that $A^{-1}(0) \neq \phi$ implies the existence of a unique solution to (3). Therefore, by the above result, (3) has a unique solution if and only if $A^{-1}(0) \neq \phi$.

Our results extend and unify previous results by Mitidieri [**15**] and Morosanu [**17**] who studied the case $c = 0$ by assuming that A is maximal monotone and $A^{-1}(0) \neq \phi$, as well as previous results of Véron [**25, 26**] who studied the existence and uniqueness of solutions to (3) for $c > 0$, as well as their asymptotic behavior by assuming A to be strongly monotone. We refer in particular to [**3, 4, 16**] and the references therein for examples of applications of these results.

2. Preliminaries. Here we recall and introduce some notations and definitions we shall use in the sequel.

Definition 2.1. A curve u in H is a function $u \in C([0, +\infty[\,, H)$. We denote $\sigma_T := 1/T \int_0^T u(t)\,dt$ for $T > 0$.

Definition 2.2. The curve u is (weakly) asymptotically regular if

$$\lim_{t\to+\infty} (u(t+h) - u(t)) = 0$$
$$(\text{resp. } w - \lim(u(t+h) - u(t)) = 0) \text{ for all } h > 0.$$

Definition 2.3. Given a bounded curve u in H, the asymptotic center c of u is defined as follows (see [**9**]): for every $q \in H$, let $\varphi(q) = \lim_{t\to+\infty} \sup |u(t) - q|^2$.

Then φ is a continuous strictly convex function on H, satisfying $\varphi(q) \to +\infty$ as $|q| \to +\infty$. Thus, φ achieves its minimum on H at a unique point c, called the asymptotic center of the curve u.

Definition 2.4. The curve u in H is called nonexpansive if

$$|u(t+h) - u(s+h)| \leq |u(t) - u(s)|, \quad \text{for all } s, t, h \geq 0.$$

Definition 2.5. Let A be an operator in H. Then A is said to satisfy condition (b) if there exists a continuous function $a : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ such that for every $[x_1, y_1] \in A$, $[x_2, y_2] \in A$ we have $(x_2, y_1)+(x_1, y_2)+a(|x_1|, |x_2|)\{(x_1, y_1) + (x_2, y_2)\} \geq 0$.

When A is monotone, condition (b) is actually equivalent to the following stronger condition (a):

$$|(x_2, y_1) + (x_1, y_2)| \leq a(|x_1|, |x_2|)\{(x_1, y_1) + (x_2, y_2)\},$$

which was introduced by Bruck [**6**], and later used also by Mitidieri [**15**].

Definition 2.6. By a solution u to (3) we mean a function $u \in C([0, T]; H) \cap H^2_{loc}((0, T); H)$ for every $T > 0$ that satisfies (3) for almost every $t \in \mathbf{R}^+$.

We note that in this case u and u' are absolutely continuous functions on each compact subinterval of $\mathbf{R}^+$.

Notation 2.7. For a curve u in H, we denote $L := \{p \in H; |u(t) - p| \text{is nonincreasing}\}$.

3. Weak convergence theorems for solutions to (3). In this section we consider the evolution system (3) for $c \geq 0$ and establish the weak convergence of solutions $u(t)$ to (3) as $t \to +\infty$, to an element of $A^{-1}(0)$ which is the asymptotic center of the curve $(u(t))_{t \geq 0}$. We shall refer to (3) for the cases $c = 0$ and $c > 0$, respectively as (3.a) and (3.b).

The main result of this section is the weak convergence Theorem 3.8. In order to achieve this result, we first prove a weak ergodic theorem

(Lemma 3.4) for solutions u to (3), by showing that u is a nonexpansive curve and using our previous results on nonexpansive curves (recalled in Theorem 3.2). Then, since we do not assume $A^{-1}(0) \neq \phi$, a crucial step in the proof is Lemma 3.5, where we exhibit a point $p \in H$ (actually the asymptotic center of the curve u), that enjoys a similar property needed from an element of $A^{-1}(0)$. The last step of the proof is to show that u is asymptotically regular and then use our previous results on nonexpansive curves (recalled in Theorem 3.3), to get the weak convergence of u to p. To this aim, and to show also that $p \in A^{-1}(0)$, some estimates on u' and u'' are needed, which are proved in Lemma 3.6, respectively 3.7, for (3.a), respectively (3.b). The last part of this section describes another method used in [**7**] for proving the weak convergence of u for (3.a) (Theorem 3.9), where somehow for (3.b), i.e., the general case $c > 0$, it is shown in Theorem 3.11 that it can be applied to the case $A = \partial\varphi$, but its applicability is not known in general (Problem 3.1).

First we recall without proof the following classical lemma whose proof can be found in any textbook on elementary differential equations.

Lemma 3.1. *Assume the function $f : \mathbf{R}^+ \to \mathbf{R}$ is bounded above on $[\varepsilon, +\infty)$ for some $\varepsilon > 0$, absolutely continuous on every compact subinterval of $\mathbf{R}^+$, and satisfies $f'(t) \geq cf(t)$ for almost every $t \in \mathbf{R}^+$, for some $c > 0$. Then we have $f(t) \leq 0$, for all $t \in [\varepsilon, +\infty)$.*

Now we recall without proof the following nonlinear ergodic theorem and weak convergence theorem for nonexpansive curves in H, which are special cases of [**21**, Theorems 3.8 and 3.10], and extensions of previous results in [**1, 2**]; see also [**22**, Corollaries 3.7 and 3.10], as well as [**20**].

Theorem 3.2. *Let $(u(t))_{t\geq 0}$ be a nonexpansive curve in H, and let $\sigma_T := 1/T \int_0^T u(t)\,dt$. Then the following are equivalent:*

(i) $L \neq \phi$,

(ii) $\liminf_{T\to+\infty} |\sigma_T| < +\infty$,

(iii) *σ_T converges weakly to some $p \in H$ as $T \to +\infty$.*

Moreover, under these conditions, p is the asymptotic center of the curve $(u(t))_{t\geq 0}$.

Theorem 3.3. *Let $(u(t))_{t\geq 0}$ be a weakly asymptotically regular nonexpansive curve in H. Then the following are equivalent*:

(i) $L \neq \phi$,

(ii) $\liminf_{t\to+\infty} |u(t)| < +\infty$,

(iii) $u(t)$ *converges weakly to the asymptotic center of* u, *as* $t \to +\infty$.

By using the above results, we are also able to establish a nonlinear ergodic theorem for solutions to (3), stated in the following lemma. This lemma will subsequently be used in the proof of Lemma 3.5 leading to the weak convergence of solutions to (3) in Theorem 3.8.

Lemma 3.4. *Assume u is a solution to* (3). *Then* $\sigma_T := 1/T\int_0^T u(t)\,dt$ *converges weakly as* $T \to +\infty$ *to some* $p \in L$, *which is also the asymptotic center of the curve* $(u(t))_{t\geq 0}$.

Proof. In order to apply Theorem 3.2, we show that the curve u is nonexpansive. Let $s \geq 0$ be fixed. By the monotonicity of A, we get from (3) that

$$\left(\frac{d^2u}{dt^2}(t+s) - \frac{d^2u}{dt^2}(t), u(t+s) - u(t)\right) \\ - c\left(\frac{du}{dt}(t+s) - \frac{du}{dt}(t), u(t+s) - u(t)\right) \geq 0$$

for almost every $t \geq 0$.

This implies that:

$$\frac{d^2}{dt^2}|u(t+s)-u(t)|^2 - c\frac{d}{dt}|u(t+s)-u(t)|^2 \geq 2|u'(t+s)-u'(t)|^2 \geq 0 \tag{3.1}$$

for almost every $t \geq 0$.

If $c = 0$, since u and u' are absolutely continuous on every compact subinterval of $\mathbf{R}^+$, we deduce from (3.1) that for each $s \geq 0$ the function $t \mapsto |u(t+s) - u(t)|^2$ from $\mathbf{R}^+$ to $\mathbf{R}^+$ is convex; since it is also bounded, we conclude that it is nonincreasing, implying that u is nonexpansive.

If $c > 0$, then for each $s > 0$ let $f(t) := d/dt|u(t+s) - u(t)|^2$ and $g(t) := |u(t+s) - u(t)|^2 - c\int_0^t |u(\theta+s) - u(\theta)|^2 d\theta$ and $M = \sup_{t\geq 0}|u(t)|$.

Then (3.1) implies that g is convex; since it is also bounded above, it follows that g is nonincreasing; thus, $g'(t) \leq 0$, and therefore $f(t) \leq c|u(t+s)-u(t)|^2 \leq 4cM^2$, for all $t \in [\epsilon, +\infty)$, for every $\epsilon > 0$. On the other hand, (3.1) implies that $f'(t) \geq cf(t)$ for almost every $t \geq 0$. Hence, f satisfies the hypothesis of Lemma 3.1. Therefore, we deduce that $f(t) \leq 0$, for all $t \in [\varepsilon, +\infty)$; since $\varepsilon > 0$ is arbitrary, this implies again that the function $t \mapsto |u(t+s)-u(t)|^2$ is nonincreasing, and hence u is nonexpansive. The conclusion follows now from Theorem 3.2. $\square$

Now we proceed to prove the weak convergence of u. We shall need some lemmas.

Lemma 3.5. *Assume that u is a solution to* (3.a) *or* (3.b), *and let p be the asymptotic center of u. Then we have*:

$$\left(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t), u(t) - p\right) \geq 0, \quad \text{for a.e. } t \in \mathbf{R}^+. \tag{3.2}$$

Proof. Let $0 < \varepsilon < T$, let u be a solution to (3.a) or (3.b), and let $t \in \mathbf{R}^+$ be fixed so that $u(t)$ satisfies (3). Then, by the monotonicity of A, we have:

$$\begin{aligned}
&\left(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t), u(t) - \frac{1}{T-\varepsilon}\int_\varepsilon^T u(s)\,ds\right) \\
&= \frac{1}{T-\varepsilon}\int_\varepsilon^T \left(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t) - \frac{d^2u}{ds^2}(s) + c\frac{du}{ds}(s), u(t) - u(s)\right) ds \\
&\quad + \frac{1}{T-\varepsilon}\int_\varepsilon^T \left(\frac{d^2u}{ds^2}(s) - c\frac{du}{ds}(s), u(t) - u(s)\right) ds \\
&\geq \frac{1}{T-\varepsilon}\int_\varepsilon^T \left[\left(\frac{d^2u}{ds^2}(s), u(t) - u(s)\right) - c\left(\frac{du}{ds}(s), u(t) - u(s)\right)\right] ds \\
&= \frac{1}{T-\varepsilon}\int_\varepsilon^T \left[\frac{d}{ds}\left(\frac{du}{ds}(s), u(t) - u(s)\right) + \left|\frac{du}{ds}(s)\right|^2 \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{c}{2}\frac{d}{ds}|u(s) - u(t)|^2\right] ds \\
&\geq \frac{1}{T-\varepsilon}\left[\left(\frac{du}{dt}(T), u(t) - u(T)\right)\right.
\end{aligned}$$

$$- \left(\frac{du}{dt}(\varepsilon), u(t) - u(\varepsilon)\right)$$
$$+ \frac{c}{2}|u(T) - u(t)|^2 - \frac{c}{2}|u(\varepsilon) - u(t)|^2\Bigg].$$

Since we showed in Lemma 3.4 that u is nonexpansive, we deduce that $|du/dt(t)|$ is nonincreasing, and therefore bounded. Now the conclusion follows from Lemma 3.4 by letting $T \to +\infty$ in the above inequalities. □

Remark 3.1. If u is a solution to (3.a) and p is the asymptotic center of u, in addition to the fact that $p \in L$, we can show that the function $t \mapsto |u(t) - p|$ from $\mathbf{R}^+$ to $\mathbf{R}^+$ is convex. Indeed, we have

$$\frac{1}{2}\frac{d^2}{dt^2}|u(t) - p|^2 = \left(\frac{d^2u}{dt^2}(t), u(t) - p\right) + \left|\frac{du}{dt}(t)\right|^2 \geq 0, \text{ for a.e. } t \in \mathbf{R}^+$$

by (3.2). Since u and u' are absolutely continuous functions on each compact subinterval of $\mathbf{R}^+$, it follows that the function $t \mapsto g(t) := 1/2|u(t)-p|^2$ from $\mathbf{R}^+$ to $\mathbf{R}^+$ is convex. But, from the above inequality, we have $g''(t) \geq |u'(t)|^2$ for almost every $t \in \mathbf{R}^+$, and therefore:

$$(g'(t))^2 = [(u'(t), u(t) - p)]^2 \leq |u'(t)|^2|u(t) - p|^2 \leq 2g''(t)g(t)$$
$$\text{for a.e. } t \in \mathbf{R}^+.$$

This implies that the function $\sqrt{g}$ is convex and gives the desired result.

The estimates in our next lemma are similar to [**3**, Lemma 2.3, Chapter 5, page 322]; however, in [**3**] these are obtained by using the assumption $A^{-1}(0) \neq \phi$ in the proof, which we don't have here in our setting, and we use (3.2) instead.

Lemma 3.6. *Assume that u is a solution to* (3.a)*, and let p be the asymptotic center of u. Then the following estimates hold*:

(i) $\int_\varepsilon^{+\infty} t|du/dt(t)|^2 dt \leq 1/2|u(\varepsilon) - p|^2 - \varepsilon(u'(\varepsilon),\, u(\varepsilon) - p) < +\infty$,

(ii) $\sup_{t\geq\varepsilon} t|du/dt(t)| \leq (|u(\varepsilon)-p|^2 - 2\varepsilon(u'(\varepsilon), u(\varepsilon)-p) + \varepsilon^2|u'(\varepsilon)|^2)^{1/2} < +\infty$,

(iii) $\int_\varepsilon^{+\infty} t^3|d^2u/dt^2(t)|^2 dt \leq 3/2|u(\varepsilon) - p|^2 - 3\varepsilon(u'(\varepsilon), u(\varepsilon) - p) - \varepsilon^3(u''(\varepsilon), u'(\varepsilon)) + 3/2\varepsilon^2|u'(\varepsilon)|^2 < +\infty$

for all $\varepsilon > 0$.

Proof. Multiplying (3.2) by t we have:

$$t\left(\frac{d^2u}{dt^2}(t), u(t)-p\right) \geq 0 \quad \text{for a.e. } t \in \mathbf{R}^+.$$

Now the rest of the proof is exactly the same as in [**3**, Lemma 2.3, Chapter 5, page 322]. □

Remark 3.2. Bruck [**7**, Theorem 6] showed that (i) implies

$$t|u'(t)| \leq \left(\frac{8}{3}\int_{t/2}^{t} s|u'(s)|^2 ds\right)^{1/2} \to 0 \quad \text{as } t \to +\infty.$$

In our next lemma we obtain similar estimates for solutions to (3.b).

Lemma 3.7. *Assume that u is a solution to* (3.b), *and let p be the asymptotic center of u. Then we have*:

(i) $\sup_{t\geq\varepsilon} t|du/dt(t)|^2 < +\infty$ *for all* $\varepsilon > 0$.

(ii) $\int_{\varepsilon}^{+\infty} |d^2u/dt^2(t)|^2 dt \leq c/2|du/dt(\varepsilon)|^2 - (d^2u/dt^2(\varepsilon), du/dt(\varepsilon)) < +\infty$ *for all* $\varepsilon > 0$.

Proof. Let $0 < \varepsilon < T$. Multiplying (3.2) by t, we get:

$$\begin{aligned}
&t\left(\frac{d^2u}{dt^2}(t), u(t)-p\right) - ct\left(\frac{du}{dt}(t), u(t)-p\right) \geq 0 \quad \text{for a.e. } t \in \mathbf{R}^+ \\
&\Longrightarrow at\frac{d}{dt}\left(\frac{du}{dt}(t), u(t)-p\right) - t\left|\frac{du}{dt}(t)\right|^2 \\
&\qquad - \frac{c}{2}t\frac{d}{dt}|u(t)-p|^2 \geq 0 \\
&\Longrightarrow \frac{d}{dt}\left(t\frac{du}{dt}(t), u(t)-p\right) - \frac{c}{2}t\frac{d}{dt}|u(t)-p|^2 - \frac{1}{2}\frac{d}{dt}|u(t)-p|^2 \\
&\geq t\left|\frac{du}{dt}(t)\right|^2
\end{aligned}$$

$$\Longrightarrow \int_{\varepsilon}^{T} t|\frac{du}{dt}(t)|^2dt \le T\left(\frac{du}{dt}(T), u(T)-p\right) - \varepsilon\left(\frac{du}{dt}(\varepsilon), u(\varepsilon)-p\right)$$
$$- \frac{1}{2}|u(T)-p|^2 + \frac{1}{2}|u(\varepsilon)-p|^2 - \frac{c}{2}\int_{\varepsilon}^{T} t\frac{d}{dt}|u(t)-p|^2dt$$
$$\le \frac{1}{2}|u(\varepsilon)-p|^2 - \varepsilon\left(\frac{du}{dt}(\varepsilon), u(\varepsilon)-p\right) - \frac{c}{2}T|u(T)-p|^2$$
$$+ \frac{c}{2}\varepsilon|u(\varepsilon)-p|^2 + \frac{c}{2}\int_{\varepsilon}^{T} |u(t)-p|^2dt$$

(since $|u(t)-p|$ is nonincreasing, and therefore

$$\left(\frac{du}{dT}(T), u(T)-p) = \frac{1}{2}\frac{d}{dT}|u(T)-p|^2 \le 0\right)$$
$$\le \frac{1}{2}|u(\varepsilon)-p|^2 - \varepsilon\left(\frac{du}{dt}(\varepsilon), u(\varepsilon)-p\right)$$
$$+ \frac{c}{2}\varepsilon|u(\varepsilon)-p|^2 + \frac{c}{2}\int_{\varepsilon}^{T} |u(t)-p|^2dt.$$

Now since u is nonexpansive and thus $|du/dt(t)|$ is nonincreasing, we deduce that:

$$\frac{(T^2-\varepsilon^2)}{2}\left|\frac{du}{dt}(T)\right|^2 = \left|\frac{du}{dt}(T)\right|^2 \int_{\varepsilon}^{T} t\,dt \le \int_{\varepsilon}^{T} t\left|\frac{du}{dt}(t)\right|^2 dt$$
$$\le \frac{1}{2}|u(\varepsilon)-p|^2 - \varepsilon\left(\frac{du}{dt}(\varepsilon), u(\varepsilon)-p\right)$$
$$+ \frac{c}{2}\varepsilon|u(\varepsilon)-p|^2 + \frac{c}{2}\int_{\varepsilon}^{T} |u(t)-p|^2dt.$$

Since $\sup_{t\ge 0}|u(t)| < +\infty$, we conclude that $\sup_{T\ge\varepsilon} T|du/dt(T)|^2 < +\infty$, proving (i).

Now, to prove (ii), let $v(t) = u(t+h) - u(t)$. Then, by the monotonicity of A, we get:

$$\begin{aligned}\left(\frac{d^2v}{dt^2}(t), v(t)\right) - c\left(\frac{dv}{dt}(t), v(t)\right) &\geq 0 \\ \Longrightarrow \left|\frac{dv}{dt}(t)\right|^2 &\leq \frac{1}{2}\frac{d^2}{dt^2}|v(t)|^2 - \frac{c}{2}\frac{d}{dt}|v(t)|^2 \\ \Longrightarrow \int_\varepsilon^T \left|\frac{dv}{dt}(t)\right|^2 dt &\leq \left(\frac{dv}{dt}(T), v(T)\right) - \left(\frac{dv}{dt}(\varepsilon), v(\varepsilon)\right) \\ &\quad - \frac{c}{2}|v(T)|^2 + \frac{c}{2}|v(\varepsilon)|^2 \leq \frac{c}{2}|v(\varepsilon)|^2 - \left(\frac{dv}{dt}(\varepsilon), v(\varepsilon)\right)\end{aligned}$$

since u is nonexpansive and therefore $|v(t)|$ is nonincreasing, implying that

$$\left(\frac{dv}{dT}(T), v(T)\right) = \frac{1}{2}\frac{d}{dT}|v(T)|^2 \leq 0.$$

Now, dividing both sides of the above inequality by h^2, and letting $h \to 0$ and $T \to +\infty$, we get:

$$\int_\varepsilon^{+\infty} \left|\frac{d^2u}{dt^2}(t)\right|^2 \leq \frac{c}{2}\left|\frac{du}{dt}(\varepsilon)\right|^2 - \left(\frac{d^2u}{dt^2}(\varepsilon), \frac{du}{dt}(\varepsilon)\right), \quad \text{for all } \varepsilon > 0,$$

proving (ii). The proof is now complete. □

Now we state the main result of this section, showing the weak convergence of solutions to (3.a) or (3.b).

Theorem 3.8. *Let u be a solution to* (3.a) *or* (3.b). *Then $A^{-1}(0) \neq \phi$, and $w - \lim_{t\to+\infty} u(t) = p$, where p is the asymptotic center of u, as well as an element of $A^{-1}(0)$. Moreover, if v is a solution to* (3.a) *or* (3.b) *with the initial condition $v(0) = v_0$, and if $w - \lim_{t\to+\infty} v(t) = q$, then we have: $|p - q| \leq |u_0 - v_0|$.*

Proof. First we show that u is asymptotically regular. Since, by Lemma 3.4, u is nonexpansive, $|du/dt(t)|$ is nonincreasing. Therefore, for $h > 0$ fixed, we get:

$$|u(t+h) - u(t)| \leq \int_t^{t+h} \left|\frac{du}{ds}(s)\right| ds \leq \left|\frac{du}{dt}(t)\right| \int_t^{t+h} ds = h\left|\frac{du}{dt}(t)\right| \longrightarrow 0$$

as $t \to +\infty$ by Lemma 3.6 (ii) for (3.a), and by Lemma 3.7 (i) for (3.b). Now the weak convergence of $u(t)$, as $t \to +\infty$, to the asymptotic center p of u follows from Theorem 3.3. By Lemma 3.6 (iii) for (3.a), and by Lemma 3.7 (ii) for (3.b), there exists a sequence $t_n \to +\infty$, as $n \to +\infty$, such that $|d^2u/dt^2(t_n)| \to 0$, as $n \to +\infty$. Then, since A is maximal monotone, hence demi-closed, it follows from Lemma 3.6 (ii) for (3.a), and Lemma 3.7 (i) for (3.b), that $p \in A^{-1}(0)$, which is therefore nonempty. Finally, a similar computation as in Lemma 3.4 shows that $|u(t) - v(t)|$ is nonincreasing, and therefore $|p - q| \leq \lim_{t\to+\infty} |u(t) - v(t)| \leq |u_0 - v_0|$, completing the proof of the theorem. □

The conclusion of Theorem 3.8 for (3.a) follows also from Lemma 3.4 and the following result of Bruck [**7**, Theorem 6].

Theorem 3.9. *Let u be a solution to* (3.a). *Then we have*:

$$\lim_{T\to+\infty} \left| u(T) - \frac{1}{T}\int_0^T u(t)\, dt \right| = 0.$$

Problem 3.1. *However, for solutions to* (3.b) *it remains an open question whether the conclusion of Theorem* 3.9 *remains true.*

An affirmative answer to Problem 3.1 is provided when $A = \partial\varphi$, where φ is a proper, convex and lower semi-continuous function on H. We shall need the following lemma.

Lemma 3.10. *Let u be a solution to* (3.b) *with $A = \partial\varphi$, where φ is a proper, convex and lower semi-continuous function on H. Then we have*:

$$\int_\varepsilon^{+\infty} t|u'(t)|^2 dt < +\infty, \quad \textit{for all } \varepsilon > 0.$$

Proof. From Lemma (3.7) (ii), it follows that there exists a sequence $t_n \to +\infty$ such that $\lim_{n\to+\infty} u''(t_n) = 0$. From Lemma (3.7) (i), we know that we also have $\lim_{n\to+\infty} u'(t_n) = 0$. Since A is maximal monotone, therefore demi-closed, and since $[u(t_n), u''(t_n)-cu'(t_n)] \in A$, and $w-\lim_{t\to+\infty} u(t) = p$ by Theorem 3.8, it follows that $[p,0] \in A$, i.e., $p \in A^{-1}(0)$.

Replacing φ by $\widetilde{\varphi}(x) := \varphi(x) - \varphi(p)$, we may assume without loss of generality that $\varphi(x) \geq \varphi(p) = 0$, for all $x \in H$.

Multiplying both sides of (3.b) by $u'(t)$, and using [**16**, Lemma 2.2, page 57], we get:

$$(u''(t), u'(t)) - c|u'(t)|^2 = (\partial\varphi(u(t)), u'(t)) = \frac{d}{dt}\varphi(u(t)),$$

where by $\partial\varphi(u(t))$ we mean any element belonging to this set. Since u is nonexpansive and therefore $|u'(t)|$ is nonincreasing, it follows from the above equality that:

$$c|u'(t)|^2 \leq -\frac{d}{dt}\varphi(u(t)).$$

Multiplying both sides of the above inequality by t and integrating on the interval $[\varepsilon, T]$, we get:

$$\begin{aligned}
c\int_\varepsilon^T & t|u'(t)|^2 dt \\
&\leq \int_\varepsilon^T -t\frac{d}{dt}\varphi(u(t))\,dt \\
&= -T\varphi(u(T)) + \varepsilon\varphi(u(\varepsilon)) + \int_\varepsilon^T \varphi(u(t))\,dt \\
&\leq \varepsilon\varphi(u(\varepsilon)) + \int_\varepsilon^T (\partial\varphi(u(t)), u(t)-p)\,dt \\
&= \varepsilon\varphi(u(\varepsilon)) \\
&\quad + \int_\varepsilon^T \left(\frac{d}{dt}(u'(t), u(t)-p) - |u'(t)|^2 - \frac{c}{2}\frac{d}{dt}|u(t)-p|^2\right) dt \\
&\leq \varepsilon\varphi(u(\varepsilon)) + (u'(T), u(T)-p) - (u'(\varepsilon), u(\varepsilon)-p) \\
&\quad - \frac{c}{2}|u(T)-p|^2 + \frac{c}{2}|u(\varepsilon)-p|^2 \\
&\leq \varepsilon\varphi(u(\varepsilon)) - (u'(\varepsilon), u(\varepsilon)-p) + \frac{c}{2}|u(\varepsilon)-p|^2 \\
&:= M(\varepsilon).
\end{aligned}$$

Letting $T \to +\infty$, we get:

$$c\int_{\varepsilon}^{+\infty} t|u'(t)|^2 dt \leq M(\varepsilon) < +\infty, \quad \text{for all } \varepsilon > 0,$$

as required. □

Theorem 3.11. *Let u be a solution to* (3.b) *with $A = \partial\varphi$, where φ is a proper, convex and lower semi-continuous function on H. Then*

$$\lim_{T\to+\infty} \left| u(T) - \frac{1}{T}\int_0^T u(t)\,dt \right| = 0.$$

Proof. Let $\varepsilon > 0$ be given. By Lemma 3.10 we know that $\int_{\varepsilon}^{+\infty} t|u'(t)|^2 dt < +\infty$. Since $|u'(t)|$ is nonincreasing, we have:

$$\int_{T/2}^{T} t|u'(t)|^2 dt \geq |u'(T)|^2 \int_{T/2}^{T} t\,dt = \frac{3}{8}T^2|u'(T)|^2.$$

This implies that:

$$T|u'(T)| \leq \left(\frac{8}{3}\int_{T/2}^{T} t|u'(t)|^2 dt\right)^{1/2} \longrightarrow 0 \quad \text{as } T \to +\infty.$$

Hence:

$$\begin{aligned}
&\left| u(T) - \frac{1}{T}\int_0^T u(t)\,dt \right| \\
&\qquad \leq \left| \frac{1}{T}\int_0^{\varepsilon} u(t)\,dt \right| + \left| u(T) - \frac{1}{T}\int_{\varepsilon}^{T} u(t)\,dt \right| \\
&\qquad = \left| \frac{1}{T}\int_0^{\varepsilon} u(t)\,dt \right| \\
&\qquad\quad + \left| u(T) - \frac{1}{T}(Tu(T) - \varepsilon u(\varepsilon) - \int_{\varepsilon}^{T} tu'(t)\,dt \right| \\
&\qquad \leq \left| \frac{1}{T}\int_0^{\varepsilon} u(t)\,dt \right| + \frac{\varepsilon}{T}|u(\varepsilon)| \\
&\qquad\quad + \frac{1}{T}\int_{\varepsilon}^{T} t|u'(t)|\,dt \longrightarrow 0 \quad \text{as } T \to +\infty. \qquad \square
\end{aligned}$$

4. Strong convergence theorems for solutions to (3). In this section, we show that with additional assumptions on A, we can get the strong convergence of solutions u to (3). These conditions include the cases where $(I + A)^{-1}$ is compact (Theorem 4.1), A is strongly monotone (Theorem 4.2), as well as A satisfying condition (b) and a positivity condition (Theorem 4.3). The main results of this section are Theorem 4.3, and its corollaries 4.4 and 4.5, where the strong convergence of u is proved under more general conditions than previously known, even for the more general case of $c > 0$.

Our first strong convergence theorem is simple and extends [**16**, Theorem 2.2, p.111], where it is assumed that $A^{-1}(0) \neq \phi$.

Theorem 4.1. *Assume that the operator A in* (3) *is maximal monotone and $(I+A)^{-1}$ is compact, and let u be a solution to* (3.a) *or* (3.b). *Then $\lim_{t\to+\infty} u(t) = p$, where p is the asymptotic center of u, as well as an element of $A^{-1}(0)$.*

Proof. It follows from Lemma 3.6 (ii) and (iii) and Lemma 3.7 (i) and (ii), that in either case for (3.a) or (3.b), there exists a sequence $t_n \to +\infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} u'(t_n) = 0$ and $\lim_{n\to+\infty} u''(t_n) = 0$. Now since $\{u(t_n)+u''(t_n)-cu'(t_n)\}$ is bounded, by the compactness of $(I + A)^{-1}$, there exists a subsequence of $\{t_n\}$ which we denote again by $\{t_n\}$ for simplicity, such that $\lim_{n\to+\infty}u(t_n) = p$, where by Theorem 3.8 p is the asymptotic center of u and $p \in A^{-1}(0)$. Since we know by Lemma 3.4 that $p \in L$, it follows that $\lim_{t\to+\infty}u(t) = p$. □

In our next theorem, we prove the strong convergence of u by assuming A to be strongly monotone. To the best of our knowledge, this result is new even with the additional assumption $A^{-1}(0) \neq \phi$.

Theorem 4.2. *Assume that the operator A in* (3) *is strongly monotone, and let u be a solution to* (3.a) *or* (3.b). *Then $u(t)$ converges strongly as $t \to +\infty$ to the asymptotic center p of u, and $p \in A^{-1}(0)$.*

Proof. Assume $(y_2 - y_1, x_2 - x_1) \geq \alpha|x_2 - x_1|^2$, for all $[x_i, y_i] \in A$, $i = 1, 2$ and for some $\alpha > 0$. Then by using the strong monotonicity of A, the same proof as in Lemma 3.5 gives the following inequality:

$$\begin{aligned}
&\left(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t), u(t) - \frac{1}{T-\varepsilon}\int_\varepsilon^T u(s)\,ds\right)\\
&\quad\geq \frac{1}{T-\varepsilon}\int_\varepsilon^T \alpha|u(t)-u(s)|^2\,ds\\
&\qquad + \frac{1}{T-\varepsilon}\bigg[\left(\frac{du}{dt}(T), u(t)-u(T)\right)\\
&\qquad - \left(\frac{du}{dt}(\varepsilon), u(t)-u(\varepsilon)\right) + \frac{c}{2}|u(T)-u(t)|^2 - \frac{c}{2}|u(\varepsilon)-u(t)|^2\bigg]
\end{aligned}$$

Letting $T \to +\infty$ in the above inequality, we get:

$$\begin{aligned}
\left(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t), u(t) - p\right) &\geq l \liminf_{T\to+\infty} \frac{1}{T-\varepsilon}\int_\varepsilon^T |u(t)-u(s)|^2 ds\\
&\geq \alpha \liminf_{s\to+\infty} |u(t)-u(s)|^2\\
&\geq \alpha|u(t)-p|^2,
\end{aligned}$$

by Theorem 3.8.

Now integrating both sides of the above inequality on $[\partial, T]$, we get:

$$\begin{aligned}
&\alpha\int_\varepsilon^T |u(t)-p|^2 dt\\
&\quad\leq \int_\varepsilon^T \left(\frac{d}{dt}\left(\frac{du}{dt}(t), u(t)-p\right) - \left|\frac{du}{dt}(t)\right|^2 - \frac{c}{2}\frac{d}{dt}|u(t)-p|^2\right) dt\\
&\quad\leq (u'(T), u(T)-p) - (u'(\varepsilon), u(\varepsilon)-p)\\
&\qquad - \frac{c}{2}|u(T)-p|^2 + \frac{c}{2}|u(\varepsilon)-p|^2\\
&\quad\leq -(u'(\varepsilon), u(\varepsilon)-p) + \frac{c}{2}|u(\varepsilon)-p|^2
\end{aligned}$$

This implies that

$$\int_\varepsilon^{+\infty} |u(t)-p|^2 dt < +\infty;$$

hence, $\liminf_{t\to+\infty} |u(t)-p| = 0$. Since $|u(t)-p|$ is nonincreasing, we conclude that $\lim_{t\to+\infty} |u(t)-p| = 0$ as desired. □

Now we prove the strong convergence of solutions to (3.a) or (3.b) without even the monotonicity assumption on A, but assuming A to satisfy only condition (b) and some positivity condition.

Theorem 4.3. *Assume that the* (*not necessarily monotone*) *operator* A *in* (3) *satisfies* $(y,x)\ge 0$, *for all* $[x,y]\in A$, *as well as condition* (b), *and let* u *be a solution to* (3.a) *or* (3.b)*. Then* $u(t)$ *converges strongly as* $t\to+\infty$ *to the asymptotic center* p *of* u.

Proof. For $h\ge 0$, let $M:=\max\{1,\sup_{t\ge 0}a(|u(t)|,|u(t+h)|)\}$ and

$$F(t):=\frac{M}{2}|u(t)|^2+\frac{M}{2}|u(t+h)|^2+(u(t),u(t+h)).$$

Then, by the assumption on A, we have:

$$\begin{aligned}
\frac{d^2}{dt^2}F(t) &= M\bigg(\frac{d^2u}{dt^2}(t),u(t)\bigg)+M\bigg(\frac{d^2u}{dt^2}(t+h),u(t+h)\bigg)\\
&\quad+M\bigg|\frac{du}{dt}(t)\bigg|^2+M\bigg|\frac{du}{dt}(t+h)\bigg|^2\\
&\quad+\bigg(\frac{d^2u}{dt^2}(t),u(t+h)\bigg)+\bigg(u(t),\frac{d^2u}{dt^2}(t+h)\bigg)\\
&\quad+2\bigg(\frac{du}{dt}(t),\frac{du}{dt}(t+h)\bigg)\\
&\ge M\bigg(\frac{d^2u}{dt^2}(t)-c\frac{du}{dt}(t),u(t)\bigg)\\
&\quad+M\bigg(\frac{d^2u}{dt^2}(t+h)-c\frac{du}{dt}(t+h),u(t+h)\bigg)\\
&\quad+\bigg(u(t),\frac{d^2u}{dt^2}(t+h)-c\frac{du}{dt}(t+h)\bigg)\\
&\quad+\bigg(\frac{d^2u}{dt^2}(t)-c\frac{du}{dt}(t),u(t+h)\bigg)+Mc\bigg(\frac{du}{dt}(t),u(t)\bigg)\\
&\quad+Mc\bigg(\frac{du}{dt}(t+h),u(t+h)\bigg)+c\bigg(u(t),\frac{du}{dt}(t+h)\bigg)\\
&\quad+c\bigg(\frac{du}{dt}(t),u(t+h)\bigg)
\end{aligned}$$

$$\geq c\frac{d}{dt}\bigg(\frac{M}{2}|u(t)|^2 + \frac{M}{2}|u(t+h)|^2 + (u(t), u(t+h))\bigg)$$
$$= c\frac{d}{dt}F(t).$$

Hence we proved: $d^2/dt^2F(t) \geq c(d/dt)F(t)$ for almost every $t \in \mathbf{R}^+$.

If $c = 0$, then F is convex, and since it is also bounded, it follows that F is nonincreasing. If $c > 0$, let $g(t) = F(t) - c\int_0^t F(s)\,ds$ and $K = \sup_{t\geq 0}|u(t)|$. Then g is convex. Since $M \geq 1$, we have: $F(t) \geq 1/2|u(t)|^2 + 1/2|u(t+h)|^2 + (u(t), u(t+h)) = 1/2|u(t) + u(t+h)|^2 \geq 0$. Hence, $g(t) \leq F(t) \leq (M+1)K^2$. Therefore, g is also bounded above; hence, g is nonincreasing. Therefore, $g'(t) \leq 0$, and thus $F'(t) \leq cF(t) \leq c(M+1)K^2$, for all $t \in [\epsilon, +\infty)$, for every $\epsilon > 0$. On the other hand, we have $F''(t) \geq cF'(t)$ for almost every $t \geq 0$. Therefore, F' satisfies the hypothesis of Lemma 3.1, and thus $F'(t) \leq 0$, for all $t \in [\epsilon, +\infty)$, for every $\epsilon > 0$. Hence, again F is nonincreasing on $[\varepsilon, +\infty)$. Therefore, in both cases we have $F(t) \leq F(s)$, for all $t \geq s \geq \varepsilon$, which implies that:

(4.1)
$$(u(t), u(t+h)) \leq (u(s), u(s+h)) + \frac{M}{2}\left[|u(s)|^2 - |u(t)|^2 + |u(s+h)|^2 - |u(t+h)|^2\right] \quad \text{for all } t \geq s \geq \varepsilon.$$

Now let's show that $|u(t)|$ is nonincreasing; hence, $\lim_{t\to+\infty}|u(t)|$ exists. Indeed, we have:

$$\frac{1}{2}\frac{d^2}{dt^2}|u(t)|^2 = \bigg(\frac{d^2u}{dt^2}(t), u(t)\bigg) + \bigg|\frac{du}{dt}(t)\bigg|^2$$
$$= \bigg(\frac{d^2u}{dt^2}(t) - c\frac{du}{dt}(t), u(t)\bigg) + \frac{c}{2}\frac{d}{dt}|u(t)|^2 + \bigg|\frac{du}{dt}(t)\bigg|^2$$
$$\geq \frac{c}{2}\frac{d}{dt}|u(t)|^2$$

by the positivity assumption on A, which implies by a similar argument as above, and then by Lemma 3.1, that $|u(t)|^2$ is nonincreasing.

Then (4.1) and this fact imply that u satisfies the assumptions (2.0) and (2.1) in [**23**]. Hence by [**23**, Theorem 4.2 and Lemma 4.3 (ii)], it already follows that σ_T converges strongly as $T \to +\infty$ to the asymptotic center p of u. To prove the strong convergence of u, we show first that u is asymptotically regular. In fact by the positivity assumption on A, we have:

$$\left|\frac{du}{dt}(t)\right|^2 \le \frac{d}{dt}\left(\frac{du}{dt}(t), u(t)\right) - \frac{c}{2}\frac{d}{dt}|u(t)|^2.$$

Integrating both sides of this inequality on $[\varepsilon, T]$, we get:

$$\begin{aligned}\int_\varepsilon^T \left|\frac{du}{dt}(t)\right|^2 dt &\le \left(\frac{du}{dt}(T), u(T)\right) \\ &\quad - \left(\frac{du}{dt}(\varepsilon), u(\varepsilon)\right) \\ &\quad - \frac{c}{2}|u(T)|^2 + \frac{c}{2}|u(\varepsilon)|^2 \\ &\le -\left(\frac{du}{dt}(\varepsilon), u(\varepsilon)\right) + \frac{c}{2}|u(\varepsilon)|^2 < +\infty\end{aligned}$$

since $|u(t)|^2$ is nonincreasing. Therefore, for $h > 0$ fixed, we have:

$$\begin{aligned}|u(t+h) - u(t)| &\le \int_t^{t+h} \left|\frac{du}{ds}(s)\right| ds \\ &\le \sqrt{h}\left(\int_t^{t+h} \left|\frac{du}{ds}(s)\right|^2 ds\right)^{1/2} \longrightarrow 0 \text{ as } t \to +\infty.\end{aligned}$$

Hence, we showed that u is asymptotically regular. Now, using the polarization identity, and rearranging the terms, it follows from (4.1) that:

$$\begin{aligned}|u(s+h) - u(s)|^2 &\le |u(t+h) - u(t)|^2 + (M+1) \\ &\quad \times \left[|u(s)|^2 - |u(t)|^2 + |u(s+h)|^2 - |u(t+h)|^2\right] \\ &\quad \text{for all } t \ge s \ge \varepsilon, \text{ for all } h \ge 0.\end{aligned}$$

For $\eta > 0$ arbitrary, let's choose $t_0 \ge \varepsilon$ so that:

$$(M+1)\left[|u(s)|^2 - |u(t)|^2 + |u(s+h)|^2 - |u(t+h)|^2\right] \le \eta,$$
$$\text{for all } t \ge s \ge t_0, \text{ for all } h \ge 0.$$

Then we get:

$|u(s+h)-u(s)|^2 \le |u(t+h)-u(t)|^2+\eta,$ for all $h \ge 0,$ for all $t \ge s \ge t_0;$

letting $t \to +\infty$ in the above inequality and using the asymptotic regularity of u we get:

$$|u(s+h) - u(s)|^2 \le \eta, \text{ for all } h \ge 0, \text{ for all } s \ge t_0.$$

Hence, $\{u(t); t \ge 0\}$ is a Cauchy net in H, and therefore strongly convergent as $t \to +\infty$, to the asymptotic center p of u. □

Our next corollary gives a partial affirmative answer to problem 3.1.

Corollary 4.4. *Assume that the operator A in* (3) *is maximal monotone and satisfies condition* (b)*. If u is a solution to* (3.a) *or* (3.b)*, then $u(t)$ converges strongly as $t \to +\infty$ to the asymptotic center p of u, and $p \in A^{-1}(0)$.*

Proof. In fact, it is easy to see that if A is monotone and satisfies condition (b), then it also satisfies condition (a), from which, by taking $x_2 = x_1$ and $y_2 = y_1$, it follows that $(y_1, x_1) \ge 0$, for all $[x_1, y_1] \in A$. Therefore, in both cases of (3.a) and (3.b), we deduce from Theorem 4.3 that $\lim_{t\to+\infty} u(t) = p$. □

Remark 4.1. Corollary 4.4 extends [**15**, Theorem 2.1], for solutions to (3.a), where A is assumed to be maximal monotone. In the proof of Theorem 4.3, we noticed the strong convergence of σ_T as $T \to +\infty$, by using our strong nonlinear ergodic theorem [**23**, Theorem 4.2], which is more general than [**5**, Theorem 2], used in the proof of [**15**, Theorem 2.1], but it is not applicable in our case. Moreover, the proof of Theorem 4.3 shows that the strong convergence of u may be proven without using this strong nonlinear ergodic theorem.

Corollary 4.5. *Let u be a solution to* (3.a) *or* (3.b) *with $A = \partial\varphi$, where φ is a proper, convex and lower semi-continuous function on H satisfying the following condition* (a$_1$)*. $D(\varphi) = -D(\varphi)$ and $\varphi(x) - \varphi(0) \ge a(|x|)(\varphi(-x)-\varphi(0))$ for all $x \in D(\varphi)$, where $a : \mathbf{R}^+ \to (0, +\infty)$*

is a continuous function. Then $u(t)$ converges strongly as $t \to +\infty$ to the asymptotic center p of u, and $p \in A^{-1}(0)$.

Proof. In fact, it is easy to show that if φ satisfies the Condition (a_1), then $A = \partial\varphi$ satisfies condition (a), which implies condition (b). Hence, by Corollary 4.4, we know that $\lim_{t\to+\infty} u(t) = p$. □

Remark 4.2. Condition (a_1) for φ was first introduced by Okochi [**18**], to study the strong convergence of solutions to monotone type first-order evolution equations.

Remark 4.3. In the proofs of all the theorems in the paper, the maximality of A was only used to show that $p \in A^{-1}(0)$, and hence $A^{-1}(0) \neq \phi$.

5. Example. Let $H = L^2(\Omega)$ where $\Omega \subseteq \mathbf{R}^n$ is a bounded domain with smooth boundary Γ. Let $j : \mathbf{R} \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function, and $\beta = \partial j$. We assume for simplicity that $0 \in \beta(0)$. Define

$$Au = -\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

with

$$D(A) = \left\{u \in H^2(\Omega), \frac{-\partial u}{\partial \eta}(x) \in \beta(u(x)), \text{ a.e. on } \Gamma\right\},$$

where $(\partial u/\partial\eta(x))$ is the outward normal derivative to Γ at $x \in \Gamma$. It is known that $A = \partial\phi$, where $\phi : L^2(\Omega) \to (-\infty, +\infty]$, is the functional:

$$\phi(u) = \begin{cases} 1/2 \int_\Omega |\nabla u|^2 dx & \\ \qquad + \int_\Gamma \beta(u(x))\, d\sigma & \text{if } u \in H^1(\Omega) \text{ and } \beta(u) \in L^1(\Gamma) \\ +\infty & \text{otherwise.} \end{cases}$$

Consider the following equation

$$\begin{cases} \partial^2 u/\partial t^2(t,x) - c\partial u/\partial t(t,x) & \\ \qquad + \sum_i \partial^2 u/\partial x_i^2(t,x) = 0 & \text{a.e. on } \mathbf{R}^+ \times \Omega \\ -\partial u/\partial\eta(t,x) \in \beta u(t,x) & \text{a.e. on } \mathbf{R}^+ \times \Gamma \\ u(0,x) = u_0(x) & \text{a.e. on } \Omega \end{cases}$$

where $c \geq 0$. Then Theorem 3.8 implies the weak convergence of $u(t,.)$ to a minimizer of ϕ.

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Department of Mathematical Sciences, University of Texas at El Paso, 500 W. University Ave., El Paso, TX 79968
Email address: behzad@math.utep.edu

Department of Mathematics, Tarbiat Modares University, P.O. Box 14115-175, Tehran, Iran
Email address: Khatibh@modares.ac.ir