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THE THREE BODY PROBLEM WITH A RIGID BODY: EULERIAN EQUILIBRIA AND STABILITY

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ABSTRACT. We consider the noncanonical Hamiltonian dynamics of a rigid body in the three body problem. By means of geometric-mechanics methods we will study the approximate dynamics that arise when we develop the potential in a series of Legendre and truncate the series to the second harmonics. Working in the reduced problem, we will study the existence of equilibria that will dominate Euler in analogy with classic results on the topic. In this way, we generalize the classical results on equilibria of the three-body problem and many of those obtained by other authors using more classic techniques for the case of rigid bodies. The instability of Eulerian equilibria is proven in this approximate dynamics if the rigid body is close to the sphere.

1. Introduction. In the study of configurations of relative equilibria by differential geometry methods or by more classical ones we will mention here the papers of Wang et al. [**8**] in regards to the problem of a rigid body in a central Newtonian field and Maciejewski [**3**] in regards to the problem of two rigid bodies in mutual Newtonian attraction.

For the problem of three rigid bodies we would like to mention that Vidyakin [**7**] and Dubochine [**1**] proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries; Zhuravlev and Petrutskii [**10**] made a review of the results up to 1990. These works use canonical variables for the deduction of their results.

In Vera [**4**] and a recent paper of Vera and Vigueras [**6**] we study the noncanonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution

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of mass or material points and the other body is a triaxial rigid body. Using the symmetries of the system, we carried out two reductions, giving in each step the Poisson structure of the reduced space. Then, we obtained the equations of motion, the Casimir function of the system and the equations that determine the equilibria and global conditions for the existence of the same ones.

This paper is a concrete application of the general methods of [**6**] to the study of certain types of equilibria of a triaxial rigid body in Newtonian attraction with two spherical bodies (or mass points). We describe the approximate dynamics that arise in a natural way when we take the Legendre development of the potential function and truncate this to the second harmonics. This approximate dynamics is a good description of the full dynamics of the problem supposing that the involved bodies are at much bigger mutual distances than the individual dimensions of the same ones.

We will see global conditions on the existence of relative equilibria and in analogy with classic results on the topic, see [**9**] for details, we will study the existence of relative equilibria that we will denominate of Euler for the case in which S_1, S_2 are spherical or punctual bodies and S_0 is a triaxial rigid body. We will obtain necessary and sufficient conditions for their existence, and we will give explicit expressions of these relative equilibria, useful for the later study of the stability of the same ones. The instability of Eulerian relative equilibria is proven in approximate dynamics if the rigid body is close to the sphere.

This analysis was done in vectorial form, giving to this problem a very compact treatment which avoids the use of canonical variables (Eulerian or Andoyer-Deprit variables) and the tedious expressions associated with them. This is a typical characteristic of the classic literature [**1, 7**] on these systems that the paper overcomes with this vectorial approach. Contrary to the canonical variables, this analysis is free of singularities.

We should notice that the system studied has potential interest both in astrodynamics (dealing with spacecrafts) as well as in the understanding of the evolution of planetary systems recently found (and more to appear), where some of the planets may be modeled like a triaxial rigid body rather than a point mass. In fact, the equilibria reported might well be compared with the ones taken for the 'parking areas' of the space missions (GENESIS, SOHO, DARWIN, etc.) around the Eulerian points of the Sun–Earth and the Earth–Moon systems, see [**2**] for details.

To finish this introduction, we describe the structure of the article. The paper is organized in six sections, one appendix and the bibliography. In these sections we study the equations of motion, Casimir function and integrals of the system, the relative equilibria and the existence of Eulerian equilibria; in particular, we study the bifurcations of Eulerian equilibria in this approximate dynamics.

Approximate Poisson dynamics. Following the line of Vera and Vigueras [**6**] let S_0 be a rigid body of mass m_0 and S_1, S_2 two spherical rigid bodies of masses m_1 and m_2. We use the following notation. For $\mathbf{u}$, $\mathbf{v} \in \mathbf{R}^3$, $\mathbf{u} \cdot \mathbf{v}$ is the dot product, $|\mathbf{u}|$ is the Euclidean norm of the vector $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ is the cross product. $\mathbf{I}_{\mathbf{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order three. We consider $\mathbf{I} = \operatorname{diag}(A, B, C)$, $A \neq B \neq C$, the diagonal tensor of inertia of the rigid body with A, B and C the principal inertia moments of S_0.

The vector $\mathbf{z} = (\mathbf{\Pi}, \boldsymbol{\lambda}, \mathbf{p}_{\boldsymbol{\lambda}}, \boldsymbol{\mu}, \mathbf{p}_{\mu}) \in \mathbf{R}^{15}$ is a generic element of the twice reduced problem obtained using the symmetries of the system. We consider $\mathbf{\Omega}$ the angular velocity of S_0, $\mathbf{\Pi} = \mathbf{I}\mathbf{\Omega}$ the total rotational angular momentum vector of the rigid body in the body frame $\mathfrak{J}$, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of S_0. The elements $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{p}_{\boldsymbol{\lambda}}$ and $\mathbf{p}_{\mu}$ are respectively the barycentric coordinates (or Jacobi coordinates) and the linear momenta expressed in the body frame $\mathfrak{J}$.

The twice reduced Hamiltonian of the system, obtained by the action of the group $\mathbf{SE}(3)$, has the following expression

$$\mathcal{H}(\mathbf{z}) = \frac{|\mathbf{p}_{\boldsymbol{\lambda}}|^2}{2g_1} + \frac{|\mathbf{p}_{\boldsymbol{\mu}}|^2}{2g_2} + \frac{1}{2}\mathbf{\Pi}\mathbf{I}^{-1}\mathbf{\Pi} + \mathcal{V} \tag{1}$$

since

$$M_2 = m_1 + m_2, \qquad M_1 = m_1 + m_2 + m_0,$$
$$g_1 = \frac{m_1 m_2}{M_2}, \qquad g_2 = \frac{m_0 M_2}{M_1}.$$

The potential function $\mathcal{V}$ is given by the formula

(2) $\mathcal{V}(\boldsymbol{\lambda}, \boldsymbol{\mu})$

$$= -\Big(\frac{Gm_1m_2}{|\boldsymbol{\lambda}|} + \int_{S_0} \frac{Gm_1 dm(\mathbf{Q})}{|\mathbf{Q} + \boldsymbol{\mu} - (m_2/M_2)\boldsymbol{\lambda}|} + \int_{S_0} \frac{Gm_2 dm(\mathbf{Q})}{|\mathbf{Q} + \mu + (m_1/M_2)\boldsymbol{\lambda}|}\Big)$$

with G the gravitational constant.

Let $\mathbf{M} = \mathbf{R}^{15}$, and we consider the Poisson manifold $(\mathbf{M}, \{\ ,\ \}, \mathcal{H})$, with Poisson brackets $\{\ ,\ \}$ defined by means of the Poisson tensor

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\mathbf{\Pi}} & \widehat{\boldsymbol{\lambda}} & \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & \widehat{\boldsymbol{\mu}} & \widehat{\mathbf{p}_{\mu}} \\ \widehat{\boldsymbol{\lambda}} & \mathbf{0} & \mathbf{I}_{\mathbf{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & -\mathbf{I}_{\mathbf{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbf{R}^3} \\ \widehat{\mathbf{p}}_{\mu} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbf{R}^3} & \mathbf{0} \end{pmatrix} \tag{3}$$

In $\mathbf{B}(\mathbf{z})$, $\widehat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbf{R}^3$ by the standard isomorphism between the Lie algebras $\mathbf{R}^3$ and $\mathfrak{so}(3)$, i.e.,

$$\widehat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

The equations of motion are

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\} = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H} \tag{4}$$

with $\boldsymbol{\nabla}_{\mathbf{u}}\mathcal{V}$ the gradient of $\mathcal{V}$ with respect to an arbitrary vector $\mathbf{u}$.

Developing $\{\mathbf{z}, \mathcal{H}(\mathbf{z})\}$, we obtain the following group of vectorial equations of the motion

$$\begin{aligned} &\frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi} \times \mathbf{\Omega} + \boldsymbol{\lambda} \times \boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V} + \boldsymbol{\mu} \times \boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V}, \\ &\frac{d\boldsymbol{\lambda}}{dt} = \frac{\mathbf{p}_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda} \times \mathbf{\Omega}, \quad \frac{d\mathbf{p}_{\boldsymbol{\lambda}}}{dt} = \mathbf{p}_{\boldsymbol{\lambda}} \times \mathbf{\Omega} - \boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V}, \\ &\frac{d\boldsymbol{\mu}}{dt} = \frac{\mathbf{p}_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu} \times \mathbf{\Omega}, \quad \frac{d\mathbf{p}_{\boldsymbol{\mu}}}{dt} = \mathbf{p}_{\boldsymbol{\mu}} \times \mathbf{\Omega} - \boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V}. \end{aligned} \tag{5}$$

Important elements of $\mathbf{B}(\mathbf{z})$ are the associate Casimir functions. We consider the total angular momentum $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{\Pi} + \boldsymbol{\lambda} \times \mathbf{p}_{\boldsymbol{\lambda}} + \boldsymbol{\mu} \times \mathbf{p}_{\boldsymbol{\mu}}. \tag{6}$$

Then the following result is verified, see [**6**] for details.

If φ is a real smooth not constant function, then $\varphi(|\mathbf{L}|^2/2)$ is a Casimir function of the Poisson tensor $\mathbf{B}(\mathbf{z})$. Moreover, $\operatorname{Ker}\mathbf{B}(\mathbf{z}) = \langle \nabla_{\mathbf{z}}\varphi \rangle$. *Also, we have $d\mathbf{L}/dt = 0$, that is to say, the total angular momentum vector remains constant.*

It is outstanding that the integrals of the potential $\mathcal{V}$, except for some geometries of the rigid body S_0, show important difficulties for the calculation. It arises in a natural way to consider the multipolar development of these potentials, supposing that the involved bodies are at much more mutual distances than the individual dimensions of the same ones. Under additional hypotheses we will be able to develop the potential in quickly convergent series. Considering the series truncated until the second harmonics, then we will be able to study the approximated Poisson dynamics.

For a triaxial rigid body at great distance the following formula is verified with great accuracy

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 \tag{7}$$

where

$$\begin{aligned}
\mathcal{V}_1 &= -\left(\frac{Gm_1m_2}{|\boldsymbol{\lambda}|} + \frac{Gm_1m_0}{|\boldsymbol{\mu} - (m_2/M_2)\boldsymbol{\lambda}|} + \frac{Gm_2m_0}{|\boldsymbol{\mu} + (m_1/M_2)\boldsymbol{\lambda}|} \right), \\
\mathcal{V}_2 &= -\frac{1}{2}\left(\frac{Gm_1\alpha}{|\boldsymbol{\mu} - (m_2/M_2)\boldsymbol{\lambda}|^3} + \frac{Gm_2\alpha}{|\boldsymbol{\mu} + (m_1/M_2)\boldsymbol{\lambda}|^3} \right) \\
&\quad + \frac{3}{2}\left(\frac{Gm_1\beta_1}{|\boldsymbol{\mu} - (m_2/M_2)\boldsymbol{\lambda}|^5} + \frac{Gm_2\beta_2}{|\boldsymbol{\mu} + (m_1/M_2)\boldsymbol{\lambda}|^5} \right)
\end{aligned}$$

and

$$\begin{aligned}
\alpha &= A + B + C, \\
\beta_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \boldsymbol{\mu} \cdot \mathbf{I}\boldsymbol{\mu} - \frac{2m_2}{M_2}\boldsymbol{\lambda} \cdot \mathbf{I}\boldsymbol{\mu} + \left(\frac{m_2}{M_2}\right)^2 \boldsymbol{\lambda} \cdot \mathbf{I}\boldsymbol{\lambda}, \\
\beta_2(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \boldsymbol{\mu} \cdot \mathbf{I}\boldsymbol{\mu} + \frac{2m_1}{M_2}\boldsymbol{\lambda} \cdot \mathbf{I}\boldsymbol{\mu} + \left(\frac{m_1}{M_2}\right)^2 \boldsymbol{\lambda} \cdot \mathbf{I}\boldsymbol{\lambda}
\end{aligned}$$

with A, B and C the principal moments of inertia of S_0 in the appropriate orientation of the body frame $\mathfrak{J}$.

We call differential equations of motion given by the following expression

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}^0(\mathbf{z})\} = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}^0$$

approximate dynamics of order zero since

$$\mathcal{H}^0(\mathbf{z}) = \frac{|\mathbf{p}_{\boldsymbol{\lambda}}|^2}{2g_1} + \frac{|\mathbf{p}_{\boldsymbol{\mu}}|^2}{2g_2} + \frac{1}{2}\boldsymbol{\Pi}\mathbf{I}^{-1}\boldsymbol{\Pi} + \mathcal{V}_1(\boldsymbol{\lambda}, \boldsymbol{\mu}).$$

Similarly, the *approximate dynamics of order one* is given by $(\mathbf{M}, \{\ ,\ \}, \mathcal{H}^1)$ with $\mathcal{H}^1 = \mathcal{H}^0 + \mathcal{V}_2$.

On the other hand, it is easy to verify that

$$\nabla_{\mathbf{z}}(|\boldsymbol{\Pi}|^2)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}^0 = 0$$

and similarly when the rigid body is of revolution

$$\nabla_{\mathbf{z}}(\boldsymbol{\pi}_3)\mathbf{B}(z)\nabla_{\mathbf{z}}\mathcal{H}^0 = 0$$

where $\boldsymbol{\pi}_3$ is the third component of the rotational angular momentum of the rigid body. In what continues $\mathcal{H} = \mathcal{H}^1$.

2.1 Relative equilibria. The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in Vera and Vigueras [**6**] for the case $n = 2$. If we denote by $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ a generic relative equilibrium of an approximate dynamics of order one, then this verifies the equations

$$(8)\qquad \begin{aligned} &\boldsymbol{\Pi}_e \times \boldsymbol{\Omega}_e + \boldsymbol{\lambda}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e + \boldsymbol{\mu}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e = \mathbf{0},\\ &\frac{\mathbf{p}^e_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda}^e \times \boldsymbol{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}^e_{\boldsymbol{\lambda}} \times \boldsymbol{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e,\\ &\frac{\mathbf{p}^e_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu}^e \times \boldsymbol{\Omega}_e = \mathbf{0}, \qquad \mathbf{p}^e_{\boldsymbol{\mu}} \times \boldsymbol{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e. \end{aligned}$$

Also, by virtue of the relationships obtained in Vera and Vigueras [**6**], we have the following result.

If $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_\mu)$ *is a relative equilibrium of an approximate dynamics of order one the following relationships are verified*

$$
\begin{aligned}
|\mathbf{\Omega}_e|^2 \, |\boldsymbol{\lambda}^e|^2 - (\boldsymbol{\lambda}^e \cdot \mathbf{\Omega}_e)^2 &= \frac{1}{g_1}(\boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}} \mathcal{V})_e), \\
|\mathbf{\Omega}_e|^2 \, |\boldsymbol{\mu}^e|^2 - (\boldsymbol{\mu}^e \cdot \mathbf{\Omega}_e)^2 &= \frac{1}{g_2}(\boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}} \mathcal{V})_e).
\end{aligned} \tag{9}
$$

The last two identities will be used to obtain necessary conditions for the existence of relative equilibria in this approximate dynamics.

We will study certain relative equilibria in the approximate dynamics supposing that the vectors $\mathbf{\Omega}_e$, $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ satisfy special geometric properties.

We say that $\mathbf{z}_e$ is an *Eulerian equilibrium* in an approximate dynamics of order one when $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ are proportional and $\mathbf{\Omega}_e$ is perpendicular to the straight line that these generate.

From the equations of motion, after some calculations with $\mathcal{V}$, the following property is deduced.

In an Eulerian equilibrium for any approximate dynamics, moments are not exercised on the rigid body. The vector $\boldsymbol{\lambda}^e$ *is an eigenvector of the tensor of inertia* $\mathbf{I}$.

Next we obtain necessary and sufficient conditions for the existence of Eulerian relative equilibria.

3. Eulerian relative equilibria. According to the relative position of the rigid body S_0 with respect to S_1 and S_2 there are three possible equilibrium configurations: a) $S_0S_2S_1$, b) $S_2S_0S_1$ and c) $S_2S_1S_0$, see Figure 1.

3.1 Necessary condition of existence. If $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_\mu)$ is a relative equilibrium of Euler type, then for the configuration $S_0S_2S_1$ we have

$$
\left|\boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| = |\boldsymbol{\lambda}^e| + \left|\boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right|.
$$

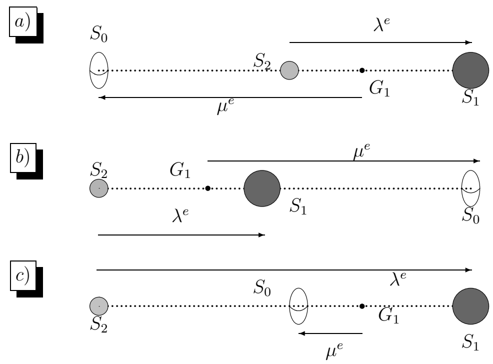

FIGURE 1. Eulerian configurations a), b) and c).

In a similar way, for the configuration $S_2S_0S_1$ we have

$$|\boldsymbol{\lambda}^e| = \left|\boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| + \left|\boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right|.$$

Finally, for the configuration $S_2S_1S_0$ we have

$$\left|\boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e\right| = \left|\boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e\right| + |\boldsymbol{\lambda}^e|.$$

The previous equations are deduced from the definition of the Jacobi coordinates, see [**6**] for details.

Next we study necessary conditions for the existence of relative equilibria of Euler type for the previous configurations. If $\mathbf{z}_e$ is a relative equilibrium of Euler type in an approximate dynamics of order one, using (9), we have

$$g_1|\boldsymbol{\Omega}_e|^2\,|\boldsymbol{\lambda}^e|^2 = \boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e,$$
$$g_2|\boldsymbol{\Omega}_e|^2\,|\boldsymbol{\mu}^e|^2 = \boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e.$$

On the other hand,

$$\boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e = \rho\boldsymbol{\lambda}^e, \qquad \boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e = (1+\rho)\boldsymbol{\lambda}^e,$$
$$\boldsymbol{\mu}^e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2}\boldsymbol{\lambda}^e$$

where $\rho \in (0, +\infty)$ in case a), $\rho \in (-1, 0)$ in case b) and $\rho \in (-\infty, -1)$ in case c).

After some calculations it is possible to obtain the following expressions

$$(\nabla_{\boldsymbol{\lambda}} \mathcal{V})_e = f_1(\rho)\boldsymbol{\lambda}^e, \qquad (\nabla_{\boldsymbol{\mu}} \mathcal{V})_e = f_2(\rho)\boldsymbol{\lambda}^e$$

where

$$f_1(\rho) = \frac{Gm_1m_2}{|\boldsymbol{\lambda}^e|^3}\frac{Gm_1m_2}{M_2}\left(\frac{m_0}{|\boldsymbol{\lambda}^e|^3}\left(\frac{1+\rho}{|1+\rho|^3} - \frac{\rho}{|\rho|^3}\right) + \frac{\beta_1}{|\boldsymbol{\lambda}^e|^5}\left(\frac{1+\rho}{|1+\rho|^5} - \frac{\rho}{|\rho|^5}\right)\right), \tag{10}$$

$$f_2(\rho) = \frac{Gm_0}{|\boldsymbol{\lambda}^e|^3}\left(\frac{m_1(1+\rho)}{|1+\rho|^3} + \frac{m_2\rho}{|\rho|^3}\right) + \frac{G\beta_1}{|\boldsymbol{\lambda}^e|^5}\left(\frac{m_1(1+\rho)}{|1+\rho|^5} + \frac{m_2\rho}{|\rho|^5}\right). \tag{11}$$

Remark 1. The parameter β_1 takes the following values

$$\beta_1 = \frac{3(A+B-2C)}{2}, \qquad \beta_1 = \frac{3(A+C-2B)}{2}, \qquad \beta_1 = \frac{3(B+C-2A)}{2}$$

according to the orientation of the body frame $\mathfrak{J}$.

Now, from the identities

$$\boldsymbol{\lambda}^e \cdot (\nabla_{\boldsymbol{\lambda}} \mathcal{V})_e = |\boldsymbol{\lambda}^e|^2 f_1(\rho),$$
$$\boldsymbol{\mu}^e \cdot (\nabla_{\boldsymbol{\mu}} \mathcal{V})_e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2}|\boldsymbol{\lambda}^e|^2 f_2(\rho)$$

and using (9), we deduce the following equations

$$|\boldsymbol{\Omega}_e|^2 = \frac{(m_1+m_2)f_1(\rho)}{m_1m_2},$$
$$|\boldsymbol{\Omega}_e|^2 = \frac{(m_0+m_1+m_2)f_2(\rho)}{m_0\left((1+\rho)m_1 + \rho m_2\right)}.$$

Then for a relative equilibrium of Euler type ρ must be a real root of the following equation

$$\frac{m_0(m_1+m_2)\left((1+\rho)m_1+\rho m_2\right)f_2(\rho)}{m_1m_2(m_0+m_1+m_2)}=f_1(\rho). \tag{12}$$

We summarize all these results in the following property.

If $\mathbf{z}_e=(\mathbf{\Pi}_e,\boldsymbol{\lambda}^e,\mathbf{p}^e_{\boldsymbol{\lambda}},\boldsymbol{\mu}^e,\mathbf{p}^e_{\mu})$ *is an Eulerian relative equilibrium in the configurations* a), b) *or* c) *the equation* (12) *has, at least, a real root. The functions* $f_1(\rho)$ *and* $f_2(\rho)$ *are given by* (10) *and* (11). *The modulus of the angular velocity of the rigid body is*

$$|\mathbf{\Omega}_e|^2=\frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3}h_1(\rho)$$

with

$$h_1(\rho)=1+\frac{m_0}{m_1+m_2}\left(\frac{1+\rho}{|1+\rho|^3}-\frac{\rho}{|\rho|^3}\right)+\frac{\beta_1}{(m_1+m_2)|\boldsymbol{\lambda}^e|^2}\left(\frac{1+\rho}{|1+\rho|^5}-\frac{\rho}{|\rho|^5}\right).$$

3.2 Sufficient condition of existence. The following result indicates how to find solutions of equation (8).

Fix $|\boldsymbol{\lambda}^e|$ *and let* ρ *be a solution of equation* (12). *Then* $\mathbf{z}_e=(\mathbf{\Pi}_e,\boldsymbol{\lambda}^e,\mathbf{p}^e_{\boldsymbol{\lambda}},\boldsymbol{\mu}^e,\mathbf{p}^e_{\mu})$ *given by*

$$\begin{aligned}\boldsymbol{\lambda}^e&=(\lambda^e,0,0), & \boldsymbol{\mu}^e&=(\mu^e,0,0),\\ \mathbf{p}^e_{\boldsymbol{\lambda}}&=(0,\pm g_1\omega_e\lambda^e,0), & \mathbf{p}^e_{\boldsymbol{\mu}}&=(0,\pm g_2\omega_e\mu^e,0),\\ \mathbf{\Omega}_e&=(0,0,\pm\omega_e), & \mathbf{\Pi}_e&=(0,0,\pm C\omega_e)\end{aligned} \tag{13}$$

or

$$\begin{aligned}\boldsymbol{\lambda}^e&=(\lambda^e,0,0), & \boldsymbol{\mu}^e&=(\mu^e,0,0),\\ \mathbf{p}^e_{\boldsymbol{\lambda}}&=(0,0,\mp g_1\omega_e\lambda^e), & \mathbf{p}^e_{\boldsymbol{\mu}}&=(0,0,\mp g_2\omega_e\mu^e),\\ \mathbf{\Omega}_e&=(0,\pm\omega_e,0), & \mathbf{\Pi}_e&=(0,\pm C\omega_e,0)\end{aligned} \tag{14}$$

where

$$\boldsymbol{\mu}^e=\frac{((1+\rho)m_1+\rho m_2)}{M_2}\boldsymbol{\lambda}^e,\qquad \omega_e^2=\frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3}h_1(\rho)$$

is a solution of relative equilibrium of Euler type in an approximate dynamic of order one *in configurations* a), b) *or* c). *The total angular momentum of the system is given by*

$$\mathbf{L} = (0, 0, \pm C\omega_e \pm g_1\omega_e\lambda^e \pm g_2\omega_e\mu^e)$$

or

$$\mathbf{L} = (0, \pm C\omega_e \mp g_1\omega_e\lambda^e \mp g_2\omega_e\mu^e, 0).$$

Let us see the existence and number of solutions for the approximate dynamics of order zero and one respectively. For superior order it is possible to use a similar technique.

4. Eulerian relative equilibria in an approximate dynamics of order zero and one. The following property gathers the results about relative equilibria of Euler type in an approximate dynamics of order zero in any of the cases previously mentioned a), b) or c). These results generalize the classical ones of [**9**].

If ρ is the unique positive root of the equation

$$\begin{aligned} p_0(\rho) = (m_1 + m_2)\rho^5 + (3m_1 + 2m_2)\rho^4 \\ + (3m_1 + m_2)\rho^3 - (3m_0 + m_2)\rho^2 \\ - (3m_0 + 2m_2)\rho - (m_0 + m_2) = 0 \end{aligned}$$

with

$$\begin{aligned} |\mathbf{\Omega}_e|^2 &= \frac{G(m_1 + m_2)}{|\boldsymbol{\lambda}_e|^3} h_0(\rho), \\ h_0(\rho) &= 1 + \frac{m_0}{m_1 + m_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right) \end{aligned}$$

then $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$, *given by* (13) *or* (14) *is a relative equilibrium of Euler type in the configuration* $S_0S_2S_1$.

If $\rho \in (-1, 0)$ is the unique root of the equation

$$\begin{aligned} p_0(\rho) = (m_1 + m_2)\rho^5 + (3m_1 + 2m_2)\rho^4 \\ + (3m_1 + m_2)\rho^3 + (3m_0 + 2m_1 + m_2)\rho^2 \\ + (3m_0 + 2m_2)\rho + (m_0 + m_2) = 0 \end{aligned}$$

with

$$|\mathbf{\Omega}_e|^2 = \frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3} h_0(\rho),$$
$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2} - \frac{1}{(1+\rho)^2}\right)$$

then $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$, *given by* (13) *or* (14) *is a relative equilibrium of Euler type in the configuration* $S_2S_0S_1$.

If $\rho \in (-\infty, -1)$ *is the unique root of the equation*

$$\begin{aligned} p_0(\rho) = (m_1+m_2)\rho^5 + (3m_1+2m_2)\rho^4 \\ + (2m_0+3m_1+m_2)\rho^3 + (3m_0+m_2)\rho^2 \\ + (3m_0+2m_2)\rho + (m_0+m_2) = 0 \end{aligned}$$

with

$$|\mathbf{\Omega}_e|^2 = \frac{G(m_1+m_2)}{|\boldsymbol{\lambda}_e|^3} h_0(\rho),$$
$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2} + \frac{1}{(1+\rho)^2}\right)$$

then $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$, *given by* (13) *or* (14) *is a relative equilibrium of Euler type in the configuration* $S_2S_1S_0$.

If $m_0 \to 0$, then $|\mathbf{\Omega}_e|^2 = (G(m_1+m_2))/|\boldsymbol{\lambda}_e|^3$ and the equations that determine the Eulerian equilibria are the same as the ones of the restricted three body problem, see [**9**].

4.1 Bifurcation of Eulerian relative equilibria in an approximate dynamics of order one. For the approximate dynamics of order one, after carrying out the appropriate calculations, equation (1) corresponding to the configuration $S_0S_2S_1$ is reduced to the study of the positive real roots of the polynomial

$$p_1(\rho) = m_0 a^2 \rho^2 (\rho+1)^2 p_0(\rho) - \beta_1 q_0(\rho)$$

where $a = |\boldsymbol{\lambda}_e|$ and $\beta_1 = 3(A+B-2C)/2$, $3(A+C-2B)/2$ or $3(B+C-2A)/2$.

The polynomial q_0 comes determined by the following expression

$$\begin{aligned} q_0(\rho) = (m_1+m_2+5m_0)\rho^4 + (4m_2+10m_0)\rho^3 \\ + (6m_2+10m_0)\rho^2 + (4m_2+5m_0)\rho + (m_0+m_2). \end{aligned}$$

The polynomial p_1 has degree nine and generalizes the classic quintic equation which determines the Eulerian equilibrium, see [**1, 7**] for details.

To study the positive real roots of p_1, we will analyze the rational function R_1

$$\beta_1 = R_1(\rho) = \frac{p_0(\rho)}{q_0(\rho)}$$

since p_0 is the polynomial of degree five that determines the relative equilibria in the approximate dynamics of order zero. The rational function $R_1(\rho)$, for any value of m_0, m_1, m_2, always presents a minimum ξ_1 located between 0 and ρ_0, since this last value is the only one positive zero of the polynomial $p_0(\rho)$. By virtue of these statements the following result is obtained.

In the approximate dynamics of order one, if $\beta_1 < 0$*, we have*:

• $\beta_1 < R_1(\xi_1)$; *then relative equilibria of Euler type don't exist.*

• $\beta_1 = R_1(\xi_1)$; *then there exists a unique relative equilibrium of Euler type.*

• $R_1(\xi_1) < \beta_1 < 0$; *then two* 1*-parametric families of relative equilibria of Euler type exist.*

• *If* $\beta_1 > 0$*, then there exists a unique* 1*-parametric family of relative equilibria of Euler type.*

For the configurations $S_2S_0S_1$ and $S_2S_1S_0$ it is possible to obtain similar results.

5. Stability of Eulerian relative equilibria. The tangent flow of equation (5) in the equilibrium $\mathbf{z}_e$ comes given by

$$\frac{d\delta\mathbf{z}}{dt} = \mathfrak{U}(\mathbf{z}_e)\delta\mathbf{z}$$

with $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ and $\mathfrak{U}(\mathbf{z}_e)$ the Jacobian matrix of (5) in $\mathbf{z}_e$.

The characteristic polynomial of $\mathfrak{U}(\mathbf{z}_e)$ has the following expression

$$p(T) = (T^2 + \Phi^2)(T^4 + mT^2 + n)h(T) \tag{15}$$

since

$$h(T) = T^8 + pT^6 + qT^4 + rT^2 + s$$

with $\Phi = (\beta_1 \omega_e + l)/A$, where the coefficients that intervene in the previous polynomial are functions of the parameters of the problem and ρ, since ρ is the root of equation (12).

5.1 Order zero approximate dynamics. The characteristic polynomial (15) of $\mathfrak{U}(\mathbf{z}_e)$ simplifies to

$$p = T^3(T^2 + \Phi^2)(T^2 + \omega_e^2)^2(T^2 + p)(T^4 + qT^2 + r) \tag{16}$$

with coefficients expressed in Appendix A.

If $p \geq 0$, $q \geq 0$, $r \geq 0$, $q^2 - 4r \geq 0$, then $\mathbf{z}_e$ is spectrally stable. These conditions are not verified since $r < 0$.

If $\mathbf{z}_e$ is an equilibrium in the configuration $S_0 S_2 S_1$ of the zero order approximate dynamics, then it is unstable.

5.2 Order one approximate dynamics. We will analyze the case where the rigid body is close to a sphere. In this case $\beta_1 \approx 0$ then applying the implicit function theorem $\mathbf{z}_e$ is unstable.

If β_1 is not close to zero, the coefficients of polynomial (15) have very complicated expressions. Numeric calculations prove that there exist, for certain values of the parameter β_1, linear stable Eulerian relative equilibria, see Vera [**4**] for details.

These results are equally valid for the configurations $S_2 S_0 S_1$ and $S_2 S_1 S_0$.

6. Conclusions and future works. The approximate dynamics of a rigid body in Newtonian interaction with two spherical or punctual rigid bodies is considered. For orders zero and one approximate dynamics, a complete study of Eulerian relative equilibria is made. The results obtained generalize those of [**1, 7, 9**]. Moreover, other not previously considered results have been studied. The bifurcations of the Eulerian relative equilibria are completely determined for an approximate dynamic of order one. The instability of these relative equilibria is made in zero order approximate dynamics and order one approximate dynamics if the rigid body S_0 is close to a sphere. Diverse results, which had been obtained by means of classic methods in previous works, have been obtained and generalized in a different way. The methods employed in this work are susceptible to being used in similar problems. Numerous problems are open, and among them it is necessary to consider the study of the "inclined" relative equilibria.

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APPENDIX

A. Coefficients of the characteristic polynomial in Eulerian relative equilibria. The coefficients of the characteristic polynomial (16) are

$$\begin{aligned}
\omega_e^2 &= \frac{G\,((m_2+m_1)\,\rho^4+(2\,m_1+2\,m_2)\,\rho^3}{\lambda_e^3(1+\rho)^2\rho^2} \\
&\quad+\frac{(m_2+m_1)\,\rho^2-2\,m_0\,\rho-m_0)}{\lambda_e^3(1+\rho)^2\rho^2}, \\
p &= \frac{G((m_2+4m_0+m_1)\rho^3+(3m_2+6m_0)\rho^2}{(1+\rho)^3\rho^3\lambda_e^3} \\
&\quad+\frac{(4m_0+3m_2)\rho+m_0+m_2)}{(1+\rho)^3\rho^3\lambda_e^3}, \\
q &= \frac{G((-2m_1m_2\rho^4}{((1+\rho)^3\rho^3\lambda_e^3)} \\
&\quad+\frac{(-2m_0m_1+m_1^2+m_2^2-2m_1m_2-2m_0m_2)\rho^3}{((1+\rho)^3\rho^3\lambda_e^3)} \\
&\quad+\frac{(3m_2^2+m_1m_2-6m_0m_1)\rho^2}{((1+\rho)^3\rho^3\lambda_e^3)} \\
&\quad+\frac{(-m_1m_2+3m_2^2+2m_0m_2-4m_0m_1)\rho}{((1+\rho)^3\rho^3\lambda_e^3)} \\
&\quad+\frac{(m_2^2-m_0m_1+m_0m_2-m_1m_2))}{((1+\rho)^3\rho^3\lambda_e^3)},
\end{aligned}$$

$$r = \frac{G^2(a_1\rho^4 + a_2\rho^4 + a_3\rho^2 + a_4\rho + a_5)}{((1+\rho)^8\,\rho^8\,\lambda_e^9)}.$$

A.1 Coefficients a_i, $i = 1, \ldots, 5$.

$$\begin{aligned}
a_1 = &-42m_2^7m_1 - 48m_2^7m_0 - 147m_2^6m_1^2\\
&- 207m_2^5m_1^3 - 782m_2^5m_1^2m_0 - 673m_2^5m_1m_0^2\\
&- 869m_2^4m_1^3m_0 - 1325m_2^4m_1^2m_0^2 - 378m_2^4m_1m_0^4\\
&- 513m_2^3m_1^4m_0 - 1270m_2^3m_1^3m_0^2 - 702m_2^3m_1^2m_0^3\\
&- 165m_2^2m_1^5m_0 - 610m_2^2m_1^4m_0^2 - 648m_2^2m_1^3m_0^3\\
&- 119m_2m_1^5m_0^2 - 297m_2m_1^4m_0^3 + 2m_1^6m_0^2 - 64m_2^3m_1^5\\
&- 336m_2^6m_1m_0 - 129m_2^6m_0^2 - 81m_2^5m_0^3 - 14m_2^2m_1^6\\
&- 150m_2^4m_1^4 - 24m_2m_1^6m_0 - 54m_1^5m_0^3,
\end{aligned}$$

$$\begin{aligned}
a_2 = &-60m_2^7m_1 - 54m_2^7m_0 - 243m_2^6m_1^2\\
&- 399m_2^5m_1^3 - 1345m_2^5m_1^2m_0 - 999m_2^5m_1m_0^2\\
&- 1846m_2^4m_1^3m_0 - 2223m_2^4m_1^2m_0^2 - 648m_2^4m_1m_0^2\\
&- 1364m_2^3m_1^4m_0 - 2506m_2^3m_1^3m_0^2 - 1242m_2^3m_1^2m_0^3\\
&- 536m_2^2m_1^5m_0 - 1530m_2^2m_1^4m_0^2 - 1188m_2^2m_1^3m_0^3\\
&- 477\,m_2m_1^5m_0^2 - 567m_2m_1^4m_0^3 - 56m_1^6m_0^2\\
&- 474m_2^6m_1m_0 - 173m_2^6m_0^2 - 329m_2^4m_1^4\\
&- 135m_2^5m_0^3 - 138m_2^3m_1^5 - 24m_2^2m_1^6\\
&- 90m_2m_1^6m_0 - 108m_1^5m_0^3,
\end{aligned}$$

$$\begin{aligned}
a_3 = &-42m_2^7m_1 - 36m_2^7m_0 - 183m_2^6m_1^2\\
&- 349m_2^5m_1^3 - 1097m_2^5m_1^2m_0 - 630m_2^5m_1m_0^2\\
&- 1776m_2^4m_1^3m_0 - 166\,m_2^4m_1^2m_0^2 - 405m_2^4m_1m_0^3\\
&- 1614m_2^3m_1^4m_0 - 2256m_2^3m_1^3m_0^2 - 810m_2^3m_1^2m_0^3\\
&- 827m_2^2m_1^5m_0 - 1683m_2^2m_1^4m_0^2 - 810m_2^2m_1^3m_0^3\\
&- 228m_2m_1^6m_0 - 666m_2m_1^5m_0^2 - 405m_2m_1^4m_0^3\\
&- 81m_1^5m_0^3 - 111m_1^6m_0^2 - 93m_2^6m_0^2 - 30m_1^7m_0
\end{aligned}$$

$$
\begin{aligned}
&- 342m_2^6m_1m_0 - 81m_2^5m_0^3 - 358m_2^4m_1^4\\
&- 189m_2^3m_1^5 - 31m_2^2m_1^6 - 6m_2\, m_1^7,
\end{aligned}
$$

$$
\begin{aligned}
a_4 = &-12m_2^7m_1 - 12m_2^7m_0 - 56m_2^6m_1^2 - 24m_1^6m_0^2\\
&- 130m_2^5m_1^3 - 387m_2^5m_1^2m_0 - 162m_2^5m_1m_0^2\\
&- 687m_2^4m_1^3m_0 - 432m_2^4m_1^2m_0^2 - 140m_2^3m_1^5\\
&- 588m_2^3m_1^3m_0^2 - 52m_2^2m_1^6 - 387m_2^2m_1^5m_0\\
&- 108m_2m_1^6m_0 - 162m_2m_1^5m_0^2 - 12m_1^7m_0\\
&- 114m_2^6m_1m_0 - 24m_2^6m_0^2 - 179m_2^4m_1^4\\
&- 693m_2^3m_1^4m_0 - 432m_2^2m_1^4m_0^2 - 6m_2m_1^7,
\end{aligned}
$$

$$
\begin{aligned}
a_5 = &-(m_0 + m_2)(18m_0m_2^6 + 12m_1m_2^6\\
&+ 81m_2^4m_0^2m_1 + 168m_2^4m_0m_1^2 + 42m_2^4m_1^3\\
&+ 27m_2^3m_1^4 + 15m_2^2m_1^5 + 31m_2^2m_0m_1^4\\
&+ 54m_2m_0^2m_1^4 + 12m_2m_0m_1^5 + 5m_2m_1^6 + 7m_1^6m_0\\
&+ 144m_2^3m_0^2m_1^2 + 36m_1^2m_2^5 + 18m_0^2m_2^5 + 9m_0^2m_1^5\\
&+ 128m_2^3m_0m_1^3 + 94m_2^5m_0m_1 + 126m_2^2m_0^2m_1^3).
\end{aligned}
$$

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