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AFFINE ISOPERIMETRIC INEQUALITIES FOR L_p-INTERSECTION BODIES

LU FENGHONG AND MAO WEIHONG

ABSTRACT. An L_p-analog of the Busemann intersection inequality and an L_p-dual analog of the L_p-Petty projection inequality for the L_p-intersection body ($p \leq -1$) are established. Moreover, the Busemann-Petty problem is studied and inequalities for the volume of an L_p-intersection body ($p \leq -1$) are proved.

1. Introduction. Intersection bodies were first explicitly defined and named by Lutwak in the important paper [**11**]. The closure of the class of intersection bodies was studied by Goody et al. [**8**]. The intersection operator and the class of intersection bodies played a critical role in Zhang's [**20**] and Gardner's [**5**] solution of the famous Busemann-Petty problem. (See also Gardner et al. [**7**].) The study of projection bodies has a long and complicated history. Projection bodies go back to Minkowski [**6, 19**]. An extensive article that details this is by Bolker [**1**]. After the appearance of Bolker's article, projection bodies have received considerable attention, see, e.g., [**2, 6, 10, 19**]. As Lutwak [**11**] shows (and as is further elaborated in Gardner's book [**6**]), there is a duality between projection and intersection bodies. A number of important results regarding these notions were proved, in particular, two fundamental inequalities: the Busemann intersection inequality ([**3**]) and the Petty projection inequality ([**17**]).

In recent years Lutwak in [**12, 13**], using Firey's p-sum [**4**], extended the Brunn-Minkowski theory to the so called L_p-Brunn-Minkowski theory. In the L_p-Brunn-Minkowski theory, Lutwak, Yang and Zhang introduced the notion of the L_p-projection body and established the following L_p-Petty projection inequality (1.1), see [**15**].

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Theorem A*. *If K is a convex body that contains the origin in its interior in $\mathbf{R}^n$, then, for $p \geq 1$,*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leq \omega_n^{n/p}, \tag{1.1}$$

with equality if and only if K is an ellipsoid centered at the origin.

Haberl and Ludwig in [**9**] define the L_p-intersection body and establish some properties of L_p-intersection bodies.

One purpose of this paper is to establish the following star dual analog of the above L_p-Petty projection inequality.

Theorem A. *If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then*

$$V(K)^{(n-p)/p} V(I_p^\circ K) \geq \omega_n^{n/p}, \tag{1.2}$$

with equality if and only if K is a ball centered at the origin.

In fact, in Section 3 we will establish the L_p-analog of the Busemann intersection inequality (Theorem 3.3) and the star dual analog of L_p-Petty projection inequality for L_p-intersection body ($p \leq -1$).

The other aim of this paper is to study the Busemann-Petty problem and to establish some inequalities for volume of L_p-intersection bodies ($p \leq -1$), Section 4.

2. Notation and preliminaries.

2.1. Support function, radial function and polar body. Let $\mathcal{K}^n$ denote the set of convex bodies (compact,convex subsets with nonempty interiors) in the Euclidean space $\mathbf{R}^n$; for the set of convex bodies containing the origin in their interiors in $\mathbf{R}^n$, write $\mathcal{K}_o^n$. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbf{R}^n \to \mathbf{R}$, is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbf{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y. The Hausdorff distance, $\delta(K, L)$, between $K, L \in \mathcal{K}^n$, can be defined by

$\delta(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbf{R}^n$, which is star-shaped (about the origin), is its radial function, $\rho_K = \rho(K,\cdot) : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$, defined by

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbf{R}^n \setminus \{0\}. \tag{2.2}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbf{R}^n$. Two star bodies K and L are said to be dilates (of each other) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, its polar body K^*, is defined by

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.3}$$

It is easy to verify that $(K^*)^* = K$. From definition (2.3), it follows that if $K \in \mathcal{K}_o^n$, then the support function and the radial function of K^* satisfy

$$h_{K^*} = \frac{1}{\rho_K} \quad \text{and} \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.4}$$

2.2. L_p-dual mixed volume. For $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, $\varepsilon > 0$, the L_p-harmonic radial combination $K +_p \varepsilon \cdot L$ is defined (see [**13**]) as the star body whose radial function is given by

$$\rho(K +_p \varepsilon \cdot L, \cdot)^{-p} = \rho(K,\cdot)^{-p} + \varepsilon \rho(L,\cdot)^{-p}.$$

The L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ of star bodies K, L, for $p \geq 1$, was defined in [**13**] by

$$-\frac{n}{p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.5}$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ of star bodies K, L, for $p \geq 1$ ([**13**, Proposition 1.9])

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,v)^{n+p}\rho(L,v)^{-p}\, dS(v), \tag{2.6}$$

where integration is with respect to the spherical Lebesgue measure S on S^{n-1}.

From the definition of the L_p-dual mixed volume, it follows immediately that for every $K \in \mathcal{S}_o^n$,

$$\widetilde{V}_{-p}(K,K) = V(K). \tag{2.7}$$

We shall also need a basic inequality for the L_p-dual mixed volumes. For $p \geq 1$, the L_p-Minkowski inequality for the L_p-dual mixed volumes states that for star bodies K and L, see [**13**],

$$\widetilde{V}_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{2.8}$$

with equality if and only if K and L are dilates.

2.3. L_p-projection body and L_p-centroid body. If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then the L_p-projection body $\Pi_p K$ of K is the origin-symmetric convex body whose support function is given by ([**14**])

$$h(\Pi_p K, u)^p = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p \, dS_p(K,v), \tag{2.9}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}},$$

and ω_n denotes the n-dimensional volume of the unit ball B in $\mathbf{R}^n$, namely,

$$\omega_n = \pi^{n/2} / \Gamma\left(1 + \frac{n}{2}\right).$$

If $K \in \mathcal{S}_o^n$ and $p \geq 1$, then the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function is given by

$$h(\Gamma_p K, u)^p = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx, \tag{2.10}$$

where the integration is with respect to the Lebesgue measure (see [**14, 15**]).

2.4. Star dual of a star body. Associated with a star body $L \in \mathcal{S}_o^n$ is its star dual L°, which was introduced by Moszyńska [**16**]. Let i be the inversion of $\mathbf{R}^n \backslash \{0\}$, with respect to S^{n-1}:

$$i(x) := \frac{x}{||x||^2}.$$

Then the star dual L° of a star body $L \in \mathcal{S}_o^n$ is defined by

$$L^\circ = \mathrm{cl}\,(\mathbf{R}^n \backslash i(L)).$$

It is easy to verify (see [**16**]) that for every $u \in S^{n-1}$,

$$\rho(L^\circ, u) = \frac{1}{\rho(L, u)}. \tag{2.11}$$

3. L_p-analog of the Busemann intersection inequality. The intersection body of a star body is defined by Lutwak in [**11**]. If $K \in \mathcal{S}_o^n$, then the intersection body IK of K is the origin-symmetric star body whose radial function, restricted to S^{n-1}, is given by

$$\rho(IK, u) = v(K \cap u^\perp),$$

where $v(K \cap u^\perp)$ denotes the $(n-1)$-dimensional volume of the section of K by the linear hyperplane orthogonal to u, see [**11**].

In [**9**] Haberl and Ludwig introduced the notion of the L_p-intersection body I_pK of a star body K for $p < 1$. If $K \in \mathcal{S}_o^n$ and $p < 1$, then the L_p-intersection body I_pK of K is the origin-symmetric star body whose radial function, for $u \in S^{n-1}$, is given by

$$\rho(I_pK, u)^p = \int_K |u \cdot x|^{-p}\, dx. \tag{3.1}$$

Further, Haberl and Ludwig [**9**] established the following relation between the intersection body and the L_p-intersection body:

$$(1-p)I_pK \longrightarrow IK \text{ as } p \longrightarrow 1^-.$$

In the present paper, we only discuss the case $p \leq -1$. For $p \leq -1$, we modify slightly definition (3.1): If $K \in \mathcal{S}_o^n$ and $p \leq -1$; then the L_p-intersection body I_pK of K is the origin-symmetric star body whose radial function, for $u \in S^{n-1}$, is given by

$$\rho(I_pK, u)^p = \frac{1}{c_{n,-p}\,\omega_n} \int_K |u \cdot x|^{-p}\, dx. \tag{3.2}$$

The normalization above is chosen so that for the standard unit ball B in $\mathbf{R}^n$, we have $I_pB = B$.

From equality (2.4) and definitions (2.10) and (3.2), we can immediately get

Theorem 3.1. *If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then*

$$I_pK = \left(\frac{V(K)}{\omega_n}\right)^{1/p} \Gamma^*_{-p}K. \tag{3.3}$$

Equality (3.3) shows that, up to a factor, the L_p-intersection body is just the polar of the L_{-p}-centroid body when $p \leq -1$.

Theorem 3.2. *If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then for $\phi \in SL(n)$,*

$$I_p\phi K = \phi^{-t} I_pK, \tag{3.4}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

Proof. Note that, if $\phi \in SL(n)$, then $\Gamma_p\phi K = \phi\Gamma_pK$ (see [**14, 15**]). Thus, by equality (3.3), we immediately obtain the result. □

One of the classical affine isoperimetric inequalities is the Busemann intersection inequality:

Theorem 3.3$_1$ [**3**]. *Let K be a star body in $\mathbf{R}^n$. Then*

$$V(K)^{1-n}V(IK) \leq \omega_n^{2-n}$$

with equality if and only if K is an ellipsoid.

We will establish the L_p-analog of the Busemann intersection inequality.

Theorem 3.3. *If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then*

$$V(K)^{(p-n)/p} V(I_p K) \leq \omega_n^{(2p-n)/p}, \tag{3.5}$$

with equality if and only if K is an ellipsoid centered at the origin.

The following statement is a "star dual" version of the L_p-Busemann intersection inequality, which may also be considered as a "dual" version of the L_p-Petty projection inequality (1.1), concerning the polar duals of convex bodies.

Theorem 3.4. *If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then*

$$V(K)^{(n-p)/p} V(I_p^\circ K) \geq \omega_n^{n/p} \tag{3.6}$$

with equality if and only if K is a ball centered at the origin.

In order to prove Theorems 3.3 and 3.4, we need the following lemma.

Lemma 3.1 [**15**].*If $K \in \mathcal{S}_o^n$ and $p \leq -1$, then*

$$V(K)V(\Gamma^*_{-p}K) \leq \omega_n^2$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 3.3. From Theorem 3.1 and Lemma 3.1, for $p \leq -1$, we have

$$V(K)V\left(\left(\frac{\omega_n}{V(K)}\right)^{1/p} I_p K\right) \leq \omega_n^2.$$

By the volume formula,

$$V(K)V(I_p K)V(K)^{-n/p} \leq \omega_n^2 \omega_n^{-n/p},$$

that is,

$$V(K)^{(p-n)/p} V(I_p K) \leq \omega_n^{(2p-n)/p}$$

with equality if and only if K is an ellipsoid centered at the origin. □

Proof of Theorem 3.4. From equality (2.11), the Hölder inequality ([**6, 19**]) and the polar coordinate formula for volume, we have

$$\begin{aligned}\omega_n &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n/2}\rho(K,u)^{-n/2}dS(u)\\ &= \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n/2}\rho(K^\circ,u)^{n/2}dS(u)\\ &\le \left(\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n\, dS(u)\right)^{1/2}\left(\frac{1}{n}\int_{S^{n-1}} \rho(K^\circ,u)^n\, dS(u)\right)^{1/2}\\ &= V(K)^{1/2}V(K^\circ)^{1/2},\end{aligned}$$

that is,

$$V(K)V(K^\circ) \ge \omega_n^2, \tag{3.7}$$

According to the equality condition of the Hölder inequality, we know that equality in inequality (3.7) holds if and only if K is a centered ball.

Combining inequality (3.5) with inequality (3.7), we have

$$V(K)^{(n-p)/p}V(I_p^\circ K) \ge \omega_n^{n/p}.$$

According to the equality conditions of inequality (3.5) and inequality (3.7), we know that equality in inequality (3.6) holds if and only if K is a ball centered at the origin. □

4. Monotonicity of volume for L_p-intersection bodies. The work of Lutwak [**11**] represents the beginning of Busemann-Petty problem's ([**6, 19**]) eventual solution. In fact, Lutwak's result (Theorem 10.1) can be formulated as follows. Let I^n denote the set of intersection bodies of star bodies.

Theorem 4.1$_1$ [**11**]. *Let $K \in I^n$, and let L be a star body in $\mathbf{R}^n$. If*

$$IK \subseteq IL,$$

then

$$V(K) \leq V(L),$$

with equality if and only if $K = L$.

We will establish a similar result for L_p-intersection bodies. Let I_p^n denote the set of L_p-intersection bodies of star bodies.

Theorem 4.1. *Let* $K \in I_p^n$*, and let* L *be a star body in* $\mathbf{R}^n$*. If* $p \leq -1$ *and*

$$I_p K \subseteq I_p L,$$

then

$$V(K) \geq V(L), \tag{4.1}$$

with equality if and only if $K = L$.

Theorem 4.1 is just an L_p-version of Busemann-Petty problem's solution for the L_p-intersection body, which is the dual analog of Shephard problem's solution for the L_p-projection body, which was studied by Ryabogin and Zvavitch [**18**].

Moreover, we establish the following inequality for L_p-intersection bodies.

Theorem 4.2. *Let* $K, L \in \mathcal{S}_o^n$ *and* $p \leq -1$*. If, for every star body* Q *in* $\mathbf{R}^n$*,* $\widetilde{V}_p(K,Q) \leq \widetilde{V}_p(L,Q)$*, then*

$$V(I_p K) \geq V(I_p L), \tag{4.2}$$

with equality if and only if $I_p K = I_p L$.

We need the following lemma in order to prove Theorems 4.1 and 4.2.

Lemma 4.1. *If* $K, L \in \mathcal{S}_o^n$ *and* $p \leq -1$*, then*

$$\widetilde{V}_p(L, I_p K) = \widetilde{V}_p(K, I_p L).$$

Proof. From definitions (2.6) and (3.2), combined with Fubini's theorem, it follows that

$$\begin{aligned}
\widetilde{V}_p(L, I_pK) &= \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-p}\rho(I_pK,u)^p\,dS(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-p}\frac{1}{nc_{n-2,p}\omega_n}\\
&\qquad\cdot\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,v)^{n-p}\,dS(v)\,dS(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,v)^{n-p}\frac{1}{nc_{n-2,p}\omega_n}\\
&\qquad\cdot\int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,u)^{n-p}\,dS(u)\,dS(v)\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(K,v)^{n-p}\rho(I_pL,v)^p\,dS(v)\\
&= \widetilde{V}_p(K, I_pL). \qquad \square
\end{aligned}$$

Proof of Theorem 4.1. Let $p \le -1$. Since $I_pK \subseteq I_pL$, using definition (2.6), we have

$$\widetilde{V}_p(Q, I_pK) \ge \widetilde{V}_p(Q, I_pL),$$

for any $Q \in \mathcal{S}_o^n$. From Lemma 4.1, we get

$$\widetilde{V}_p(K, I_pQ) \ge \widetilde{V}_p(L, I_pQ), \tag{4.3}$$

with equality if and only if $K = L$. Since $K \in I_p^n$, taking $I_pQ = K$ in (4.3) and using equality (2.7) and inequality (2.8), we obtain

$$V(K) = \widetilde{V}_p(K,K) \ge \widetilde{V}_p(L,K) \ge V(L)^{(n-p)/n}V(K)^{p/n}. \tag{4.4}$$

Therefore,

$$V(K) \ge V(L).$$

According to the equality conditions of the L_p-Minkowski inequality (2.8) and inequality (4.3), we know that equality in (4.1) holds if and only if $K = L$. $\square$

Proof of Theorem 4.2. Let $p \leq -1$. Since $K, L \in \mathcal{S}_o^n$ and

$$\widetilde{V}_p(K,Q) \leq \widetilde{V}_p(L,Q),$$

for any $Q \in \mathcal{S}_o^n$. Taking $Q = I_pM$ for any $M \in \mathcal{S}_o^n$, we get

$$\widetilde{V}_p(K, I_pM) \leq \widetilde{V}_p(L, I_pM). \tag{4.5}$$

By Lemma 4.1,

$$\widetilde{V}_p(M, I_pK) \leq \widetilde{V}_p(M, I_pL). \tag{4.6}$$

Let $M = I_pL$; by (2.7) and (2.8),

$$V(I_pL) \geq \widetilde{V}_p(I_pL, I_pK) \geq V(I_pL)^{(n-p)/n} V(I_pK)^{p/n}; \tag{4.7}$$

therefore,

$$V(I_pK) \geq V(I_pL).$$

Because inequalities (4.5) and (4.6) are equivalent, but $\widetilde{V}_p(M, I_pK) = \widetilde{V}_p(M, I_pL)$ for any $M \in \mathcal{S}_o^n$ if and only if $I_pK = I_pL$, and equality in inequality (4.7) holds if and only if I_pK and I_pL are dilates. Therefore, equality in inequality (4.2) holds if and only if $I_pK = I_pL$.

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Department of Mathematics, Shanghai University of Electric Power, Shanghai, 200090, P.R. China
Email address: lufengred@163.com

Department of Mathematics, Jiangsu University, Jiangsu, 212013, P.R. China
Email address: maoweihong1029@sina.com