ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 1, 2010

STABILITY CLASSES OF SECOND-ORDER LINEAR RECURRENCES MODULO 2^k II

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ABSTRACT. We classify the 2^k-blocks of second-order recurrence sequences with parameter $b \equiv 5 \pmod 8$ and identify stability classes modulo 2.

1. Introduction. Let $w(a,b)$ denote the second-order recursive sequence (w_i) determined by integer parameters a and b, initial integer terms w_0 and w_1, and recursion

$$w_i = aw_{i-1} + bw_{i-2}. \tag{1.1}$$

For each integer m, let $(\overline{w}_i)$ denote the corresponding sequence of residues modulo m. The residue sequence $(\overline{w}_i)$ is periodic, and purely periodic when m is relatively prime to b. Despite their simple definition, such sequences remain a source of many interesting open questions, among them the determination of the period, restricted period, and residue frequency distribution.

Considerable progress has been made in understanding the frequency distributions of sequences $(\overline{w}_i)$ when the modulus m is a prime power, much of it motivated by pioneering work of Eliot Jacobson beginning with [**10**]. Suppose that $m = p^k$ is a power of a prime p. Let $\lambda_k = \lambda_w(p^k)$ be the (least) period of $(\overline{w}_i)$ and, for each residue d modulo p^k, $\nu_w(d,p^k)$ the number of times that the residue d appears in a single cycle of the recurrence $(\overline{w}_i)$. Let $\Omega_w(p^k)$ be the image of the frequency distribution function $\nu_w(d,p^k)$, i.e.,

$$\Omega_w(p^k) = \{\nu_w(d,p^k) \mid d \in \mathbf{Z}\}.$$

In [**10**], Jacobson observed that when $w(a,b)$ is the Fibonacci sequence, the sets $\Omega_w(2^k)$ are eventually constant as a function of k, and hence

2010 AMS *Mathematics subject classification.* Primary 11B39, 11B50, Secondary 11B37, 11K36.
Keywords and phrases. Lucas, Fibonacci, distribution, stability.
Received by the editors on December 21, 2006, and in revised form on August 27, 2007.
DOI:10.1216/RMJ-2010-40-1-85

computation of the residue distribution of the Fibonacci sequence for all powers of 2 requires only a finite computational procedure. A similar analysis applies to any sequence that is known to be *p-stable*, as defined below.

Definition 1.1. A sequence (w) is *stable modulo* p, or simply *p-stable*, if there is a positive integer N such that $\Omega_w(p^k) = \Omega_w(p^N)$ for all $k \geq N$.

There is considerable mathematical literature on the stability of second-order recurrence sequences. In a series of papers [**1**–**4**], Carlip and Jacobson examined the 2-stability of the generalized Fibonacci sequences $u(a,b)$, whose initial terms are $u_0 = 0$ and $u_1 = 1$. In [**11**], Morgan examined the 2-stability of the Lucas sequence $v(1,1)$, whose initial terms are $v_0 = 2$ and $v_1 = a = 1$, providing an important example of a sequence that fails to be 2-stable. Stability modulo odd primes p has been examined by Carroll, Jacobson, and Somer in [**9**], Carlip, Jacobson, and Somer in [**5**] and Carlip and Somer in [**6, 8**]. The last two of these papers lay the groundwork for a general theory of stability of second-order recurrence sequences, not limited to the generalized Fibonacci and Lucas sequences studied in the earlier papers. In particular, the treatise [**6**] provided a broad generalization to *blocks* of second-order recursive sequences.

In this paper, the second in a series of papers in which we apply the methods of [**1, 6**] to sequences modulo powers of 2, we examine the 2-stability of sequences satisfying (1.1) for which a is odd and $b \equiv 5 \pmod 8$. In particular, in Section 3, we describe the block structure of the family of regular second-order recurrences, $\mathcal{F}(a,b)$. In Section 4, we offer our main results, a characterization of the 2-stability of blocks in terms of a parameter t, the *transition degree*, introduced in Section 3.1. Finally, in Section 5, we examine the sequences for which the transition degree is not defined.

2. Background and definitions.

2.1. Regular sequences. Throughout this paper we are interested in 2-regular recurrence sequences.

Definition 2.1. A second-order recurrence $w(a,b)$ is said to be 2-*regular* if

$$\begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix} = w_0 w_2 - w_1^2 \not\equiv 0 \pmod{2}. \tag{2.1}$$

Equivalently, $w(a,b)$ is 2-regular if the initial state vectors (w_0, w_1) and (w_1, w_2) are linearly independent modulo 2.

Lemma 2.2 *If a and b are odd, then $w(a,b)$ fails to be 2-regular if and only if $2 \mid (w_0, w_1)$.*

Proof. This follows immediately from (1.1) and (2.1). □

Note that this characterization is true only for $p = 2$. See [**6**] for a discussion of the more complicated situation when p is odd.

Definition 2.3. Let $\mathcal{F}(a,b)$ denote the set of all 2-regular recurrences $w(a,b)$.

We single out two important classes of second-order recurrences, the generalized Fibonacci and Lucas sequences, also known as the Lucas Sequences of the First Kind (LSFK) and Lucas Sequences of the Second Kind (LSSK).

Definition 2.4. The generalized Fibonacci sequence $u = u(a,b)$ is the second-order recurrence that obeys (1.1) and has initial terms $u_0 = 0$ and $u_1 = 1$. The generalized Lucas sequence $v = v(a,b)$ is the second-order recurrence that obeys (1.1) and has initial terms $v_0 = 2$ and $v_1 = a$.

The generalized Fibonacci sequence $u(a,b)$ is always 2-regular, and the generalized Lucas sequence $v(a,b)$ is 2-regular when the parameter a is odd.

2.2. Periods and restricted periods.

Definition 2.5. Let $\lambda_w(2^k)$ denote the *period* of $w(a,b)$ modulo 2^k, that is, the least positive integer λ such that for all n,

$$w_{n+\lambda} \equiv w_n \pmod{2^k}.$$

Similarly, let $h_w(2^k)$ denote the *restricted period* of $w(a,b)$ modulo 2^k, that is, the least positive integer h such that for some integer M and for all n,

$$w_{n+h} \equiv M w_n \pmod{2^k}.$$

The integer $M = M_w(2^k)$, defined up to congruence modulo 2^k, is called the *multiplier* of $w(a,b)$ modulo 2^k.

It is well known that $h_w(2^k) \mid \lambda_w(2^k)$ and that $E_w(2^k) = \lambda_w(2^k)/h_w(2^k)$ is the multiplicative order in $(\mathbf{Z}/2^k\mathbf{Z})^*$ of the multiplier $M_w(2^k)$. Moreover, if $h = h_w(2^k)$ and $M = M_w(2^k)$, then for all n,

$$w_{n+ih} \equiv M^i w_n \pmod{2^k}. \tag{2.2}$$

Lemma 2.6. *For each $k \geq 1$, all recurrences $w(a,b) \in \mathcal{F}(a,b)$ have the same period modulo 2^k and all recurrences $w(a,b) \in \mathcal{F}(a,b)$ have the same restricted period modulo 2^k.*

Proof. This follows from the regularity by a simple argument using Cramer's rule. See, e.g., [**6**, page 695]. □

It is not unusual to find parameters like $\lambda_w(2^k)$ and $h_w(2^k)$ that depend only upon the defining parameters a and b and not on the initial terms of the sequence. We refer to such parameters as *global parameters* and think of them as parameters of the family $\mathcal{F}(a,b)$ as well as of the individual sequences.

The periods and restricted periods modulo 2^k, for $k \geq 1$, of the regular second-order recurrences for which a is odd and $b \equiv 5 \pmod 8$ are determined in Theorems 3.10 and 3.11 below.

3. The class a odd and $b \equiv 5 \pmod 8$.

3.1. The transition degree. The main goal of this section is to compute the period and restricted period of the sequence $u(a, b)$, under our general hypotheses that a is odd and $b \equiv 5 \pmod 8$. These are characterized in terms of a global parameter t, the *transition degree*, which is useful for describing properties of sequences in the family $\mathcal{F}(a, b)$ and was originally defined in [**1**].

Definition 3.1. The *transition degree*, t, is the unique nonnegative integer satisfying

$$2^{t+1} \parallel u_6. \tag{3.1}$$

Clearly, if $u_6 = 0$, then parameter t is undefined. In this case, we write $t = \infty$. The sequences for which $t = \infty$ are analyzed in Section 5.

While the transition degree t is defined using the generalized Fibonacci sequence $u(a, b)$, its value depends only upon the parameters a and b and is hence a global parameter of the family $\mathcal{F}(a, b)$.

In order to handle subtle differences in the values of the terms of $u(a, b)$ that depend upon the congruence class of a, Carlip and Jacobson introduced parameters θ and ξ in [**3**]. For reference, we reproduce the definitions we require.

Definition 3.2. For all $b \equiv 5 \pmod 8$, define

$$\theta = \begin{cases} 5 & \text{if } a \equiv 1 \pmod{16} \text{ or } a \equiv 15 \pmod{16}, \\ 3 & \text{if } a \equiv 3 \pmod{16} \text{ or } a \equiv 13 \pmod{16}, \\ 7 & \text{if } a \equiv 5 \pmod{16} \text{ or } a \equiv 11 \pmod{16}, \\ 1 & \text{if } a \equiv 7 \pmod{16} \text{ or } a \equiv 9 \pmod{16}. \end{cases} \tag{3.2}$$

For all $b \equiv 5 \pmod{16}$, define

$$\xi = \begin{cases} 0 & \text{if } a \equiv 1 \pmod 8 \text{ or } a \equiv 7 \pmod 8 \\ 2 & \text{if } a \equiv 3 \pmod 8 \text{ or } a \equiv 5 \pmod 8, \end{cases} \tag{3.3}$$

and for $b \equiv 13 \pmod{16}$, define

$$\xi = \begin{cases} 0 & \text{if } a \equiv 3 \pmod 8 \text{ or } a \equiv 5 \pmod 8 \\ 2 & \text{if } a \equiv 1 \pmod 8 \text{ or } a \equiv 7 \pmod 8. \end{cases} \tag{3.4}$$

For much of the remainder of the paper, we fix the following hypothesis.

Hypothesis 3.3. *Assume that* $w(a,b) \in \mathcal{F}(a,b)$*, that* a *is odd and* $b \equiv 5 \pmod 8$*, and that* t*,* θ *and* ξ *are given by* (3.1), (3.2), (3.3) *and* (3.4)*, respectively.*

Our computation of the periods and restricted periods of sequences satisfying Hypothesis 3.3 require several preliminary results. We begin with an extremely useful lemma, which we often refer to as the *Intertwining Lemma*.

Lemma 3.4 (Intertwining Lemma). *For any nonnegative integers* i *and* j*,*

$$w_{i+j} = bw_{i-1}u_j + w_i u_{j+1}.$$

Proof. The lemma follows easily by induction on j. □

Two consequences of the Intertwining Lemma and the defining recursion (1.1) are the useful relations

$$u_{2i+1} = b(u_i)^2 + (u_{i+1})^2 \quad \text{and} \tag{3.5}$$

$$u_{2i} = 2u_i u_{i+1} - a(u_i)^2. \tag{3.6}$$

The Intertwining Lemma can also be used to show that $t \geq 3$ when our standard hypothesis holds.

Theorem 3.5. *If* $w(a,b)$ *satisfies Hypothesis* 3.3*, then* $t \geq 3$.

Proof. By the Intertwining Lemma,

$$\begin{aligned} u_6 &= bu_2u_3 + u_3u_4 = u_3(bu_2 + u_4) \\ &= u_3(bu_2 + au_3 + bu_2) = au_3(a^2 + 3b). \end{aligned}$$

But $u_3 = a^2 + b \equiv 6 \pmod 8$, so $2 \parallel u_3$, while $a^2 + 3b \equiv 1 + 15 \equiv 0 \pmod 8$, so $8 \mid a^2 + 3b$. It follows that $2^4 \mid u_6$, and hence $t \geq 3$. □

Before we turn to the computation of the period and restricted period, we present several additional lemmas that describe properties of $u(a,b)$.

Lemma 3.6. *Assume Hypothesis* 3.3. *Then, for all* $k \geq 4$,

$$u_{3\cdot 2^{k-3}} \equiv \xi 2^{k-1} \pmod{2^{k+1}}, \tag{3.7}$$

$$u_{3\cdot 2^{k-2}+1} \equiv (1+2^{k-1})^{\theta} \pmod{2^{k+1}} \tag{3.8}$$

$$\equiv 1+\theta 2^{k-1} \pmod{2^{k+1}}. \tag{3.9}$$

Proof. We prove (3.7) and (3.8) simultaneously by induction on k. The induction is begun by explicit computation for $k=4$ and $k=5$. Note that this computation is finite because there are only finitely many distinct two-term recurrence sequences modulo 2^5 and 2^6. We leave it to the reader to write the elementary computer programs needed to verify this first step of the induction.

Now assume that (3.7) and (3.8) hold for some $k \geq 5$ and also for $k-1$.

By (3.6), the induction hypothesis, and the binomial theorem,

$$\begin{aligned} u_{3\cdot 2^{k-2}} &= u_{2(3\cdot 2^{k-3})} = 2(u_{3\cdot 2^{k-3}} u_{3\cdot 2^{k-3}+1}) - a(u_{3\cdot 2^{k-3}})^2 \\ &\equiv 2\xi 2^{k-1}(1+2^{k-2})^{\theta} - a(\xi 2^{k-1})^2 \pmod{2^{k+2}} \\ &\equiv \xi 2^{k}(1+\theta 2^{k-2}) \pmod{2^{k+2}} \\ &\equiv \xi 2^{k} \pmod{2^{k+2}}, \end{aligned}$$

as desired. This proves (3.7).

Next, by (3.7), (3.5), the induction hypothesis, and the binomial theorem,

$$\begin{aligned} u_{3\cdot 2^{k-1}+1} &= u_{2(3\cdot 2^{k-2})+1} = b(u_{3\cdot 2^{k-2}})^2 + (u_{3\cdot 2^{k-2}+1})^2 \\ &\equiv b(\xi 2^{k})^2 + ((1+2^{k-1})^{\theta})^2 \pmod{2^{k+2}} \\ &\equiv (1+2\cdot 2^{k-1})^{\theta} \pmod{2^{k+2}} \\ &\equiv (1+2^{k})^{\theta} \pmod{2^{k+2}}, \end{aligned}$$

as desired. This proves (3.8), and (3.9) follows by the binomial formula and the observation that $2k-2 \geq k+1$ when $k \geq 3$. □

Lemma 3.7. *Assume Hypothesis* 3.3. *Then, for all* $n \geq 0$ *and all* $k \geq 2$,

$$u_{n+3\cdot 2^{k-1}} \equiv u_n(1+2^k)^\theta \pmod{2^{k+1}}.$$

Proof. Consider first the case that $k \geq 3$. By the Intertwining Lemma and Lemma 3.6,

$$\begin{aligned} u_{n+3\cdot 2^{k-1}} &= bu_{n-1}u_{3\cdot 2^{k-1}} + u_n u_{3\cdot 2^{k-1}+1} \\ &\equiv bu_{n-1}\xi 2^{k+1} + u_n(1+2^k)^\theta \pmod{2^{k+1}} \\ &\equiv u_n(1+2^k)^\theta \pmod{2^{k+1}}. \end{aligned}$$

Unfortunately, we cannot apply Lemma 3.6 when $k = 2$, so we treat this case separately. We know that $u_3 = a^2 + b \equiv 6 \pmod 8$ and Theorem 3.5 implies that $u_6 \equiv 0 \pmod 8$. Since u_4 is odd, (3.5) shows that $u_7 = bu_3^2 + u_4^2 \equiv 5 \pmod 8$ and, since θ is odd, $5^\theta \equiv 5 \pmod 8$. Finally, by the Intertwining Lemma,

$$u_{n+3\cdot 2^{2-1}} = u_{n+6} = bu_{n-1}u_6 + u_n u_7 \equiv u_n(1+2^2)^\theta \pmod{2^{2+1}},$$

and hence the lemma is true when $k = 2$. □

Lemma 3.8. *Assume Hypothesis* 3.3. *If* $k \geq 1$ *is a nonnegative integer, then* $2^{t+k} \parallel u_{3\cdot 2^k}$.

Proof. This follows immediately from Lemma 3.5 of [**3**]. □

Lemma 3.9. *Assume Hypothesis* 3.3. *Then* $u_\ell \equiv 0 \pmod 2$ *if and only if* $3 \mid \ell$. *Moreover, for all positive integers* m, *if* $3 \mid \ell$ *and* $2^m \parallel \ell$, *then* $2^{t+m} \parallel u_\ell$.

Proof. The first assertion follows from the observation that a complete cycle of $u(a, b)$ modulo 2 is given by $0, 1, 1$.

To prove the second assertion, write $\ell = 3 \cdot 2^m \cdot d$, with d odd, and assume that d has base two expansion

$$d = d_0 + d_1 2 + d_2 2^2 + \cdots + d_n 2^n,$$

with $d_0 \neq 0$ and $d_n \neq 0$. Proceed by induction on n. If $n = 0$, then the result is simply Lemma 3.8. Now suppose that $n \geq 1$. Then the Intertwining Lemma yields

$$\begin{aligned} u_\ell = u_{3\cdot 2^m \cdot d} &= u_{3\cdot 2^m \cdot (d_0 + d_1 2 + d_2 2^2 + \cdots + d_n 2^n)} \\ &= u_{3\cdot 2^m + 3\cdot 2^{m+1}\cdot(d_1 + d_2 2 + \cdots + d_n 2^{n-1})} \\ &= u_{3\cdot 2^m + 1} u_{3\cdot 2^{m+1}\cdot(d_1 + d_2 2 + \cdots + d_n 2^{n-1})} \\ &\quad + b u_{3\cdot 2^m} u_{3\cdot 2^{m+1}\cdot(d_1 + d_2 2 + \cdots + d_n 2^{n-1}) - 1}. \end{aligned}$$

The induction hypothesis and the first assertion of the lemma imply that

$$2^{t+m+1} \mid u_{3\cdot 2^m + 1} u_{3\cdot 2^{m+1}\cdot(d_1 + \cdots + d_n 2^{n-1})},$$

while

$$2^{t+m} \parallel b u_{3\cdot 2^m} u_{3\cdot 2^{m+1}\cdot(d_1 + \cdots + d_n 2^{n-1}) - 1}.$$

It follows that $2^{t+m} \parallel u_\ell$, as desired. □

With these facts in hand, we turn to the question of computing the period and restricted period of sequences satisfying Hypothesis 3.3. We begin with the period.

Theorem 3.10. *Suppose that* $w(a, b)$ *satisfies Hypothesis* 3.3. *Then* $\lambda_w(2^k) = 3 \cdot 2^{k-1}$ *when* $k \geq 1$.

Proof. By Lemma 2.6, it suffices to prove the theorem for the Lucas sequence $u(a, b)$. It is a simple matter to computationally verify the period modulo 2^k when $k = 1$, 2, and 3. For $k > 3$, we proceed by induction.

By way of induction, assume that $\lambda_w(2^k) = 3 \cdot 2^{k-1}$ for some $k \geq 3$. Clearly, $\lambda_w(2^k)$ divides $\lambda_w(2^{k+1})$, so

$$3 \cdot 2^{k-1} \mid \lambda_w(2^{k+1}). \tag{3.10}$$

By Lemma 3.6, $u_{3\cdot 2^k} \equiv \xi 2^{k+2} \pmod{2^{k+4}}$, and therefore $u_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$. Similarly, Lemma 3.6 implies that $u_{3\cdot 2^k + 1} \equiv (1 + 2^{k+1})^\theta \pmod{2^{k+3}}$, and hence, $u_{3\cdot 2^k+1} \equiv 1 \pmod{2^{k+1}}$. It follows that

$$\lambda_w(2^{k+1}) \mid 3 \cdot 2^k. \tag{3.11}$$

On the other hand, $u_{3\cdot 2^{k-1}+1} \equiv 1 + \theta 2^k \pmod{2^{k+2}}$, so $u_{3\cdot 2^{k-1}+1} \equiv 1 + \theta 2^k \pmod{2^{k+1}}$. Since θ is odd, $u_{3\cdot 2^{k-1}+1} \equiv 1 + 2^k \pmod{2^{k+1}}$. Therefore $\lambda_w(2^{k+1}) \neq 3 \cdot 2^{k-1}$.

We now conclude from (3.10) and (3.11) that $\lambda_w(2^{k+1}) = 3 \cdot 2^k$, and the induction is complete. □

Next we compute the restricted period. Notice that the restricted period $h_w(2^k)$ is constant for k greater than one and less than or equal to the transition degree, but doubles for each increment of k when k has passed the transition degree. This difference in behavior when k surpasses t is one motivation for the name *transition degree.*

Theorem 3.11. *Assume Hypothesis* 3.3. *Then $h_w(2^k) = 3 \cdot 2^{k-t}$ when $k > t$, $\lambda_w(2^k) = 6$ when $1 < k \leq t$, and $h_w(2^k) = 3$ when $k = 1$.*

Proof. By Lemma 2.6, it suffices to prove the theorem for $u(a, b)$. Moreover, since $u_0 = 0$, the restricted period of $u(a, b)$ modulo 2^k is simply the rank of appearance of 2^k, i.e., the least index $m > 0$ such that $2^k \mid u_m$.

Since $2 \parallel u_3$ and $2^{t+1} \parallel u_6$, the first assertion of Lemma 3.9 implies that $h_w(2^k) = 3$ when $k = 1$ and $h_w(2^k) = 6$ when $1 < k \leq t$. Moreover, if $k > t$, Lemma 3.9 implies that the first term of $u(a, b)$ that is divisible by 2^k is $u_{3\cdot 2^{k-t}}$, as desired. □

3.2. Blocks. One of the primary tools used in [**6**] to study second-order recurrence sequences is an equivalence relation used to partition the set of recurrence sequences into equivalence classes known as *blocks.* In this section we examine the ramifications of this theory for sequences modulo powers of 2.

Definition 3.12. For each $k \geq 1$, we define a relation on the set of second-order recurrences with fixed parameters a and b by $w(a, b) \sim_k w'(a, b)$ if $w'(a, b)$ is an odd (hence invertible modulo 2) multiple of a translation of $w(a, b)$ modulo 2^k, that is, $w(a, b) \sim_k w'(a, b)$ if and only if there exist integers m and c such that $2 \nmid c$ and, for all nonnegative integers n,

$$w'_n \equiv cw_{n+m} \pmod{2^k}. \tag{3.12}$$

Since c is odd, it is easy to see that $\sim_k$ defines an equivalence relation on the set of two-term recurrence relations with fixed parameters a and b. The equivalence classes of $\sim_k$ are called the 2^k-*blocks*, or simply the *blocks* when k is evident.

It is also clear that regularity is preserved by $\sim_k$, i.e., if $w(a,b) \sim_k w'(a,b)$, then $w(a,b)$ is 2-regular if and only if $w'(a,b)$ is 2-regular. Thus, the equivalence relation $\sim_k$ restricts to $\mathcal{F}(a,b)$ and, without risk of confusion, we can refer to *regular blocks*. Furthermore, if $w(a,b)$ and $w'(a,b)$ satisfy (3.12), and d is any integer, then $\nu_w(d, 2^k) = \nu_{w'}(cd, 2^k)$, and therefore, for every n,

$$\Omega_w(2^k) = \Omega_{w'}(2^k). \tag{3.13}$$

In the following theorem we determine the number of regular 2^k-blocks in $\mathcal{F}(a,b)$. Once again, the behavior changes when k surpasses the transition degree t.

Theorem 3.13. *Assume Hypothesis* 3.3. *Then for all $k > t$, there are exactly 2^{t-1} regular 2^k-blocks and each 2^k-block contains $3 \cdot 2^{2k-t-1}$ distinct sequences modulo 2^k.*

Proof. Since a second-order recurrence $w(a,b)$ is determined by its two initial terms, there are $(2^k)^2$ distinct recurrences modulo 2^k. Of these, exactly $(2^{k-1})^2$ satisfy $2 \mid (w_0, w_1)$ and, by Lemma 2.2, are not 2-regular. It follows that $\mathcal{F}(a,b)$ contains $(2^k)^2 - (2^{k-1})^2 = 3 \cdot 2^{2(k-1)}$ distinct sequences modulo 2^k.

If $w(a,b)$ is a 2-regular recurrence, then, modulo 2^k, $w(a,b)$ has 2^{k-1} distinct odd multiples, each of which has $h(2^k)$ distinct translations that are not nontrivial multiples of each other. It follows that the 2^k-block containing $w(a,b)$ has exactly $2^{k-1}h(2^k)$ distinct sequences modulo 2^k. Moreover, since, by Theorem 3.11, $h(2^k) = 3 \cdot 2^{k-t}$, a regular 2^k-block contains $3 \cdot 2^{2k-t-1}$ recurrences.

We now see that the number of regular 2^k-blocks in $\mathcal{F}(a,b)$ is

$$T_k = \frac{3 \cdot 2^{2(k-1)}}{3 \cdot 2^{2k-t-1}} = 2^{t-1}. \qquad \square$$

Remark. By the argument of the previous theorem, when $1 < k \le t$, there are 2^{k-2} 2^k-blocks, each containing $3 \cdot 2^k$ sequences, and, when $k = 1$, there is only one block, which contains $2^{k-1} \cdot 3 = 3$ sequences. It follows that, as k increases from 2 to $t+1$, the number of 2^k-blocks doubles at each step, but the number of blocks remains constant once k passes the transition degree.

Corollary 3.14. *For all $k > t$, $w(a, b)$ and $w'(a, b)$ lie in the same 2^k-block if and only if they lie in the same 2^{t+1}-block.*

Proof. If $w(a, b)$ and $w'(a, b)$ lie in the same 2^k-block, it is obvious that they lie in the same 2^{t+1}-block. The converse follows from Theorem 3.13. □

Corollary 3.15. *If $w(a, b)$ and $w'(a, b)$ lie in the same 2^{t+1}-block, then $w(a, b)$ is 2-stable if and only if $w'(a, b)$ is 2-stable.*

Proof. This follows from Corollary 3.14 and (3.13). □

In view of Corollary 3.15, we may, with no ambiguity, refer to *stable blocks* of sequences.

3.3. Principal divisors. One of the consequences of Lemma 3.9 is that we can identify the precise power of two that divides each element of $u(a, b)$. Since every sequence $w(a, b)$ in the same 2^k-block as $u(a, b)$ is a translation of an odd multiple of $u(a, b)$, we can easily determine the exact power of two that divides each element of $w(a, b)$. In particular, for every $k > t$, there is an element of $u(a, b)$, and hence of $w(a, b)$, that is exactly divisible by 2^k. Moreover, if a sequence $w(a, b)$ contains a term that is divisible by 2^{t+1}, Corollary 3.14 tells us that $w(a, b)$ lies in the same 2^k-block as $u(a, b)$ for all $k > t$, and hence $w(a, b)$ contains terms divisible by 2^k for all $k > t$.

For sequences $w(a, b)$ that lie outside of the 2^{t+1}-block containing $u(a, b)$, the situation is strikingly different. In particular, no element of $w(a, b)$ can be divisible by 2^{t+1}, since otherwise $w(a, b)$ would be a multiple of a translation of $u(a, b)$ modulo 2^{t+1}. It follows that sequences in $\mathcal{F}(a,b)$ may be characterized by the largest power of two dividing any element in the sequence. This motivates the following definitions.

Definition 3.16. Let m be a nonnegative integer and $w(a,b) \in \mathcal{F}(a,b)$. We say that m is a *divisor* of the sequence $w(a,b)$ if $m \mid w_n$ for some n. We say that 2^k is the *principal* 2-*divisor*, or simply the *principal divisor* of $w(a,b)$ when the prime 2 is evident, if 2^k is the largest power of 2 dividing any element of $w(a,b)$. If $w(a,b)$ contains terms divisible by 2^k for unbounded values of k, for example when $w(a,b)$ contains a term that is identically zero, as $u(a,b)$ does, we say that the principal divisor does not exist or, informally, that the principal divisor is infinite.

It is self-evident that any two sequences in the same block have the same principal divisor. It is often convenient to choose a block representative $w(a,b)$ with the property that w_0 is divisible by the principal divisor. Clearly, this is always possible when a principal divisor exists, and we refer to such a representative as a *principal representative* of the block.

Armed with these definitions we can now describe precisely the powers of two that divide each term in a sequence $w(a,b)$ lying outside the 2^{t+1}-block of $u(a,b)$.

Lemma 3.17. *Assume Hypothesis* 3.3*, and suppose that* $k > t$*. If* $w(a,b)$ *is a principal representative of a* 2^k*-block that does not contain* $u(a,b)$*, then* $w(a,b)$ *has principal divisor* 2^m *for some* m *satisfying* $2 \le m \le t$*. Moreover,* $2^m \parallel w_{6n}$ *and* $2 \parallel w_{6n+3}$ *for all nonnegative integers* n*.*

Proof. That $m \le t$ follows from the fact that $w(a,b)$ does not lie in the same block as $u(a,b)$. Moreover, since a and b are odd, two consecutive odd terms of $w(a,b)$ must be followed by an even term, so $m > 0$.

To show $m \ge 2$, suppose instead that $m = 1$, i.e., $2 \parallel w_0$. Two applications of (1.1) show that

$$w_3 = a(aw_1 + bw_0) + bw_1 = (a^2 + b)w_1 + (ab)w_0. \tag{3.14}$$

Since $a^2 + b \equiv 6 \pmod 8$ and w_1 is odd, we know that $2 \parallel (a^2 + b)w_1$. By our hypothesis and the fact that ab is odd, we also know that $2 \parallel (ab)w_0$. But then $4 \mid w_3$, contrary to the maximality of m.

To show that $2^m \parallel w_{6n}$ for all $n \geq 0$, proceed by induction on n. For $n = 0$, the result follows from the definition of a principal representative of a block. By way of induction, assume that $2^m \parallel w_{6n}$ for some $n \geq 0$. It is easy to see that a complete cycle of $w(a, b)$ modulo 2 is given by $0, 1, 1$, and therefore that w_{6n+1} is odd. By the Intertwining Lemma,

$$w_{6n+6} = w_{(6n+1)+5} = bw_{6n}u_5 + w_{6n+1}u_6.$$

Since bu_5 and w_{6n+1} are odd, $2^{t+1} \parallel w_{6n+1}u_6$, while $2^m \parallel bw_{6n}u_5$. Since $m < t + 1$, we conclude that $2^m \parallel w_{6n+6}$.

Finally, we prove that $2 \parallel w_{6n+3}$ for all $n \geq 0$. First note that (3.14), together with the fact that $m > 1$, implies that $2 \parallel w_3$. Now, for each $n \geq 0$, the Intertwining Lemma yields

$$w_{6n+3} = w_{(6n+1)+2} = bw_{6n}u_2 + w_{6n+1}u_3.$$

Since bu_2 is odd, the previous paragraph shows that $2^m \parallel bw_{6n}u_2$. On the other hand, w_{6n+1} is odd, so $2 \parallel w_{6n+1}u_3$. Since $m > 1$, it follows that $2 \parallel w_{6n+3}$. □

While sequences in the same block always have the same principal divisor, the converse is not true.

Theorem 3.18. *If $k > t$, then there is a unique 2^k-block having principal divisor infinity, namely the block containing $u(a, b)$. If $2 \leq m \leq t$, then there are exactly 2^{t-m} distinct 2^k-blocks with principal divisor 2^m.*

Proof. First note that if a sequence $w(a, b)$ has infinite principal divisor, then some term of $w(a, b)$ is congruent to zero modulo 2^k, and it is clear that $w(a, b)$ is a multiple of a translation of $u(a, b)$. The first statement of the theorem follows immediately.

Now suppose that $m \leq t$. Since, by Theorem 3.13, each 2^k-block contains $3 \cdot 2^{2k-t-1}$ sequences, we need only count the number of distinct sequences modulo 2^k that have principal divisor 2^m. Since any two consecutive terms determine the entire sequence modulo 2^k and, by Lemma 3.17, one of the first six terms must be exactly divisible by 2^m and followed by an odd term, the number of sequences modulo 2^k with principal divisor 2^m is $6 \cdot 2^{k-m-1} \cdot 2^{k-1} = 3 \cdot 2^{2k-m-1}$. It now follows that the number of 2^k-blocks having principal divisor 2^m is

$$\frac{3 \cdot 2^{2k-m-1}}{3 \cdot 2^{2k-t-1}} = 2^{t-m}. \qquad \square$$

Note that, since $1+1+2+4+\cdots+2^{t-2} = 2^{t-1}$, Theorem 3.18 yields an alternate proof of Theorem 3.13.

4. Stability.

4.1. Stability classes. In this section we begin our study of the stability of sequences for which a is odd and $b \equiv 5 \pmod 8$ and state our three main results, which characterize the stability of each block of $\mathcal{F}(a,b)$. By Corollary 3.14, it is sufficient to examine the 2^{t+1}-blocks. The first theorem concerns the block containing the generalized Fibonacci sequence $u(a,b)$, the block with infinite principal divisor, and is proven in this section. The next two theorems examine in turn the stability of blocks with principal divisors $m \neq 2^{t-1}$ and $m = 2^{t-1}$, and are proven in Sections 4.2 and 4.3, respectively.

In our first result, we observe that Theorem 1.2 and Corollary 1.5 (a) of [**3**] apply to each sequence in the 2^{t+1}-block of the generalized Fibonacci sequence $u(a,b)$, and therefore this block is 2-stable.

Theorem 4.1. *Assume Hypothesis* 3.3. *Then the* 2^{t+1}*-block of* $u(a,b)$ *is* 2*-stable.*

Proof. By Corollary 3.15, if $w(a,b)$ lies in the same 2^{t+1}-block as $u(a,b)$, then $w(a,b)$ is 2-stable if and only if $u(a,b)$ is 2-stable. By Corollary 1.5 (a) of [**3**], $u(a,b)$ is, indeed, 2-stable. $\square$

The blocks that do not contain $u(a,b)$ require a little more work. By Lemma 3.17, if $k > t$, the 2^k-block of a sequence $w(a,b)$ lying outside the block of $u(a,b)$ has principal divisor 2^m with $2 \le m \le t$. If $m \ne t-1$, we obtain the following theorem, which is proven in Section 4.2.

Theorem 4.2. *Assume Hypothesis* 3.3, *and suppose that $k > t$. Then each sequence $w(a,b) \in \mathcal{F}(a,b)$ that lies in a 2^k-block having principal* 2*-divisor 2^m, with $m \ne t-1$, is* 2*-stable.*

When $m = t-1$, the stability classification is more complicated. By Lemma 3.18 there are exactly two 2^{t+1}-blocks having principal divisor 2^{t-1}. The following definition provides canonical representatives of these two blocks.

Definition 4.3. Let $x(a,b)$ and $y(a,b)$ denote the two sequences in $\mathcal{F}(a,b)$ that satisfy $(x_0, x_1) = (2^{t-1}, 1)$ and $(y_0, y_1) = (2^{t-1}, 3)$, respectively.

Theorem 4.4. *Assume Hypothesis* 3.3, *and suppose that $k > t$. Then there are exactly two 2^k-blocks with principal divisor 2^{t-1}, each containing $3 \cdot 2^{2k-t-1}$ distinct sequences modulo 2^k. The sequences $x(a,b)$ and $y(a,b)$ are principal representatives of these two blocks.*

Proof. The first assertion of the theorem follows immediately from Theorems 3.13 and 3.18, and it suffices to show that the sequences $x(a,b)$ and $y(a,b)$ lie in different blocks with principal divisor 2^{t-1}.

Since 2^{t-1} divides both x_0 and y_0, and $2^{t-1} > 2$, Lemma 3.17 implies that both $x(a,b)$ and $y(a,b)$ have principal divisor 2^{t-1}.

Suppose now that $x(a,b)$ and $y(a,b)$ lie in the same 2^k-block. Then, by Corollary 3.14, $x(a,b)$ and $y(a,b)$ lie in the same 2^{t+1}-block. By Theorem 3.11, $x(a,b)$ has restricted period 6 modulo 2^{t+1}. Since $y(a,b)$ is in the same 2^{t+1}-block as $x(a,b)$, there must be an odd integer α with the property that $\alpha(2^{t-1}, 1) \equiv (2^{t-1}, 3) \pmod{2^{t+1}}$. But then $\alpha 2^{t-1} - 2^{t-1} \equiv 0 \pmod{2^{t+1}}$, so $\alpha - 1 \equiv 0 \pmod{4}$, while $\alpha - 3 \equiv 0 \pmod{2^{t+1}}$, so $\alpha - 3 \equiv 0 \pmod{4}$. Subtracting these congruences yields $2 \equiv 0 \pmod{4}$, a contradiction, and it follows that $x(a,b)$ and $y(a,b)$ lie in different 2^k-blocks. $\square$

Finally, the following theorem, which is proven in Section 4.3, describes the stability of the blocks with principal divisor 2^{t-1}.

Theorem 4.5. *Assume Hypothesis* 3.3*, and suppose that* $k > t$*. Then one of the two* 2^k*-blocks with principal divisor* 2^{t-1} *is stable. In particular, if the congruence class of* (b,a) *modulo* 16 *is in the set*

$$\{(5,3),(5,5),(5,7),(5,9),(13,3),(13,5),(13,9),(13,13),(13,15)\},$$

then the block that contains the sequence $x(a,b)$ *is* 2*-stable, and if the congruence class of* (b,a) *modulo* 16 *is in the set*

$$\{(5,1),(5,11),(5,13),(5,15),(13,1),(13,7),(13,11)\},$$

then the block that contains the sequence $y(a,b)$ *is* 2*-stable.*

4.2. Principal divisors not equal to 2^{t-1}. In this section we prove Theorem 4.2, the stability classification of blocks with principal divisor 2^m for m finite and not equal to $t-1$. We require several lemmas that repeatedly apply the Intertwining Lemma to obtain congruences that are used in the stability analysis. First we augment our main hypothesis.

Hypothesis 4.6. *Assume Hypothesis* 3.3*. Let* $w(a,b) \in \mathcal{F}(a,b)$ *be a principal representative of a block that does not contain the generalized Fibonacci sequence* $u(a,b)$ *and has principal divisor* 2^m *with* $1 \le m \le t$*.*

Lemma 4.7. *Assume Hypothesis* 4.6*, and suppose that* $k > t$*. Then*

(a) $w_{n+3\cdot 2^{k-1}} \equiv w_n + 2^k \pmod{2^{k+1}}$ *if* $3 \nmid n$*,*

(b) $w_{n+3\cdot 2^{k-2}} \equiv w_n + 2^k \pmod{2^{k+1}}$ *if* $n \equiv 3 \pmod 6$*,*

(c) $w_{n+3\cdot 2^{k-m-1}} \equiv w_n + 2^k \pmod{2^{k+1}}$ *if* $n \equiv 0 \pmod 6$ *and* $m < t-1$*, and*

(d) $w_{n+3\cdot 2^{k-t}} \equiv w_n + 2^k \pmod{2^{k+1}}$ *if* $n \equiv 0 \pmod 6$ *and* $m = t$*.*

Proof. (a) Since $w(a,b)$ is a principal representative of its block and $3 \nmid n$, we know that w_n is odd. By Lemma 3.8, $2^{t+k-1} \parallel u_{3\cdot 2^{k-1}}$ and, since $t \geq 3$, $u_{3\cdot 2^{k-1}} \equiv 0 \pmod{2^{k+1}}$. By the Intertwining Lemma, Lemma 3.7, and the binomial equation,

$$\begin{aligned} w_{n+3\cdot 2^{k-1}} &= bw_{n-1}u_{3\cdot 2^{k-1}} + w_n u_{3\cdot 2^{k-1}+1} \\ &\equiv 0 + w_n(1+2^k)^\theta \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(b) Since $w(a,b)$ is a principal representative of its block and $n \equiv 3 \pmod 6$, Lemma 3.17 implies that $2 \parallel w_n$, while w_{n-1} is odd. By Lemma 3.8, $2^{t+k-2} \parallel u_{3\cdot 2^{k-2}}$ and, since $t \geq 3$, $u_{3\cdot 2^{k-2}} \equiv 0 \pmod{2^{k+1}}$. By the Intertwining Lemma, Lemma 3.7, and the binomial equation,

$$\begin{aligned} w_{n+3\cdot 2^{k-2}} &= bw_{n-1}u_{3\cdot 2^{k-2}} + w_n u_{3\cdot 2^{k-2}+1} \\ &\equiv 0 + w_n(1+2^{k-1})^\theta \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(c) Since $w(a,b)$ is a principal representative of its block and $n \equiv 0 \pmod 6$, Lemma 3.17 implies that $2^m \parallel w_n$, while w_{n-1} is odd. By Lemma 3.8, $2^{t+k-m-1} \parallel u_{3\cdot 2^{k-m-1}}$ and, since $m+1 < t$, $u_{3\cdot 2^{k-m-1}} \equiv 0 \pmod{2^{k+1}}$. By the Intertwining Lemma, Lemma 3.7, and the binomial equation,

$$\begin{aligned} w_{n+3\cdot 2^{k-m-1}} &= bw_{n-1}u_{3\cdot 2^{k-m-1}} + w_n u_{3\cdot 2^{k-m-1}+1} \\ &\equiv 0 + w_n(1+2^{k-m})^\theta \pmod{2^{k+1}} \\ &\equiv w_n + 2^k \pmod{2^{k+1}}. \end{aligned}$$

(d) Since $w(a,b)$ is a principal representative of its block, $n \equiv 0 \pmod 6$, and $m = t$, Lemma 3.17 implies that $2^t \parallel w_n$, while w_{n-1} is odd. By Lemma 3.8, $2^k \parallel u_{3\cdot 2^{k-t}}$. By the Intertwining Lemma, Lemma 3.7, and the binomial equation,

$$\begin{aligned} w_{n+3\cdot 2^{k-t}} &= bw_{n-1}u_{3\cdot 2^{k-t}} + w_n u_{3\cdot 2^{k-t}+1} \\ &\equiv 2^k + w_n(1+2^{k-t+1})^\theta \pmod{2^{k+1}} \\ &\equiv 2^k + w_n \pmod{2^{k+1}}. \qquad \square \end{aligned}$$

We can now prove Theorem 4.2. The method used is the same as that used in [**1, 7**].

Proof of Theorem 4.2. Assume Hypothesis 4.6, and suppose that $m \neq t-1$. By Lemma 4.7 (a) and (b), and either Lemma 4.7 (c) or (d), if $k > t$, then

$$\begin{aligned} w_{n+3\cdot 2^{k-1}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{if } 3 \nmid n \\ w_{n+3\cdot 2^{k-2}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{if } n \equiv 3 \pmod 6 \\ w_{n+3\cdot 2^{k-m-1}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{if } n \equiv 0 \pmod 6 \text{ and } m < t-1 \\ w_{n+3\cdot 2^{k-t}} &\equiv w_n + 2^k \pmod{2^{k+1}} && \text{if } n \equiv 0 \pmod 6 \text{ and } m = t. \end{aligned}$$

It follows that each instance of a residue d in one cycle of $w(a,b)$ modulo 2^{k+1} can be paired with a unique instance of the residue $d + 2^k$ in the same cycle. Therefore, $\nu_w(d, 2^{k+1}) = \nu_w(d + 2^k, 2^{k+1})$. On the other hand, by Theorem 3.10, the period of $w(a,b)$ modulo 2^{k+1} is double the period modulo 2^k. It follows that $\nu_w(d, 2^k) = \nu_w(d, 2^{k+1}) = \nu_w(d + 2^k, 2^{k+1})$. Finally, we conclude that no new residue frequencies occur modulo 2^{k+1}, that is $\Omega_w(2^k) = \Omega_w(2^{k+1})$. Since this is true for all sufficiently large k, it follows that $w(a,b)$ is 2-stable. □

4.3. The principal divisor 2^{t-1}. Finally, in this section, we examine the stability of sequences $w(a,b) \in \mathcal{F}(a,b)$ that have principal divisor 2^{t-1} and establish Theorem 4.5. Since, by Theorem 4.4, there are exactly two blocks with principal divisor 2^{t-1}, and these have principal representatives $x(a,b)$ and $y(a,b)$ given in Definition 4.3, we can restrict our attention to these two sequences.

To begin, we observe that the argument of Theorem 4.2 does not work for blocks with principal divisor $t-1$ simply because we have provided no analogue of Lemma 4.7(c) and (d) in this case. It is easy to see that the stability of these blocks can be reduced to a study of the subsequences x_{6n} and y_{6n}. By Lemma 3.17, each of the terms of these two subsequences is exactly divisible by 2^{t-1}, so we can study the sequences $x_{6n}/2^{t-1}$ and $y_{6n}/2^{t-1}$. For convenience, we fix the following notation:

$$x'_n = x_{6n}, \quad \widehat{x}_n = \frac{x'_n}{2^{t-1}}, \quad y'_n = y_{6n}, \quad \widehat{y}_n = \frac{y'_n}{2^{t-1}}.$$

We observe that x', $\widehat{x}$, y', and $\widehat{y}$ are second-order recursive sequences.

Theorem 4.8. *If z is any of the sequences x', $\widehat{x}$, y' or $\widehat{y}$, then z is a second-order recursive sequence that satisfies the recursion*

$$z_i = a'z_{i-1} + b'z_{i-2}, \tag{4.1}$$

where $a' = v_6$, for $v = v(a,b) \in \mathcal{F}(a,b)$, and $b' = -(-b)^6$.

Proof. That z satisfies (4.1) follows immediately from Lemma 2.10 of [**6**]. □

Lemma 4.9. *Suppose that a is odd. If $b \equiv 5 \pmod 8$, then $a' \equiv 10 \pmod{16}$ and $b' \equiv 7 \pmod{16}$. Moreover, if $b \equiv 5 \pmod{16}$, then $b' \equiv 23 \pmod{32}$ and $a' \equiv 26 \pmod{32}$, while if $b \equiv 13 \pmod{16}$, then $b' \equiv 7 \pmod{32}$ and $a' \equiv 10 \pmod{32}$.*

Proof. By definition, $v_6 = a^6 + 6a^4b + 9a^2b^2 + 2b^3$, and it is easy to compute that this expression is congruent to 26 modulo 32 when $b \in \{5, 21\}$ and $a \in \{2t+1 \mid 0 \le t \le 15\}$. Now suppose that $b = 5+16k$. Then $b^2 = 25 + 160k + 256k^2 \equiv 25 \pmod{32}$ and $b^4 \equiv 625 \equiv 17 \pmod{32}$. Therefore, $b^6 \equiv 425 \equiv 9 \pmod{32}$ and $b' = -(-b)^6 \equiv 23 \pmod{32}$.

Similarly, it is easy to compute that v_6 is congruent to 10 modulo 32 when $b \in \{13, 29\}$ and $a \in \{2t+1 \mid 0 \le t \le 15\}$. Write $b = 13 + 16k$. Then $b^2 = 169 + 416k + 256k^2 \equiv 9 \pmod{32}$ and $b^4 \equiv 81 \equiv 17 \pmod{16}$. Therefore, $b^6 \equiv 153 \equiv 25 \pmod{32}$ and $b' = -(-b)^6 \equiv 7 \pmod{32}$. □

Lemma 4.10. *If a is odd and $b \equiv 5 \pmod 8$, then one of $\widehat{x}_1$ or $\widehat{y}_1$ is congruent to 1 modulo 16 and the other is congruent to 9 modulo 16.*

Proof. Let $w = x$ or y and $\nu = w_1$, so that $\nu \in \{1, 3\}$. An easy computation shows that

$$\begin{aligned} w_6 &= 2^{t-1}(a^4b + 3a^2b^2 + b^3) + \nu(a^5 + 4a^3b + 3ab^2) \\ &= 2^{t-1}(a^4b + 3a^2b^2 + b^3) + \nu u_6. \end{aligned} \tag{4.2}$$

Since $b \equiv 5 \pmod 8$, $a^2 \equiv 1 \pmod 8$, and $2^{t+1} \parallel u_6$, we obtain

$$\begin{aligned} w_6/2^{t-1} &\equiv a^4b + 3a^2b^2 + b^3 + 4 \pmod 8 \\ &\equiv 5 + 75 + 125 + 4 \pmod 8 \\ &\equiv 1 \pmod 8. \end{aligned}$$

Moreover, using the expression for w_6 with both $\nu = 1$ and $\nu = 3$, we have

$$\begin{aligned} x_1' + y_1' &= 2^t(a^4b + 3a^2b^2 + b^3) + 4(a^5 + 4a^3b + 3ab^2) \\ &\equiv 2^t(a^4b + 3a^2b^2 + b^3) \pmod{2^{t+3}} \\ &\equiv 5 \cdot 2^t \pmod{2^{t+3}}, \end{aligned}$$

and therefore

$$\widehat{x}_1 + \widehat{y}_1 \equiv 10 \pmod{16}.$$

Since $\widehat{x}_1$ and $\widehat{y}_1$ are both congruent to either 1 or 9 modulo 16, it follows that one of them is congruent to 1 and the other to 9. □

Explicit computation using (4.2) provides even more detail.

Theorem 4.11. *Assume that* $b \equiv 5 \pmod 8$ *and* a *is odd. Then* $(\widehat{x}_1, \widehat{y}_1) \equiv (1, 9) \pmod{16}$ *when*

$$(b, a) \in \{(5, 3), (5, 5), (5, 7), (5, 9), (13, 3), (13, 5), (13, 9), (13, 13), (13, 15)\}$$

modulo 16 *and* $(\widehat{x}_1, \widehat{y}_1) \equiv (9, 1) \pmod{16}$ *when*

$$(b, a) \in \{(5, 1), (5, 11), (5, 13), (5, 15), (13, 1), (13, 7), (13, 11)\}$$

modulo 16.

Proof. This result is obtained by explicit computation using (4.2). □

To continue our study, we require a deeper analysis of the generalized Fibonacci sequence $u(a', b') \in \mathcal{F}(a', b')$. We begin with the period, which was computed in [**4**].

Theorem 4.12. *Suppose that $u(a',b')$ is the generalized Fibonacci sequence in the family $\mathcal{F}(a',b')$, where (b',a') is congruent to $(7,10)$* modulo 16*. Then $\lambda_u(2^k) = 2^k$ for all $k > 1$.*

Proof. This theorem follows immediately from Theorem 3.1 of [**4**]. □

It follows from Theorem 4.12 that $u_{2^k} \equiv 0 \pmod{2^k}$ and $u_{2^k+1} \equiv 1 \pmod{2^k}$ for all $k > 1$. Accordingly, we can define integer-valued functions $\zeta(k)$ and $\eta(k)$ as follows.

Definition 4.13. If $u(a',b') \in \mathcal{F}(a',b')$ and $k \geq 4$, define $\zeta(k)$ and $\eta(k)$ by

$$u_{2^{k-3}} = \zeta(k)2^{k-3} \quad \text{and} \quad u_{2^{k-3}+1} = \eta(k)2^{k-3} + 1.$$

In the next theorem, we show that, if k is sufficiently large, then, up to congruence modulo a power of 2, ζ and η are independent of k.

Theorem 4.14. *For all integers $s \geq 1$, $\zeta(k) \equiv \zeta(\ell) \pmod{2^s}$ and $\eta(k) \equiv \eta(\ell) \pmod{2^s}$ when $\ell > k \geq s + 4$.*

Proof. By (3.6) and the definition of $\zeta(k)$, $\zeta(k+1)$, and $\eta(k)$,

$$\begin{aligned} \zeta(k+1)2^{k-2} &= u_{2^{k-2}} = 2u_{2^{k-3}}u_{2^{k-3}+1} - a(u_{2^{k-3}})^2 \\ &= 2\zeta(k)2^{k-3}(\eta(k)2^{k-3} + 1) - a(\zeta(k)2^{k-3})^2 \\ &= \zeta(k)2^{k-2} + \zeta(k)\eta(k)2^{2k-5} - a\zeta^2(k)2^{2k-6}. \end{aligned}$$

Since $k \geq s + 4$, we know that $2k - 6 \geq k + s - 2$, and hence

$$\zeta(k+1)2^{k-2} \equiv \zeta(k)2^{k-2} \pmod{2^{k+s-2}},$$

and therefore

$$\zeta(k+1) \equiv \zeta(k) \pmod{2^s}.$$

It follows inductively that $\zeta(k) \equiv \zeta(\ell) \pmod{2^s}$.

By (3.5) and the definition of $\eta(k)$, $\eta(k+1)$, and $\zeta(k)$,

$$\begin{aligned}\eta(k+1)2^{k-2}+1 &= u_{2^{k-2}+1} = b(u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 \\ &= b\zeta^2(k)2^{2k-6} + (\eta(k)2^{k-3}+1)^2 \\ &= b\zeta^2(k)2^{2k-6} + \eta^2(k)2^{2k-6} + \eta(k)2^{k-2} + 1.\end{aligned}$$

As in our analysis of $\zeta(k)$, it follows that

$$\eta(k+1)2^{k-2} \equiv \eta(k)2^{k-2} \pmod{2^{k+s-2}},$$

and therefore

$$\eta(k+1) \equiv \eta(k) \pmod{2^s}.$$

Again, it follows inductively that $\eta(k) \equiv \eta(\ell) \pmod{2^s}$. □

To prove our main theorem of this section, we require the values of $\eta(k)$ and $\zeta(k)$ modulo 16. By the theorem above, the values of $\eta(k)$ and $\zeta(k)$ are constant modulo 16, when $k \geq 8$, and therefore we can determine all possible values of $\eta(k)$ and $\zeta(k)$ modulo 16, when $k \geq 8$, by examining $u_{2^{k-3}}$ and $u_{2^{k-3}+1}$ when $k = 8$, i.e., the terms u_{32} and u_{33}. Moreover, the values of $\zeta(k)$ and $\eta(k)$ modulo 16 can be determined by examining the sequence $u(a', b')$ modulo 2^{k+1}, i.e., modulo 2^9. We perform this examination in the next lemma.

Lemma 4.15. *If $k \geq 8$, then*

(4.3) $$(\zeta(k), \eta(k)) \equiv (13, 13) \pmod{16} \quad \textit{when } (b', a') \equiv (7, 10) \pmod{32}$$

(4.4) $$(\zeta(k), \eta(k)) \equiv (5, 5) \pmod{16} \quad \textit{when } (b', a') \equiv (23, 26) \pmod{32}.$$

Proof. By Theorem 4.14, it is sufficient to prove (4.3) and (4.4) when $k = 8$. First let $b' \in \{7 + 32t \mid 0 \leq t \leq 15\}$ and $a' \in \{10 + 32t \mid 0 \leq t \leq 15\}$. It is a simple computation to verify that, with these parameters,

$$u_{32} \equiv 416 \pmod{2^9} \quad \text{and} \quad u_{33} \equiv 417 \pmod{2^9}.$$

Since $416 = 2^5 \cdot 13$ and $417 = 2^5 \cdot 13 + 1$, we obtain $(\zeta(8), \eta(8)) \equiv (13, 13) \pmod{16}$ when $(b', a') \equiv (7, 10) \pmod{32}$, as desired.

Next, let $b' \in \{23 + 32t \mid 0 \le t \le 15\}$ and $a' \in \{26 + 32t \mid 0 \le t \le 15\}$. Again, a simple computation yields

$$u_{32} \equiv 160 \pmod{2^9} \quad \text{and} \quad u_{33} \equiv 161 \pmod{2^9}.$$

Since $160 = 2^5 \cdot 5$ and $161 = 2^5 \cdot 5 + 1$, we obtain $(\zeta(8), \eta(8)) \equiv (5, 5) \pmod{16}$ when $(b', a') \equiv (23, 26) \pmod{32}$, as desired. □

The next lemma will serve as an analogue of Lemma 4.7 (c) and (d) in the proof of Theorem 4.5.

Lemma 4.16. *Suppose that $z = \widehat{x}$ or $z = \widehat{y}$, $z_1 \equiv 1 \pmod{16}$, and that either $(b', a') \equiv (7, 10) \pmod{32}$ or $(b', a') \equiv (23, 26) \pmod{32}$. Then, for all $k \ge 8$,*

$$z_{n+2^{k-3}} \equiv z_n + 2^k \pmod{2^{k+1}}.$$

Proof. By the Intertwining Lemma,

$$\begin{aligned} z_{n+2^{k-3}} &= b' z_{n-1} u_{2^{k-3}} + z_n u_{2^{k-3}+1} \\ &= b' z_{n-1} \zeta(k) 2^{k-3} + z_n(\eta(k) 2^{k-3} + 1) \\ &= z_n + (b'\zeta(k) z_{n-1} + \eta(k) z_n) 2^{k-3}. \end{aligned}$$

Now, since $z_0 = 1$, $z_1 \equiv 1 \pmod{16}$, and $(b', a') \equiv (7, 10) \pmod{16}$, it is easy to verify that $z_2 = a' z_1 + b' z_0 \equiv a' + b' \equiv 1 \pmod{16}$, and therefore $z_n \equiv 1 \pmod{16}$ for all n. Applying Lemma 4.15 in the case that $(b', a') \equiv (7, 10) \pmod{32}$, we obtain

$$b'\zeta(k) z_{n-1} + \eta(k) z_n \equiv 7 \cdot 13 \cdot 1 + 13 \cdot 1 \equiv 104 \equiv 8 \pmod{16}.$$

On the other hand, in the case that $(b', a') \equiv (23, 26) \pmod{32}$, Lemma 4.15 yields

$$b'\zeta(k) z_{n-1} + \eta(k) z_n \equiv 7 \cdot 5 \cdot 1 + 5 \cdot 1 \equiv 40 \equiv 8 \pmod{16}.$$

Thus, in both cases,

$$z_{n+2^{k-3}} = z_n + (b'\zeta(k)z_{n-1} + \eta(k)z_n)2^{k-3} \equiv z_n + 2^k \pmod{2^{k+1}},$$

as desired. □

Our next goal is to compute the period of the sequences $\widehat{x}$ and $\widehat{y}$. First note that these sequences are not 2-regular. If z is either $\widehat{x}$ or $\widehat{y}$, then $z_0 \equiv z_1 \equiv 1 \pmod 8$, $a' \equiv 2 \pmod 8$, and $b' \equiv 7 \pmod 8$. It follows that $z_2 \equiv 1 \pmod 8$, and therefore $z_0 z_2 - z_1^2 \equiv 0 \pmod 8$. Thus z is not 2-regular.

Irregular sequences are usually more complicated than regular ones and require special techniques to analyze, similar to those discussed in [**8**]. It is a general property, true for most second-order recursive sequences z, that for k sufficiently large $\lambda_z(2^k) = 2 \cdot \lambda_z(2^{k-1})$. Generalizing the standard definition for odd moduli given in [**6**], we offer the following definition.

Definition 4.17. If z is a second-order recursive sequence, let f denote the smallest integer, if it exists, such that $\lambda_z(2^{k+1}) = 2 \cdot \lambda_z(2^k)$ for all $k \geq f$.

Armed with this definition we can compute the periods $\lambda_z(2^k)$, for $z = \widehat{x}$ and $z = \widehat{y}$. These depend upon the value of z_1, which, according to Lemma 4.10, is congruent to either 1 or 9 modulo 16.

We require two lemmas. For both lemmas, assume that $z = \widehat{x}$ or $z = \widehat{y}$, that $z_1 \equiv 1 \pmod{16}$, and that either $(b', a') \equiv (7, 10) \pmod{32}$ or $(b', a') \equiv (23, 26) \pmod{32}$.

Lemma 4.18. *For all* $n \geq 1$, $z_n = z_1 u_n + b' u_{n-1}$.

Proof. When $n = 1$ the result is obvious. For $n = 2$, we obtain $z_1 u_n + b' u_{n-1} = z_1 u_2 + b' u_1 = a' z_1 + b' z_0 = z_2$, as desired.

By way of induction, assume that the result is true for $n - 1$ and n. Then

$$\begin{aligned} z_{n+1} &= a'z_n + b'z_{n-1} = a'z_1u_n + a'b'u_{n-1} + b'z_1u_{n-1} + (b')^2u_{n-2} \\ &= z_1(a'u_n + b'u_{n-1}) + b'(a'u_{n-1} + b'u_{n-2}) = z_1u_{n+1} + b'u_n, \end{aligned}$$

as desired. □

Lemma 4.19. *For all $k \geq 1$,*

$$z_{2^{k-3}} \equiv u_{2^{k-3}+1} - u_{2^{k-3}} \pmod{2^k} \tag{4.5}$$

$$z_{2^{k-3}+1} \equiv z_1u_{2^{k-3}+1} - u_{2^{k-3}} \pmod{2^k}. \tag{4.6}$$

Proof. By Lemma 2.1 of [**4**], $2^{k-3} \parallel u_{2^{k-3}}$. Since $z_1 \equiv 1 \pmod{16}$, it follows that $z_1u_{2^{k-3}} \equiv u_{2^{k-3}} \pmod{2^{k+1}}$. Moreover, since $a' \equiv 10 \pmod{16}$, we obtain $(a'-1)u_{2^{k-3}} \equiv u_{2^{k-3}} \pmod{2^k}$.

Now, by Lemma 4.18,

$$\begin{aligned} z_{2^{k-3}} &= z_1u_{2^{k-3}} + b'u_{2^{k-3}-1} \\ &= z_1u_{2^{k-3}} + u_{2^{k-3}+1} - a'u_{2^{k-3}} \\ &\equiv u_{2^{k-3}} + u_{2^{k-3}+1} - a'u_{2^{k-3}} \pmod{2^k} \\ &\equiv u_{2^{k-3}+1} - (a'-1)u_{2^{k-3}} \pmod{2^k} \\ &\equiv u_{2^{k-3}+1} - u_{2^{k-3}} \pmod{2^k}. \end{aligned}$$

This verifies the first congruence.

Similarly, by Lemma 4.18, Lemma 2.1 of [**4**], and the hypothesis that $b' \equiv 7 \pmod{16}$,

$$\begin{aligned} z_{2^{k-3}+1} &= z_1u_{2^{k-3}+1} + b'u_{2^{k-3}} \\ &\equiv z_1u_{2^{k-3}+1} + (8-1)u_{2^{k-3}} \pmod{2^k} \\ &\equiv z_1u_{2^{k-3}+1} - u_{2^{k-3}} \pmod{2^k}, \end{aligned}$$

as desired. □

Theorem 4.20. *Suppose that $z = \widehat{x}$ or $z = \widehat{y}$ and $z_1 \equiv 1 \pmod{16}$. Then $\lambda_z(2^{k+1}) = 2^{k-2}$ when $k \geq 4$ and $\lambda_z(2^{k+1}) = 1$ for $0 \leq k \leq 3$. In particular, $f = 5$.*

Proof. First note that by Theorem 4.9, either $(b', a') \equiv (7, 10) \pmod{32}$ or $(b', a') \equiv (23, 26) \pmod{32}$, and in either case $(b', a') \equiv (7, 10) \pmod{16}$.

For $k \leq 4$, the result is easily verified by computation. To prove the result for $k > 4$, proceed by induction.

Assume that $\lambda_z(2^k) = 2^{k-3}$ for some $k \geq 4$. Then, in particular, by Lemma 4.19,

$$u_{2^{k-3}+1} - u_{2^{k-3}} \equiv z_{2^{k-3}} \equiv 1 \pmod{2^k}. \tag{4.7}$$

However, by Lemma 4.19, (3.5), and (3.6),

$$\begin{aligned} z_{2^{k-2}} &\equiv u_{2^{k-2}+1} - u_{2^{k-2}} \pmod{2^{k+1}} \\ &\equiv b'(u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 \\ &\quad - 2u_{2^{k-3}}u_{2^{k-3}+1} + a'(u_{2^{k-3}})^2 \pmod{2^{k+1}} \\ &\equiv (a' + b')(u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 \\ &\quad - 2u_{2^{k-3}}u_{2^{k-3}+1} \pmod{2^{k+1}}. \end{aligned}$$

Since $a' + b' \equiv 1 \pmod{16}$ and, by Lemma 2.1 of [**4**], $2^{k-3} \parallel u_{2^{k-3}}$,

$$\begin{aligned} z_{2^{k-2}} &\equiv (u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 - 2u_{2^{k-3}}u_{2^{k-3}+1} \pmod{2^{k+1}} \\ &\equiv (u_{2^{k-3}+1} - u_{2^{k-3}})^2 \pmod{2^{k+1}}. \end{aligned}$$

Finally, by (4.7), $z_{2^{k-2}} \equiv 1 \pmod{2^{k+1}}$.

Similarly, by the induction hypothesis and Lemma 4.19,

$$z_1 u_{2^{k-3}+1} - u_{2^{k-3}} \equiv z_{2^{k-3}+1} \equiv z_1 \pmod{2^k}.$$

Since $z_1 \equiv 1 \pmod{16}$, Lemma 2.1 of [**4**] implies that $z_1 u_{2^{k-3}} \equiv u_{2^{k-3}} \pmod{2^{k+1}}$, and Lemma 2.3 (a) of [**4**] implies that $(z_1 - 1)u_{2^{k-3}+1} \equiv$

$(z_1 - 1) \pmod{2^{k+1}}$. Now, by Lemma 4.19, (3.5), and (3.6),

$$\begin{aligned} z_{2^{k-2}+1} &\equiv u_{2^{k-2}+1} z_1 - u_{2^{k-2}} \pmod{2^{k+1}} \\ &\equiv [b'(u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2] z_1 \\ &\quad - 2u_{2^{k-3}} u_{2^{k-3}+1} + a'(u_{2^{k-3}})^2 \pmod{2^{k+1}} \\ &\equiv (a' + b')(u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 z_1 \\ &\quad - 2u_{2^{k-3}} u_{2^{k-3}+1} \pmod{2^{k+1}} \\ &\equiv (u_{2^{k-3}})^2 + (u_{2^{k-3}+1})^2 z_1 \\ &\quad - 2u_{2^{k-3}} u_{2^{k-3}+1} \pmod{2^{k+1}} \\ &\equiv (u_{2^{k-3}+1} - u_{2^{k-3}})^2 + (z_1 - 1)(u_{2^{k-3}+1})^2 \pmod{2^{k+1}} \\ &\equiv 1 + (z_1 - 1) \pmod{2^{k+1}} \\ &\equiv z_1 \pmod{2^{k+1}}, \end{aligned}$$

as desired.

It follows that $\lambda_z(2^{k+1}) \mid 2^{k-2}$.

On the other hand, $2^{k-3} = \lambda_z(2^k) \mid \lambda_z(2^{k+1})$. Finally, Lemma 4.6 implies that $\lambda_z(2^{k+1}) = 2^{k-2}$. □

We can now prove Theorem 4.5. Again, the method used is the same as that used in [**1, 7**].

Proof of Theorem 4.5. By Theorems 4.4 and 4.11, it is sufficient to prove stability for a sequence $w(a,b)$ equal to either $x(a,b)$ or $y(a,b)$, with the additional condition that $w_6 \equiv 1 \pmod{16}$. Assume that $k > \max(7,t)$. Then by Lemma 4.7 (a) and (b),

$$\begin{aligned} w_{n+3\cdot 2^{k-1}} &\equiv w_n + 2^k \pmod{2^{k+1}} \quad \text{if } 3 \nmid n \\ w_{n+3\cdot 2^{k-2}} &\equiv w_n + 2^k \pmod{2^{k+1}} \quad \text{if } n \equiv 3 \pmod{6}. \end{aligned}$$

By hypothesis, if $z = w_{6n}$, then z satisfies the hypotheses of Lemma 4.16, and therefore

$$z_{n+2^{k-3}} \equiv z_n + 2^k \pmod{2^{k+1}},$$

or, in terms of w,

$$w_{n+3\cdot 2^{k-2}} \equiv w_n + 2^k \pmod{2^{k+1}} \quad \text{if } n \equiv 0 \pmod{6}.$$

It follows that each instance of a residue d in one cycle of $w(a,b)$ modulo 2^{k+1} can be paired with a unique instance of the residue $d+2^k$ in the same cycle. Therefore, $\nu_w(d,2^{k+1}) = \nu_w(d+2^k,2^{k+1})$. On the other hand, by Theorem 3.10, the period of $w(a,b)$ modulo 2^{k+1} is double the period modulo 2^k. It follows that $\nu_w(d,2^k) = \nu_w(d,2^{k+1}) = \nu_w(d+2^k,2^{k+1})$. Finally, we conclude that no new residue frequencies occur modulo 2^{k+1}, that is, $\Omega_w(2^k) = \Omega_w(2^{k+1})$. Since this is true for all sufficiently large k, it follows that $w(a,b)$ is 2-stable. □

5. The case when the transition degree does not exist.

5.1. Classification. Under our hypotheses that a is odd and $b \equiv 5 \pmod 8$, there is a parametrized family of recurrence sequences $u(a,b)$ for which the transition degree t fails to exist. This unique family is given by the following theorem.

Theorem 5.1. *Suppose that $w(a,b)$ satisfies Hypothesis 3.3. Then the transition degree t fails to exist when $(a,b) \in \{(3s,-3s^2) \mid s \text{ is an odd positive integer}\}$.*

Proof. Using the definition of $u(a,b)$ an easy computation shows that $u_6 = a^5 + 4a^3b + 3ab^2$, and therefore t fails to exist if and only if $a^5 + 4a^3b + 3ab^2 = 0$.

Let s be an odd positive integer, and suppose that $(a,b) = (3s,-3s^2)$. Then certainly a is odd and $b \equiv -3 \equiv 5 \pmod 8$, and it is easy to verify that $u_6 = 243s^5 + (108s^3)(-3s^2) + (9s)(9s^4) = 0$.

Conversely, suppose that $a^5 + 4a^3b + 3ab^2 = 0$, and let $\alpha = a^2$. Then

$$0 = a^5 + 4a^3b + 3ab^2 = a(a^4 + 4a^2b + 3b^2) = a(\alpha^2 + 4\alpha b + 3b^2).$$

Under our hypotheses, a is odd, so it cannot be zero. Therefore α and b must satisfy the quadratic equation

$$\alpha^2 + 4\alpha b + 3b^2 = 0,$$

which has solutions $\alpha = -b$ and $\alpha = -3b$. However, again by hypothesis, $\alpha = a^2 \equiv 1 \pmod 8$ and $b \equiv 5 \pmod 8$. This eliminates the possible solution $\alpha = -b$, and we are left with $a^2 = \alpha = -3b$. But if $-3b$ is to be a square, we must have $b = -3s^2$ for some odd integer s, as desired. □

5.2. Stability. In this section we examine the 2-stability of the sequences in $\mathcal{F}(a, b)$ for which the transition degree t fails to exist.

Theorem 5.2. *Suppose that $w(a, b)$ satisfies Hypothesis* 3.3 *and the transition degree t does not exist. Then $w(a, b)$ is* 2*-stable if and only if $w_i \neq 0$ for all i.*

Proof. Suppose that $w(a, b)$ satisfies Hypothesis 3.3, t does not exist, and $k \geq 3$. By Theorem 3.10, w has period $\lambda_w(2^k) = 3 \cdot 2^{k-1}$ and, by Theorem 3.11, w has restricted period $h_w(2^k) = 6$. It follows that the multiplier M of each sequence has multiplicative order 2^{k-2} in the group $G_k = (\mathbf{Z}/2^k\mathbf{Z})^*$, and therefore, $H_k = \{M^0, M^1, \ldots, M^{2^{k-2}-1}\}$ is a cyclic subgroup of G_k of order 2^{k-2} and index $[G_k : H_k] = 2$. One period of the sequence $w(a, b)$ modulo 2^k consists of the residues

$$\bigcup_{\substack{0 \leq j \leq 5 \\ 0 \leq i < 2^{k-2}}} M^i w_j = \bigcup_{0 \leq j \leq 5} w_j H_k.$$

Since the parameters a and b are both odd and $2 \nmid (w_0, w_1)$, (1.1) implies that of any three consecutive terms of w, two are odd and one is even. Without loss of generality, we may suppose that w is a principal representative of its block, so that w_0 and w_3 are even, while w_1, w_2, w_4 and w_5 are odd. It follows that all of the even residues in one period of w modulo 2^k lie in the subsets $w_0 H_k$ and $w_3 H_k$ and all of the odd residues lie in $w_i H_k$ for $i = 1$, 2, 4, and 5.

Consider first the even terms. Since $h_w(2^k) = 6$, w_0 and w_3 cannot both be zero. If $w_0 = 0$, then $\nu_w(0, 2^k) = |H_k| = 2^{k-2}$, since in this case $w_3 \neq 0$ and $0 \notin w_3 H_k$. Thus, if w contains a term that is equal to zero, then w is not stable.

Now assume that neither w_0 nor w_3 is equal to zero, and that $2^m \parallel w_0$. By the argument of Lemma 3.17, $m > 1$ and $2 \parallel w_3$.

Let $w_0M^i \in w_0H_k$ and $w_3M^j \in w_3H_k$. Since $4 \mid w_0M^i$ and $2 \parallel w_3M^j$, it follows that $2 \parallel w_0M^i - w_3M^j$, and therefore, since $k \geq 3$, $w_0M^i \not\equiv w_3M^j \pmod{2^k}$.

Next consider the residues in w_3H_k. Since $2 \parallel w_3$, it follows that $w_3M^i \equiv w_3M^j \pmod{2^k}$ for exponents i and j satisfying $0 \leq i < j < 2^{k-2}$ if and only if $M^i \equiv M^j \pmod{2^{k-1}}$, i.e., if and only if $M^{j-i} \equiv 1 \pmod{2^{k-1}}$. Since M has multiplicative order 2^{k-3} modulo 2^{k-1}, this occurs exactly when $2^{k-3} \mid j - i$. It follows that $\nu_w(d, 2^k) = 2$ when $d \equiv w_3M^i$ for some i.

Now, suppose that $k > m$. Then $w_0M^i \equiv w_0M^j \pmod{2^k}$ for exponents i and j satisfying $0 \leq i < j < 2^{k-2}$ if and only if $M^i \equiv M^j \pmod{2^{k-m}}$. This occurs precisely when $2^{k-m-2} \mid j - i$, and it follows that $\nu_w(d, 2^k) = 2^m$ when $d \equiv w_0M^i$ for some i.

Finally consider the odd terms. Let $i \in \{1, 2, 4, 5\}$. Since w_i is odd, the subset w_iH_k is a left coset of the subgroup H_k in G_k, and hence the elements of w_iH_k are distinct modulo 2^k. Since the cosets of H_k partition G_k, any two of these cosets are either identical or disjoint. Moreover, since $[G_k : H_k] = 2$, there are exactly two cosets, H_k and $-H_k$. Clearly, $w_iH_k = H_k$ if and only if $w_i \equiv M^\alpha \pmod{2^k}$ for some α, and $w_iH_k = -H_k$ if and only if $w_i \equiv -M^\alpha \pmod{2^k}$ for some α. It is now easy to show that if $w_iH_k = H_k$, then $w_iH_\ell = H_\ell$ for all $\ell \geq k$, and similarly, if $w_iH_k = -H_k$, then $w_iH_\ell = -H_\ell$ for all $\ell \geq k$. It follows that the multiplicity of each coset H_k and $-H_k$ among $\{w_1H_k, w_2H_k, w_4H_k, w_5H_k\}$ is constant as a function of k, and hence $\nu_w(d, 2^k)$ is constant as a function of k whenever d is odd.

Combining our observations for odd and even terms of $w(a, b)$, we conclude that $w(a, b)$ is 2-stable if and only if $w_i \neq 0$ for all i, as desired. □

Acknowledgments. Portions of this work were completed while the first author was visiting Franklin & Marshall College.

REFERENCES

1. Walter Carlip and Eliot Jacobson, *A criterion for stability of two-term recurrence sequences modulo* 2^k, Finite Fields Appl. **2** (1996), 369–406.

2. ———, *On the stability of certain Lucas sequences modulo* 2^k, Fibonacci Quart. **34** (1996), 298–305.

3. Walter Carlip and Eliot Jacobson, *Unbounded stability of two-term recurrence sequences modulo* 2^k, Acta Arith. **74** (1996), 329–346.

4. ———, *Stability of two-term recurrence sequences with even parameter*, Finite Fields Appl. **3** (1997), 70–83.

5. Walter Carlip, Eliot Jacobson, and Lawrence Somer, *A criterion for stability of two-term recurrence sequences modulo odd primes*, in *Applications of Fibonacci numbers*, Vol. **7** (Graz, 1996), Kluwer Academic Publishers, Dordrecht, 1998.

6. Walter Carlip and Lawrence Somer, *Bounds for frequencies of residues of regular second-order recurrences modulo* p^r, in *Number theory progress*, Vol. **2** (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999.

7. ———, *Stability classes of second-order linear recurrences modulo* 2^k, Tatra Mountain Math. Publ. **20** (2000), 31–57.

8. ———, *Bounds for frequencies of residues of second-order recurrences modulo* p^r, Math. Bohem. **132** (2007), 137–175.

9. Dana Carroll, Eliot Jacobson, and Lawrence Somer, *Distribution of two-term recurrence sequences* mod p^e, Fibonacci Quart. **32** (1994), 260–265.

10. Eliot T. Jacobson, *Distribution of the Fibonacci numbers* mod 2^k, Fibonacci Quart. **30** (1992), 211–215.

11. Mark D. Morgan, *The distribution of second order linear recurrence sequences* mod 2^m, Acta Arith. **83** (1998), 181–195.

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