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ASYMPTOTIC INTEGRATION UNDER WEAK DICHOTOMIES

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ABSTRACT. In a classical result, Levinson considered perturbations of diagonal systems of differential equations. He showed that the unperturbed and the perturbed system are strongly asymptotically equivalent if the entries of the diagonal matrix satisfy a certain dichotomy condition and if the perturbation is absolutely integrable. Here we are interested in diagonal linear systems that satisfy dichotomy conditions which are weaker than Levinson's. We show that the diagonal system is still strongly asymptotically equivalent to a perturbed system provided that the perturbation is sufficiently small. We also generalize these results to perturbations of Jordan matrices. We give some corresponding results for perturbations of systems of difference equations and conclude with examples.

1. Introduction. In [**3**], we introduced a concept called *strong asymptotic equivalence* between two linear systems $x' = A(t)x$ and $y' = B(t)y$ defined for $t \geq t_0$. This means that there exist corresponding fundamental solutions related by the asymptotic equation

$$X(t) = [I + \mathrm{o}(1)]\, Y(t) \text{ as } t \to +\infty.$$

While it can be shown (see Theorem 5) that any two systems are strongly asymptotically equivalent provided $A(t) - B(t)$ is "sufficiently small," this general result does not yield a practical (close to optimal) bound. Instead, in most applications, one should think of $A(t)$ as having a certain structure (e.g. diagonal, block-diagonal, Jordan) and $B(t) = A(t) + R(t)$ as a perturbed system. Then a simpler problem is, given the structure of A and certain easily-checked properties, to determine suitable smallness conditions on R which imply strong asymptotic equivalence to the unperturbed system.

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What has become the central, classical result concerned with perturbations of diagonal systems is due to Levinson (see [**12**] or [**8**, Theorem 1.3.1]). He showed essentially that if $A(t) = \Lambda(t)$ is diagonal and satisfies certain dichotomy conditions (see below), then $R \in L^1[t_0, \infty)$ is sufficient for strong asymptotic equivalence between $x' = \Lambda(t)x$ and $y' = [\Lambda(t) + R(t)]\, y$.

This result is central because it has been shown that many other types of results on strong asymptotic equivalence can be reduced to it, namely, ones which involve stronger dichotomy conditions on the diagonal matrix and weaker conditions on the perturbation terms, see e.g., [**1, 4, 9–11**].

For a diagonal matrix $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \dots, \lambda_d(t)\}$, Levinson's dichotomy condition pertains to integrals of the form

$$\int_s^t \operatorname{Re}\, [\lambda_i(\tau) - \lambda_j(\tau)]\; d\tau$$

where $t_0 \le s \le t < \infty$. If every such integral is either bounded from above or bounded from below, then $\Lambda(t)$ is said to satisfy Levinson's dichotomy condition.

If, on the other hand, the integrals oscillate and become unbounded in both directions (toward plus and minus infinity simultaneously), then Levinson's condition fails. Examples due to Eastham [**8**, page 10] and Perron [**13**] show that in such a case $R(t) \in L^1[t_0, \infty)$ is generally not sufficient for strong asymptotic equivalence.

Here we will be concerned with weakening Levinson's dichotomy conditions and then finding admissible growth conditions on $R(t)$. We will quantify the unbounded oscillations by first assuming the existence of a positive function β on $[t_0, \infty)$ satisfying either

$$\exp\left[\int_{t_0}^t \operatorname{Re}\{\lambda_i(\tau) - \lambda_j(\tau)\}\, d\tau\right] \longrightarrow 0 \quad \text{as } t \to \infty$$

and

$$\exp\left[\int_s^t \operatorname{Re}\{\lambda_i(\tau) - \lambda_j(\tau)\}\, d\tau\right] \le \beta(s) \quad \text{for all } t_0 \le s \le t < \infty$$

or else

$$\exp\left[\int_s^t \mathrm{Re}\,[\lambda_i(\tau)-\lambda_j(\tau)]\,d\tau\right] \le \beta(s) \quad \text{for all } t_0 \le t \le s < \infty.$$

One of our main results, Theorem 2, shows that if $\Lambda(t)$ satisfies these conditions for all (i,j) and if $\beta(t)R(t) \in L^1[t_0,\infty)$, then strong asymptotic equivalence follows. We then will also show that such a function β always exists, although for applications finding an optimal, i.e. smallest, β might involve some computational difficulties.

We will derive our results for differential equations in Section 2 and some analogous results for systems of linear difference equations will be given in Section 3. In Section 4, we will present some families of examples (depending upon parameters) which explain how our results are related to the (counter) examples of Eastham and Perron, and we will see that the growth condition above is quite sharp.

Our work was motivated by results of Chiba and Kimura [**5**] who studied perturbations of systems which do not satisfy Levinson's dichotomy conditions using Hukuhara's theorem. In our approach, we are able to simplify their hypotheses and, moreover, show that strong asymptotic equivalence is always obtainable for sufficiently small perturbations.

2. Differential equations. In [**6**, page 76], Coppel considered perturbations of linear systems of differential equations which possess an ordinary dichotomy. He showed that there is a one-to-one and bicontinuous correspondence between the bounded solutions of unperturbed and perturbed systems provided that the perturbation is absolutely integrable. Our first theorem is a reformulation of this result, allowing weaker dichotomy conditions on the unperturbed system but imposing a more restrictive requirement on the perturbation $R(t)$. The proof is just a modification of Coppel's original proof, but we include it for the sake of completeness and emphasize that Coppel's result is a special case of Theorem 1 corresponding to bounded $\beta(t)$.

Throughout this section, we will assume the continuity of all coefficient matrices to have existence and uniqueness of solutions. As usual, we could also just require that coefficient matrices are locally integrable which would lead to almost everywhere differentiable solution matrices.

Theorem 1. *For continuous $d \times d$ matrices $A(t)$ and $R(t)$, consider the unperturbed system*

$$x' = A(t)x, \quad t \geq t_0, \tag{1}$$

and the perturbed system

$$y' = [A(t) + R(t)]\, y, \quad t \geq t_0. \tag{2}$$

Let $X(t)$ be a fundamental matrix of (1). Assume that there exists a projection matrix P and a continuous function $\beta(t) \geq 1$ such that

$$|X(t)PX^{-1}(s)| \leq \beta(s) \text{ for all } t_0 \leq s \leq t \tag{3}$$

and

$$|X(t)[I-P]X^{-1}(s)| \leq \beta(s) \text{ for all } t_0 \leq t \leq s. \tag{4}$$

Suppose that

$$\beta(t)R(t) \in L^1[t_0, \infty). \tag{5}$$

Then there exists a one-to-one and bicontinuous correspondence between the bounded solutions of (1) and (2). Moreover, the difference between corresponding bounded solutions of (1) and (2) tends to zero as $t \to \infty$ if $X(t)P \to 0$ as $t \to \infty$.

Proof. Fix $t_1 \geq t_0$ such that

$$\theta = \int_{t_1}^{\infty} \beta(t)|R(t)|\, dt < 1.$$

Let $\mathcal{B}$ be the Banach space of bounded d-dimensional vector-valued functions with the supremum norm

$$\|y\| = \sup_{t \geq t_1} |y(t)|.$$

For $t \geq t_1$, define an operator T acting on $\mathcal{B}$ by

$$\begin{aligned}(Ty)(t) = &\int_{t_1}^{t} X(t)PX^{-1}(s)R(s)y(s)\, ds \\ &- \int_{t}^{\infty} X(t)[I-P]X^{-1}(s)R(s)y(s)\, ds.\end{aligned} \tag{6}$$

Now (3), (4) and (5) imply that

$$|(Ty)(t)| \leq \|y\| \int_{t_1}^{\infty} \beta(s)|R(s)|\, ds \leq \theta \|y\|,$$

and, similarly,

$$|(Ty_1)(t) - (Ty_2)(t)| \leq \int_{t_1}^{\infty} \beta(s)|R(s)|\, |y_1(s) - y_2(s)|\, ds \leq \theta \|y_1 - y_2\|;$$

thus, also

$$\|Ty_1 - Ty_2\| \leq \theta \|y_1 - y_2\|.$$

Hence $T : \mathcal{B} \to \mathcal{B}$ is a contraction. Therefore, for any $x \in \mathcal{B}$, the operator equation

$$x = y - Ty$$

has a unique solution $y \in \mathcal{B}$, and it is straightforward to check that if x is a solution of (1), then $y = x + Ty$ is a solution of (2). The equation $x = y - Ty$ establishes therefore a one-to-one correspondence between bounded (for $t \geq t_1$) solutions of (1) and (2) (using (6)). The bicontinuity of the correspondence for $t \geq t_1$ follows from the inequalities

$$\frac{1}{1+\theta}\|x_1 - x_2\| \leq \|y_1 - y_2\| \leq \frac{1}{1-\theta}\|x_1 - x_2\|.$$

Using continuous dependence of solutions on initial conditions, the bicontinuity can be extended to $[t_0, \infty)$. Finally, given corresponding bounded solutions x and y of (1) and (2), respectively, and $\varepsilon > 0$, fix $t_2 = t_2(\varepsilon) > t_1$ such that

$$\int_{t_2}^{\infty} \beta(t)|R(t)|\, |y(t)|\, dt \leq \|y\| \int_{t_2}^{\infty} \beta(t)|R(t)|\, dt < \frac{\varepsilon}{2}.$$

Then, for $t \geq t_2$,

$$\begin{aligned} |y(t) - x(t)| &= |(Ty)(t)| \\ &\leq |X(t)P| \int_{t_1}^{t_2} \left|X^{-1}(s)R(s)y(s)\right|\, ds \\ &\quad + \|y\| \int_{t_2}^{\infty} \beta(s)|R(s)|\, ds \\ &\leq |X(t)P| \int_{t_1}^{t_2} |X^{-1}(s)R(s)y(s)|\, ds + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

for all t sufficiently large if $X(t)P \to 0$ as $t \to \infty$. $\square$

In Theorem 1 the existence of a function $\beta(s)$ satisfying (3) and (4) was assumed. However, it can be easily shown that such a $\beta(s)$ always exists. For example, one may put for $s \geq t_0$,

$$\beta(s) = \sup_{t_0 \leq t \leq s} \left| X(t) X^{-1}(s) \right|,$$

where $X(t)$ is the fundamental matrix of the differential system (1) satisfying $X(t_0) = I$. Then $\beta(s)$ is well-defined and continuous for $s \geq t_0$, $\beta(s) \geq 1$ for all $s \geq s_0$, and (3) and (4) hold corresponding to the projection matrix $P = 0$. It is clear, however, that this choice for $\beta(s)$ will not yield, in general, an optimal bound for the perturbation. Instead, one should take into account the special structure of $X(t)$ and choose P and β using that information.

This brings us to our first result on asymptotic integration of perturbations of diagonal systems not necessarily satisfying Levinson's dichotomy conditions.

Theorem 2. *Let $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \dots, \lambda_d(t)\}$ be a diagonal and continuous $d \times d$ matrix for $t \geq t_0$. Fix $h \in \{1, 2, \dots, d\}$. Assume that there exists a continuous function $\beta_h(t) \geq 1$ for $t \geq t_0$ such that, for each $1 \leq i \leq d$, either*

$$(7) \qquad \left.\begin{array}{l} \exp\left[\displaystyle\int_{t_0}^{t} \operatorname{Re}\{\lambda_i(\tau) - \lambda_h(\tau)\}\, d\tau\right] \longrightarrow 0 \qquad \text{as } t \to \infty \\ \qquad\qquad\qquad \text{and} \\ \exp\left[\displaystyle\int_{s}^{t} \operatorname{Re}\{\lambda_i(\tau) - \lambda_h(\tau)\}\, d\tau\right] \leq \beta_h(s) \text{ for all } t_0 \leq s \leq t \end{array}\right\}$$

or

$$(8) \qquad \exp\left[\int_{s}^{t} \operatorname{Re}\{\lambda_i(\tau) - \lambda_h(\tau)\}\, d\tau\right] \leq \beta_h(s) \quad \text{for all } t_0 \leq t \leq s.$$

Furthermore, assume that $R(t)$ is a continuous $d \times d$ matrix for $t \geq t_0$ and that

$$(9) \qquad \beta_h(t) R(t) \in L^1[t_0, \infty).$$

Then the linear differential system

$$y' = [\Lambda(t) + R(t)]\, y \tag{10}$$

has a solution satisfying

$$y_h(t) = [e_h + \mathrm{o}(1)] \exp\left[\int^t \lambda_h(s)\, ds\right] \text{ as } t \to \infty, \tag{11}$$

where e_h is the hth column of the identity matrix.

Proof. For fixed $h \in \{1, 2, \ldots, d\}$, put

$$y(t) = z(t) \exp\left[\int^t \lambda_h(\tau)\, d\tau\right]. \tag{12}$$

Then (10) implies that

$$z'(t) = [\Lambda(t) - \lambda_h(t)I + R(t)]\, z. \tag{13}$$

We also consider the unperturbed shifted system

$$w' = [\Lambda(t) - \lambda_h(t)I]\, w, \tag{14}$$

and we will apply Theorem 1 to these shifted systems. For that purpose, let $P = \mathrm{diag}\,\{p_1, p_2, \ldots, p_d\}$, where

$$p_i = \begin{cases} 1 & \text{if } (i,h) \text{ satisfies (7)} \\ 0 & \text{if } (i,h) \text{ satisfies (8).} \end{cases}$$

Then the unperturbed shifted system (14) satisfies dichotomy conditions (3) and (4) with $\beta(s)$ replaced by $\beta_h(s)$ and, moreover, $W(t)P \to 0$ as $t \to \infty$. Recall that R satisfies (9). Now, by Theorem 1 and since $w_h(t) = e_h$ is trivially a bounded solution of (14), there exists a bounded solution $z_h(t)$ of (13) with

$$z_h(t) = e_h + \mathrm{o}(1) \text{ as } t \to \infty.$$

By (12), (10) has a solution of the form (11) as $t \to \infty$. □

Remark 3. If $\beta_h(s)$ is bounded from above by a constant, then $\Lambda(t)$ satisfies Levinson's dichotomy condition, and the statement of Theorem 2 reduces to Levinson's well-known result, see e.g., [**8**, Theorem 1.3.1].

Although in Theorem 2 the existence of an appropriate function β_h was assumed, it can again be shown that such a function β_h always exists. For example, for fixed h and for each $i \in \{1, \dots, d\}$, one may put

$$\gamma_i(s) = \sup_{t_0 \le t \le s} \left(\exp \left[\int_s^t \operatorname{Re}\{\lambda_i(\tau) - \lambda_h(\tau)\}\, d\tau \right] \right),$$

and

$$\beta_h(s) = \max_{1 \le i \le d} \gamma_i(s),$$

i.e., the point-wise maximum over all γ_i. It follows that (8) holds for all $1 \le i \le d$. However, this construction of β_h should be avoided in applications, since $\beta_h(t)$ derived this way is likely neither the smallest possible choice nor easy to compute (also see Example 8).

While Theorem 2 was concerned with perturbations of diagonal systems, these results can be generalized to perturbations of Jordan matrices in a standard manner, see e.g., [**8**, Theorem 1.10.1] or [**6**, page 91]. This method uses preliminary transformations resulting in a differential system with a diagonal main matrix (with additional terms on the diagonal) and then applying Theorem 2 to this new system provided that the perturbation is sufficiently small.

More specifically, we consider the situation of M distinct eigenvalues $\lambda_i(t)$ which are grouped (for each i) into d_i Jordan blocks of size n_{ij}. For $1 \le i \le M$ and $1 \le j \le d_i$, we will use the notation

$$J_{ij}(t) = \lambda_i(t) I_{n_{ij}} + N_{n_{ij}}, \tag{15}$$

where $I_{n_{ij}}$ denotes the $n_{ij} \times n_{ij}$ identity matrix and $N_{n_{ij}}$ is the $n_{ij} \times n_{ij}$ matrix with ones on the first super-diagonal and zeroes elsewhere. We will also put for $1 \le i \le M$

(16) $$n_i = \max_{1\le j\le d_i} n_{ij}, \qquad n_0 = \max_{1\le i\le M} n_i, \qquad \widehat{n}_i = \sum_{j=1}^{d_i} n_{ij}.$$

The overall dimension of the system is $d = \sum_{i=1}^{M} \widehat{n}_i$.

Theorem 4. *For each i, $1 \le i \le M$, let $\lambda_i(t)$ be continuous scalar-valued functions for $t \ge t_0$. Let a pair of indices h, l be given such that $1 \le h \le M$ and $1 \le l \le n_{hk}$ for some $k \in \{1, 2, \dots, d_h\}$. Assume that there exists a continuous function $\beta_h(t) \ge 1$ for $t \ge t_0$ such that for each $1 \le i \le M$ either*

(17)
$$\left.\begin{aligned} &\exp\left[\int_{t_0}^{t}\left(\mathrm{Re}\,\{\lambda_i(\tau)-\lambda_h(\tau)\}+\frac{n_i-l}{\tau}\right)d\tau\right] \longrightarrow 0 \text{ as } t\to\infty \text{ and} \\ &\exp\left[\int_{s}^{t}\left(\mathrm{Re}\,\{\lambda_i(\tau)-\lambda_h(\tau)\}+\frac{n_i-l}{\tau}\right)d\tau\right] \le \beta_h(s) \text{ for all } t_0 \le s \le t \end{aligned}\right\}$$

or

(18)
$$\exp\left[\int_{s}^{t}\left(\mathrm{Re}\,\{\lambda_i(\tau)-\lambda_h(\tau)\}+\frac{1-l}{\tau}\right)d\tau\right] \le \beta_h(s) \text{ for all } t_0 \le t \le s.$$

Furthermore, assume that $R(t)$ is a continuous $d \times d$ matrix for $t \ge t_0$ and that, for all $1 \le i \le M$,

(19) $$t^{n_i-1}\beta_h(t) r_{jm}(t) \in L^1[t_0, \infty),$$

for $\widehat{n}_1 + \ldots + \widehat{n}_{i-1} + 1 \le j \le \widehat{n}_1 + \ldots + \widehat{n}_i$ and all $1 \le m \le d$, where n_i, $\widehat{n}_i$ were defined in (16). Then for each k such that $n_{hk} \ge l$, $1 \le k \le d_h$, there exists a solution y of the linear differential system

(20) $$y' = \left[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus J_{ij}(t) + R(t)\right] y,$$

satisfying

(21) $$y = [e_\kappa + \mathrm{o}(1)]\,\frac{t^{l-1}}{(l-1)!}\,\exp\left[\int^{t} \lambda_h(\tau)\,d\tau\right], \text{ as } t \to \infty,$$

where

$$\kappa = \sum_{\mu=1}^{h-1} \hat{n}_\mu + \sum_{\nu=1}^{j-1} n_{h\nu} + 1.$$

Proof. In (20), make the change of variables

$$y = \left[\sum_{i=1}^{M} \sum_{j=1}^{d_i} \oplus \Phi_{ij}(t) \right] z, \tag{22}$$

where
(23)

$$\Phi_{ij}(t) = \exp[N_{n_{ij}} t] = \begin{pmatrix} 1 & t & t^2/2! & \cdots & (t^{n_{ij}-1})/(n_{ij}-1)! \\ 0 & 1 & t & & (t^{n_{ij}-2})/(n_{ij}-2)! \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & t \\ 0 & & & 0 & 1 \end{pmatrix}.$$

Then $\Phi'_{ij}(t) = N_{n_{ij}} \Phi_{ij}(t)$. It follows from (20) that

$$z' = \left[\sum_{i=1}^{M} \oplus \lambda_i(t) I_{\widehat{n}_i} + \widetilde{R}(t) \right] z, \tag{24}$$

with

$$\widetilde{R}(t) = \left[\sum_{i=1}^{M} \sum_{j=1}^{d_i} \oplus \Phi_{ij}(t) \right]^{-1} R(t) \left[\sum_{i=1}^{M} \sum_{j=1}^{d_i} \oplus \Phi_{ij}(t) \right].$$

Noting that both $\Phi_{ij}(t)$ and $\Phi_{ij}^{-1}(t)$ are of order $t^{n_{ij}-1}$ for all i and j, it is straightforward to show that $\widetilde{R}(t) = \mathrm{O}(t^{2(n_0-1)}|R(t)|)$ as $t \to \infty$, where n_0 was defined in (16). Hence, while the leading matrix in (24) is diagonal, the magnitude of the perturbation has been increased significantly.

To reduce the magnitude of the perturbation, put

$$z = \left[\sum_{i=1}^{M} \sum_{j=1}^{d_i} \oplus D_{n_{ij}}^{-1}(t) \right] w, \tag{25}$$

where $D_{n_{ij}}(t) = \operatorname{diag}\{1, t, \dots, t^{n_{ij}-1}\}$. Since $D_{ij}(t)\Phi_{ij}(t)D_{ij}^{-1}(t) = e^{N_{ij}}$ (where $e^{N_{ij}}$ was given in (23) for $t = 1$) it follows that

$$(26)\quad w' = \left[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus \left\{\lambda_i(t) I_{n_{ij}} + \frac{1}{t}\operatorname{diag}\{0, 1, \dots, n_{ij}-1\}\right\} + \widehat{R}(t)\right] w.$$

Here $\widehat{R}$ is, up to multiplication by a constant matrix from the right and from the left, given by

$$\left[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus D_{n_{ij}}\right] R \left[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus D_{n_{ij}}\right]^{-1},$$

whose (j, m) element has order

$$\mathrm{O}\left(t^{n_i - 1} r_{jm}\right) \text{ for } \widehat{n}_1 + \dots + \widehat{n}_{i-1} + 1 \le j \le \widehat{n}_1 + \dots + \widehat{n}_i,\ 1 \le m \le d,$$

for some $1 \le i \le M$. Hence, by (19), it follows that

$$(27)\qquad \beta_h(t)\widehat{R}(t) \in L^1[t_0, \infty).$$

For the fixed pair of indices h, l with $1 \le h \le M$ and $1 \le l \le n_{hk}$ for some $k \in \{1, \dots, d_h\}$, we make in (26) the change of variables

$$(28)\qquad \begin{aligned} w(t) &= \xi(t) \exp\left[\int^t \left\{\lambda_h(\tau) + \frac{l-1}{\tau}\right\} d\tau\right] \\ &= \xi(t) t^{l-1} \exp\left[\int^t \lambda_h(\tau)\, d\tau\right]. \end{aligned}$$

Then it follows from (26) that

$$(29)\qquad \begin{aligned} \xi' = \Bigg[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus \Bigg\{ \left(\lambda_i(t) - \lambda_h(t) - \frac{l-1}{t}\right) I_{n_{ij}} \\ + \frac{1}{t}\operatorname{diag}\{0, 1, \dots, n_{ij} - 1\}\Bigg\} + \widehat{R}(t)\Bigg] \xi. \end{aligned}$$

The corresponding unperturbed system

$$(30)\quad v' = \left[\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus \left\{\left(\lambda_i(t) - \lambda_h(t) - \frac{l-1}{t}\right) I_{n_{ij}} + \frac{1}{t}\operatorname{diag}\{0, 1, \dots, n_{ij}-1\}\right\}\right] v,$$

has bounded solutions in the form of Euclidean unit vectors if $i = h$ and, moreover, for those values of j where $0 \le l-1 \le n_{hj}-1$. Then the $1/t$ term vanishes at the lth position in the (h,j)th Jordan blocks. Hence, for each j with $n_{hj} \ge l$, there exists a bounded solution of (30)

$$v = e_\rho \text{ with } \rho = \sum_{\mu=1}^{h-1} \widehat{n}_\mu + \sum_{\nu=1}^{j-1} n_{h\nu} + l,$$

and we want to apply Theorem 2 with the goal of finding an asymptotically constant solution $\xi = e_\rho + \mathrm{o}(1)$ of (29) as $t \to \infty$.

Rewriting (30) as

$$v' = \mathrm{diag}\,\{\mu_1(t), \dots, \mu_d(t)\}v, \quad \mu_\rho(t) \equiv 0,$$

one finds for the differences, $1 \le k \le d$,

$$\mu_k(t) - \mu_\rho(t) = \mu_k(t) = \lambda_i(t) - \lambda_h(t) + \frac{\nu_{ij} - (l-1)}{t},$$
$$0 \le \nu_{ij} \le n_{ij} - 1 \le n_i - 1,$$

for some $1 \le i \le M$. By (17) and (18), $\mu_k(t) - \mu_\rho(t)$ satisfies the dichotomy conditions (7) and (8), respectively; hence, Theorem 2 implies the existence of a solution ξ of (29) satisfying

$$\xi(t) = e_\rho + \mathrm{o}(1) \quad \text{as } t \to \infty.$$

Therefore, by (22), (25) and (28), there exists a solution y of (20) satisfying

$$y = \left(\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus \Phi_{ij}(t)\right)\left(\sum_{i=1}^{M}\sum_{j=1}^{d_i} \oplus D_{n_{ij}}^{-1}(t)\right) t^{l-1}$$
$$\times \exp\left[\int^t \lambda_h(\tau)\,d\tau\right][e_\rho + \mathrm{o}(1)], \text{ as } t \to \infty,$$

which by basic matrix multiplication can be rewritten as (21). □

Theorem 4 yields the same result that was first established by Chiba and Kimura [**5**, Theorem 5.1] but under simplified hypotheses. In particular, various monotonicity requirements from their result are not needed in Theorem 4.

We conclude this section with a result on general linear differential systems which do not possess any particular structure, e.g., Jordan form. Here $|\cdot|$ denotes any submultiplicative matrix norm.

Theorem 5. *Let $A(t)$ and $R(t)$ be continuous $d \times d$-matrices for $t \geq t_0$. Let $X(t)$ be the fundamental matrix of* (1) *satisfying $X(t_0) = I$. Assume that*

$$\rho(t) := |X^{-1}(t)R(t)X(t)| \in L^1[t_0, \infty) \tag{31}$$

and

$$|X(t)||X^{-1}(t)| \int_t^\infty \rho(\tau)\, d\tau \longrightarrow 0 \quad \text{as } t \to \infty. \tag{32}$$

Then (2) *has a fundamental solution matrix satisfying*

$$Y(t) = [I + \mathrm{o}(1)]\, X(t) \quad \text{as } t \to \infty. \tag{33}$$

Proof. In (2), we make the change of variables

$$y = X(t)\widetilde{y}. \tag{34}$$

Then (2) implies that

$$\widetilde{y}\,' = X^{-1}(t)R(t)X(t)\widetilde{y},$$

which we consider as a perturbation of the trivial system $z' = 0$. Since the zero matrix satisfies Levinson's dichotomy conditions, (31) and Levinson's theorem, see e.g., [**8**, Theorem 1.3.1], imply the existence of a fundamental matrix of the form $\widetilde{Y} = I + \mathrm{o}(1)$ as $t \to \infty$. However, we prefer to apply [**4**, Theorem 11] which gives the existence of a fundamental matrix of the form

$$\widetilde{Y} = I + \mathrm{O}\bigg(\int_t^\infty \rho(\tau)\, d\tau\bigg).$$

By (34), (2) has a fundamental matrix

$$Y(t) = X(t)\widetilde{Y}(t) = \left[I + X(t)\,\mathrm{O}\left(\int_t^\infty \rho(\tau)\,d\tau\right)X^{-1}(t)\right]X(t) \text{ as } t \to \infty,$$

which satisfies (33) by hypothesis (32). □

We wish to emphasize that the above conditions on $\rho(t)$ are rather restrictive. The reason is the lack of any assumptions on $A(t)$, for example, being diagonal with a fundamental matrix satisfying some dichotomy condition.

3. Difference equations. The fruitful interplay between results for systems of linear differential and difference equations is well known and has led to a better understanding of the parallel theories. Benzaid and Lutz [**2**] proved an analogue of Levinson's theorem for difference equations and also constructed an example showing that, in the absence of a Levinson type of dichotomy condition for diagonal systems, summable perturbations do not necessarily preserve strong asymptotic equivalence (also defined analogously). Here we wish to point out that some results corresponding to those in Section 2 carry over to linear difference systems satisfying dichotomy conditions weaker than Levinson's.

Theorem 6. *For $d \times d$ matrices $A(n)$ and $R(n)$, consider*

$$x(n+1) = A(n)x(n), \quad n \geq n_0 \tag{35}$$

where $A(n)$ are nonsingular for all $n \geq n_0$, and the perturbed system

$$y(n+1) = [A(n) + R(n)]\,y(n), \quad n \geq n_0. \tag{36}$$

Let $X(n)$ be the fundamental matrix of (35) *such that $X(n_0) = I$. Suppose that a projection matrix P and a scalar-valued sequence $\{\beta_n\}$ exist with $\beta(n) \geq 1$ for all $n \geq n_0$ such that*

$$\left.\begin{aligned} |X(n)PX^{-1}(k+1)| &\leq \beta(k) \quad \text{for all } n_0 \leq k < n \\ |X(n)[I-P]X^{-1}(k+1)| &\leq \beta(k) \quad \text{for all } n_0 \leq n \leq k \end{aligned}\right\}. \tag{37}$$

Suppose that

$$\sum_{n=n_0}^{\infty} \beta(n)|R(n)| < \infty. \tag{38}$$

Then there exists a one-to-one and bicontinuous correspondence between the bounded solutions of (35) *and* (36). *Moreover, the difference between corresponding bounded solutions of* (35) *and* (36) *tends to zero as* $n \to \infty$ *if* $X(n)P \to 0$ *as* $n \to \infty$.

The proof is analogous to the proof of Theorem 1, and we will just outline the necessary modifications. Let l_∞ be the Banach space of bounded d-dimensional vector-valued sequences with the supremum norm

$$\|y\| = \sup_{n \geq n_1} |y(n)|,$$

where n_1 is picked sufficiently large such that

$$\theta = \sum_{n=n_1}^{\infty} \beta(n)|R(n)| < 1.$$

For $n \geq n_1$, define an operator T_Δ acting on l_∞ by

$$\begin{aligned}(T_\Delta y)(n) = &\sum_{k=n_1}^{n-1} X(n)PX^{-1}(k+1)R(k)y(k)\\ &- \sum_{k=n}^{\infty} X(n)[I-P]X^{-1}(k+1)R(k)y(k).\end{aligned}$$

As in the proof of Theorem 1, one can show that $T_\Delta : l_\infty \to l_\infty$ is a contraction and hence, given $x \in l_\infty$,

$$x = y - T_\Delta y \tag{39}$$

has a unique solution $y \in l_\infty$ which can be shown to be a solution of (36) if x is a solution of (35). Equation (39) therefore establishes a one-to-one correspondence between the bounded solutions of (35) and (36) for $n \geq n_1$. The bicontinuity of this correspondence as well as that the difference between corresponding solutions tends to zero if $X(n)P \to 0$ as $n \to \infty$ is then shown in a similar manner to the proof of Theorem 1. □

As in the case of differential equations, the existence of such a sequence $\beta(k)$ is easily shown, e.g., (corresponding to the projection matrix $P = 0$)

$$\beta(k) := \sup_{n_0 \le n \le k+1} |X(n)X^{-1}(k+1)|.$$

We continue with a discrete version of Theorem 2.

Theorem 7. *Let $\Lambda(n) = \operatorname{diag}\{\lambda_1(n), \dots, \lambda_d(n)\}$ be a diagonal and invertible $d \times d$ matrix for $n \ge n_0$. Fix $h \in \{1, 2, \dots, d\}$. Assume that a scalar sequence $\beta_h(n) \ge 1$ exists for $n \ge n_0$ such that for each $1 \le i \le d$ either*

$$\left.\begin{array}{ll} \displaystyle\prod_{k=n_0}^{n} \left|\frac{\lambda_i(k)}{\lambda_h(k)}\right| \longrightarrow 0 & \text{as } n \to \infty \\ \text{and} & \\ \displaystyle\prod_{k=n_1+1}^{n_2-1} \left|\frac{\lambda_i(k)}{\lambda_h(k)}\right| \le \beta_h(n_1) & \text{for all } n_0 \le n_1 < n_2 \end{array}\right\} \tag{40}$$

or

$$\prod_{k=n_1}^{n_2} \left|\frac{\lambda_i(k)}{\lambda_h(k)}\right| \ge \frac{1}{\beta_h(n_2)} > 0 \quad \text{for all } n_0 \le n_1 \le n_2. \tag{41}$$

Furthermore, assume that $R(n)$ is a $d \times d$ matrix for $n \ge n_0$ such that

$$\sum_{n=n_0}^{\infty} \frac{\beta_h(n)|R(n)|}{|\lambda_h(n)|} < \infty \text{ for this fixed value of } h. \tag{42}$$

Then the linear differential system

$$y(n+1) = [\Lambda(n) + R(n)]\, y(n) \tag{43}$$

has a solution satisfying

$$y_h(n) = [e_h + \mathrm{o}(1)] \prod_{k=n_0}^{n-1} \lambda_h(k) \text{ as } n \to \infty, \tag{44}$$

where e_h is the hth column of the identity matrix.

Proof. For a fixed $h \in \{1, 2, \dots, d\}$, put

$$y(n) = z(n) \prod_{k=n_0}^{n-1} \lambda_h(k).$$

Then (43) implies that

$$z(n+1) = \frac{1}{\lambda_h(n)} [\Lambda(n) + R(n)]\, z(n). \tag{45}$$

We also consider the unperturbed shifted system

$$w(n+1) = \frac{\Lambda(n)}{\lambda_h(n)}\, w(n), \tag{46}$$

and we will apply Theorem 6 to these shifted system. For that purpose, let $P = \operatorname{diag}\{p_1, p_2, \dots, p_d\}$, where

$$p_j = \begin{cases} 1 & \text{if } (i,h) \text{ satisfies (40)} \\ 0 & \text{if } (i,h) \text{ satisfies (41).} \end{cases}$$

Then (46) satisfies the dichotomy condition (37) with $\beta(k)$ replaced by $\beta_h(k)$, and $W(n)P \to 0$ as $n \to \infty$. Now, by Theorem 6 and since $w_h(n) = e_h$ is a bounded solution of (46), there exists a bounded solution $z_h(n)$ of (45) with

$$z_h(n) = w_h + \varepsilon_h(n) = e_h + \varepsilon_h(n) \text{ and } \lim_{n\to\infty} \varepsilon_h(n) = 0.$$

Therefore, (43) has a solution of the form (44). □

In Theorem 7, such a sequence $\beta_h(n)$ always exists. For example, for this fixed value of h, one may put

$$\beta_h(n) = \sup_{1\le i\le d} \sup_{n_0\le k\le n} \prod_{l=k}^{n} \left|\frac{\lambda_h(l)}{\lambda_i(l)}\right|.$$

Then (41) holds for all $1 \le i \le d$. However, as for differential equations, this choice usually leads to a nonoptimal, i.e., too large, $\beta(n)$.

4. Examples. As we mentioned in the introduction, there are examples showing that with weaker dichotomies than Levinson's, absolutely integrable perturbations do not necessarily preserve strong asymptotic equivalence. Such an example was privately communicated to one of the authors by Devinatz [**7**] and another one can be found in Eastham [**8**, page 10]. Devinatz's example is essentially the same as one found in Coppel [**6**, page 71]. That example is also equivalent to one given by Perron [**13**], but Perron and Coppel were concerned with a related question concerning perturbations of asymptotically, but not uniformly, stable systems. There, even exponentially small perturbations of asymptotically stable systems may be unstable. The same class of examples applies to both questions because of a relation between uniform stability and Levinson's dichotomy condition for linear systems.

We will now show how our results apply to some modified examples in which we introduce parameters to determine more precisely how the rate of decay in the perturbation is related to preservation or not of strong asymptotic equivalence.

Example 8 (Modified Coppel/Devinatz/Perron system). Consider, for $t \ge t_0 \ge 1$ and positive numbers a and b,

$$y' = \begin{bmatrix} -a & 0 \\ e^{-bt} & \sin\log t + \cos\log t - 2a \end{bmatrix} y \tag{47}$$

and the unperturbed diagonal system

$$x' = \begin{bmatrix} -a & \\ & \sin\log t + \cos\log t - 2a \end{bmatrix} x. \tag{48}$$

Since

$$\text{Re}\,[\lambda_2(t) - \lambda_1(t)] = \sqrt{2}\sin(\log t + \pi/4) - a,$$

Levinson's dichotomy conditions hold for $a \geq \sqrt{2}$, and hence Levinson's theorem, see e.g., [**8**, Theorem 1.3.1], implies that (47) and (48) are for $a \geq \sqrt{2}$ strongly asymptotically equivalent for any value of $b > 0$.

We will now consider what happens when $0 < a < \sqrt{2}$. It turns out that there exist explicit constants γ, respectively $\widehat{\gamma}$, dependent upon the value of a (see (54), respectively (58)) such that strong asymptotic equivalence to the unperturbed system holds if $b > \gamma$, respectively $\widehat{\gamma}$, but not for $b < \gamma$, respectively $\widehat{\gamma}$.

Case 1. $0 < a < 1$. We first want to apply Theorem 2 with $h = 1$ to find a lower bound on b for (47) and (48) to be strongly asymptotically equivalent (note that because of the lower triangular structure of the coefficient matrix in (47), we do not need to consider the case $h = 2$). Since

$$\exp\left[\int_{t_0}^{t} \text{Re}\,\{\lambda_2(\tau) - \lambda_1(\tau)\}\, d\tau\right] = c\exp\left[t(\sin\log t - a)\right] \neq \text{o}(1) \text{ as } t \to \infty,$$

we consider (8) for $t_0 \leq t \leq s$. Looking for a $\beta_1(s)$ in the special form $\beta_1(s) = \exp[\gamma s]$ for some positive γ, (8) is satisfied if we can find a γ such that

$$t(\sin\log t - a) - s(\sin\log s - a) \leq \gamma s, \quad t_0 \leq t \leq s. \tag{49}$$

Using $\sin\log t \leq 1$ and $t \leq s$, a first estimate might be

$$\begin{aligned} &t(\sin\log t - a) - s(\sin\log s - a) \\ &\qquad \leq t(1-a) - s(\sin\log s - a) \leq s(1 - \sin\log s) \leq 2s, \end{aligned}$$

but this value $\gamma = 2$, independent of the particular value of $a \in (0, 1)$, is unnecessarily large.

To obtain a better estimate, we want to find, for fixed $s > t_0$, an upper bound for

$$f(t) := t(\sin\log t - a) \quad t_0 \leq t \leq s.$$

Now $f(t)$ takes on its maxima when

$$t = T_n = \exp\left[3\pi/4 - \arcsin\left(a/\sqrt{2}\right) + 2n\pi\right], \quad n \in \mathbf{Z}. \tag{50}$$

We put

$$\eta = \frac{3\pi}{4} - \arcsin\left(\frac{a}{\sqrt{2}}\right), \tag{51}$$

and it is straightforward to show that

$$\sin\eta - a = \frac{1}{2}\left(\sqrt{2-a^2} - a\right), \tag{52}$$

which is positive for $0 < a < 1$ and negative for $1 < a \le \sqrt{2}$. Let $t_0 = T_0$ and fix the integer $n = n(s) \ge 0$ such that $T_n < s \le T_{n+1}$. Since $f(T_n) = T_n(\sin\eta - a)$, $f(t)$ has increasing local maxima for $0 < a < 1$, and it follows that $f(t) \le \max\{f(T_n), f(s)\}$ for $t_0 \le t \le s$. Thus, $f(t) - f(s) \le \max\{f(T_n) - f(s), 0\}$ and, by (49) it is sufficient to find a positive γ such that

$$\max\{f(T_n) - f(s), 0\} \le \gamma s, \quad T_n < s \le T_{n+1}.$$

We want to find the minimal positive γ satisfying

$$f(T_n) - f(s) = T_n(\sin\eta - a) - s(\sin\log s - a) \le \gamma s, \; T_n < s \le T_{n+1}.$$

Dividing by T_n and putting

$$\tau = \frac{s}{T_n}, \tag{53}$$

we want to find the smallest γ such that

$$\sin\eta - a - \tau[\sin(\log\tau + \eta) - a] \le \gamma\tau, \quad 1 < \tau \le e^{2\pi}.$$

Hence, one can put

$$\gamma = \gamma(a) = \sup_{1<\tau\le e^{2\pi}} \frac{\sin\eta - a}{\tau} + a - \sin(\eta + \log\tau), \tag{54}$$

and we will show below, see (57), that $\gamma(a) > a+1$ and is therefore positive. Theorem 2 now implies that (47) and (48) are strongly asymptotically equivalent if $\beta_1(t)R(t) = \exp[(\gamma(a)-b)t] \in L^1[t_0,\infty)$, i.e.,

$$b > \gamma(a).$$

Next, we want to find an upper bound for the values of b for which there is no strong asymptotic equivalence between (47) and (48). For that purpose, observe that solving (47) by quadrature shows that (47) is strongly asymptotically equivalent to (48) if and only if

$$\int_T^t e^{t(\sin\log t - 2a)+s(a-b-\sin\log s)}\,ds = \mathrm{o}\left(e^{-at}\right), \quad (t\to\infty)$$

or

$$(55)\quad I(t) := \int_T^t e^{t(\sin\log t - a)+s(a-b-\sin\log s)}\,ds = \mathrm{o}(1), \quad (t\to\infty)$$

for an appropriate choice of $T \in \{t_0,\infty\}$. We also assume here that $b > a+1$, and consequently we may choose $T=\infty$ since the improper integral converges. Recalling that $f(t) = t(\sin\log t - a)$ takes on its maxima when $t = T_n$, where T_n was defined in (50), we want to find an upper bound for b such that

$$I(T_n) \neq \mathrm{o}(1) \quad \text{as } n\to\infty.$$

Now, with η and τ given in (51) and (53), respectively,

$$\begin{aligned}|I(T_n)| &= e^{T_n(\sin\eta - a)}\int_{T_n}^{\infty} e^{s(a-b-\sin\log s)}\,ds\\ &\geq e^{T_n(\sin\eta - a)}\int_{T_n}^{T_{n+1}} e^{s(a-b-\sin\log s)}\,ds\\ &= T_n\int_1^{e^{2\pi}} \exp\left[T_n\left\{\sin\eta - a + \tau(a-b-\sin(\eta+\log\tau))\right\}\right]\,d\tau.\end{aligned}$$

If there exists a $b > a+1$ such that

$$(56)\qquad \sup_{1\leq\tau\leq\exp[2\pi]}\{\sin\eta - a + \tau(a-b-\sin(\eta+\log\tau))\} > 0,$$

then there exists an $\varepsilon_0 > 0$ and $1 \leq \tau_1 < \tau_2 \leq \exp[2\pi]$ such that $\{\sin\eta - a + \tau(a - b - \sin(\eta + \log\tau))\} > \varepsilon_0$ for $\tau_1 \leq \tau \leq \tau_2$, and hence

$$\begin{aligned} |I(T_n)| &\geq T_n \int_{\tau_1}^{\tau_2} \exp\left[T_n \{\sin\eta - a + \tau(a - b - \sin(\eta + \log\tau))\}\right] d\tau \\ &\geq T_n \int_{\tau_1}^{\tau_2} \exp[T_n \varepsilon_0]\, d\tau \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, sufficient for $I(T_n) \neq \mathrm{o}(1)$ as $n \to \infty$ is that $b > a+1$ exists such that (56) holds, or, equivalently, that

$$b < \gamma(a),$$

where $\gamma(a)$ was given in (54).

To justify that such a $b \in (a+1, \gamma(a))$ exists, we note that with τ_0 defined by $\eta + \log(\tau_0) = 3\pi/2$ (hence $\tau_0 \in (\exp(3\pi/4), \exp(\pi)] \subset [1, \exp(2\pi)]$),

$$\gamma(a) \geq \frac{\sin\eta - a}{\tau_0} + a - \sin(\eta + \log\tau_0) = \frac{\sin\eta - a}{\tau_0} + a + 1, \tag{57}$$

and it suffices to show that $\sin\eta - a > 0$. But this follows directly from (52) for $0 < a < 1$, which completes our study of Case 1.

Case 2. $1 < a \leq \sqrt{2}$**.** This case is treated similarly, and we will again use T_n, η and τ as defined in (50), (51) and (53), respectively. For $a > 1$,

$$\exp\left[\int_{t_0}^{t} \mathrm{Re}\ \{\lambda_2(\tau) - \lambda_1(\tau)\}\ d\tau\right] \longrightarrow 0 \quad \text{as } t \to \infty;$$

hence, we require a function $\beta_1(s)$ such that

$$\exp\left[\int_{s}^{t} \{\lambda_2(\tau) - \lambda_1(\tau)\}\ d\tau\right] \leq \beta_1(s) \quad t_0 \leq s \leq t.$$

Choosing again $\beta_1(s)$ of the special form $\exp[\widehat{\gamma} s]$, we seek a positive number $\widehat{\gamma}$ such that

$$f(t) - f(s) = t(\sin\log t - a) - s(\sin\log s - a) \leq \widehat{\gamma} s, \quad t_0 \leq s \leq t.$$

For a fixed $s \geq t_0 = T_0$, let $T_{n-1} \leq s < T_n$ for some positive integer n. Since $f(T_n) > f(T_{n+1})$ for $a > 1$, it follows that $f(t) \leq \max\{f(s), f(T_n)\}$ for $t \geq s$, and it suffices to find a positive $\widehat{\gamma}$ such that

$$T_n(\sin\eta - a) - s(\sin\log s - a) \leq \widehat{\gamma} s,$$

or, after dividing by T_n and again putting $\tau = s/T_n$,

$$(58) \qquad \widehat{\gamma} = \widehat{\gamma}(a) = \sup_{e^{-2\pi} \leq \tau < 1} \frac{\sin\eta - a}{\tau} + a - \sin(\eta + \log\tau).$$

Therefore (47) and (48) are strongly asymptotically equivalent by Theorem 2 if

$$b > \widehat{\gamma}(a).$$

To find an upper bound for the values of b for which there is no strong asymptotic equivalence between (47) and (48), we consider $I(t)$ given in (55) with $T = t_0$. Then, for $n \geq 1$,

$$\begin{aligned} I(T_n) &= e^{T_n(\sin\eta - a)} \int_{t_0}^{T_n} e^{s(a-b-\sin\log s)}\, ds \\ &\geq e^{T_n(\sin\eta - a)} \int_{T_{n-1}}^{T_n} e^{s(a-b-\sin\log s)}\, ds \\ &= T_n \int_{e^{-2\pi}}^{1} \exp\left[T_n\left\{\sin\eta - a + \tau(a - b - \sin(\eta + \log\tau))\right\}\right] d\tau, \end{aligned}$$

and sufficient for $I(T_n) \neq \mathrm{o}(1)$ as $n \to \infty$ is that

$$b < \widehat{\gamma}(a),$$

where $\widehat{\gamma}(a)$ was given in (58).

With some more work, it can be shown that $\widehat{\gamma}(a)$ is positive for $1 < a < \sqrt{2}$ and that $\widehat{\gamma}(\sqrt{2}) = 0$, which coincides with the above-mentioned fact that Levinson's theorem implies that (47) and (48) are strongly asymptotically equivalent for any value of $b > 0$ if $a = \sqrt{2}$.

Finally, we discuss an example due to Eastham [**8**, page 10] which again shows the "necessity" of dichotomy conditions and apply our results to determine which perturbations lead to strong asymptotic equivalence.

Example 9 (Modified Eastham example). For $t \geq 0$, consider

$$y' = \left[\begin{pmatrix} 0 & \\ & \rho'(t)/\rho(t) \end{pmatrix} + \begin{pmatrix} 0 & r(t) \\ 0 & 0 \end{pmatrix}\right] y, \quad \rho(t) = t^2(1 - \sin t) + 1. \tag{59}$$

This example is due to Eastham [**8**, page 10] who showed that the unperturbed diagonal system

$$x' = \begin{bmatrix} 0 & \\ & \rho'(t)/\rho(t) \end{bmatrix} x \tag{60}$$

does not satisfy the dichotomy conditions of Levinson's theorem, see e.g., [**8**, Theorem 1.3.1], and that for the absolutely integrable perturbation $r(t) = 1/t^2$, (59) is not strongly asymptotically equivalent to (60).

To find a measure on the magnitude of the perturbation $r(t)$ to ensure strong asymptotic equivalence, it suffices to find in Theorem 2 a suitable function $\beta_2(t)$ (due to the triangular structure of the perturbation in (59), $\beta_1(t)$ is not needed).

Since $\exp[\int_0^t \operatorname{Re}\{\lambda_1(\tau) - \lambda_2(\tau)\}\, d\tau]$ does not go to zero as $t \to \infty$, one needs to find $\beta_2(t)$ such that (8) holds. Observe that, for $0 \leq t \leq s$,

$$\begin{aligned} \exp\left[\operatorname{Re}\int_s^t (\lambda_1 - \lambda_2)\, d\tau\right] &= \frac{\rho(s)}{\rho(t)} = \frac{s^2(1-\sin s)+1}{t^2(1-\sin t)+1} \\ &\leq s^2(1 - \sin s) + 1 \leq 2s^2 + 1, \end{aligned}$$

which one can choose for $\beta_2(s)$. Assuming then, for example, that

$$r(t) = t^{-p}, \quad p = 3 + \varepsilon \text{ for some } \varepsilon > 0,$$

(thus, $\beta_2(t)r(t) \in L^1[t_0, \infty)$), Theorem 2 implies that (59) is strongly asymptotically equivalent to (60). In fact, integration of (59) by quadrature (integrating from t to infinity) shows that for this choice of $r(t)$, there exists a fundamental matrix, satisfying, as $t \to \infty$,

$$Y(t) = \begin{bmatrix} 1 & \mathrm{O}(t^{-\varepsilon}) \\ 0 & \rho(t) \end{bmatrix} = [I + \mathrm{o}(1)] \begin{bmatrix} 1 & 0 \\ 0 & \rho(t) \end{bmatrix}.$$

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