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EXPLICIT ESTIMATE ON PRIMES BETWEEN CONSECUTIVE CUBES

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ABSTRACT. We give an explicit form of Ingham's theorem on primes in the short intervals and show that there is at least one prime between every two consecutive cubes x^3 and $(x+1)^3$ if $\log\log x \geq 15$.

1. Introduction. Studies about certain problems in number theory are often connected to those about the distribution of prime numbers; problems about the distribution of primes are among the central ones in number theory. One problem concerning the distribution of primes is the distribution of primes in certain intervals. For example, Bertrand's postulate asserts that there is a number B such that, for every $x > 1$, there is at least one prime number between x and Bx. If the interval $[x, Bx]$ is replaced by a "short interval" $[x, x + x^\theta]$, then the problem is more difficult.

In 1930, Hoheisel showed that there is at least one prime in the above mentioned "short interval" with $\theta = 1-(1/33,000)$ for sufficiently large x's, see [**13**]. Ingham [**15**], in 1941, proved that there is at least one prime in $[x, x+x^{3/5+\varepsilon}]$, where ε is an arbitrary positive number tending to zero whenever x is tending to infinity, for "sufficiently large" x's. This implies that there is at least one prime between two consecutive cubes if the numbers involved are "large enough." One of the better results in this direction, conjectured by using the Riemann hypothesis, is that there is at least one prime between $[x, x+x^{1/2+\varepsilon}]$ for "sufficiently large" x's. The latter has not been proved or disproved, though better results than Hoheisel's and Ingham's are available. For example, one may see [**2, 3, 12, 15, 17, 18, 19, 26, 28**].

These kinds of results would have many useful applications if they were "explicit" (with all constants being determined explicitly). For references in other directions with explicit results, one can see [**4,**

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8, 22–25]. To figure out the "sufficiently large" x's related to θ as mentioned above, one needs to investigate the proof in a "slightly different" way. As a starting step in this direction, we study the distribution of primes between consecutive cubes. In this article, we give an explicit form of Ingham's theorem; specifically, we show that there is at least one prime between consecutive cubes if the numbers involved are larger than the cubes of x_0 where $x_0 = \exp(\exp(15))$, and we also set $T_0 = \exp(\exp(18))$ throughout this paper accordingly.

Our main task is to prove the density theorem or to estimate the number of zeros in the strip $\sigma > 1/2$ for the Riemann zeta function, see the following Theorem 1. We let $\beta = \Re(\rho)$ and $I_\beta(u)$ be the unit step function at the point $u = \beta$; that is, $I_\beta(u) = 1$ for $0 \le u \le \beta$ and $I_\beta(u) = 0$ for $\beta < u \le 1$. One defines $N(u,T) := \sum_{0 \le \Im(\rho) < T} I_\beta(u)$ and $N(T) := N(0,T)$.

Theorem 1. *Let* $5/8 \le \sigma < 1$ *and* $T \ge T_0$. *One has*

$$N(\sigma,T) \le C_D T^{(8(1-\sigma))/3} \log^5 T,$$

where $C_D := 453472.54$.

Theorem 2. *Let* $x \ge x_0$, $h \ge 3x^{2/3}$ *and* C_D *be defined as in Theorem* 1. *Then*

$$\psi(x+h) - \psi(x) \ge (h/\log x)(1 - \varepsilon(x)),$$

where

$$|\varepsilon(x)| := 3192.34 \exp\left(-\frac{1}{273.79}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right).$$

Theorem 3. *Let* $x \ge \exp(\exp(45))$ *and* $h \ge 3x^{2/3}$. *Then*

$$\pi(x+h) - \pi(x) \ge h\left(1 - 3192.34 \exp\left(-\frac{1}{283.79}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)\right).$$

Corollary. *Let $x \geq \exp(\exp(15))$. Then there is at least one prime between each pair of consecutive cubes x^3 and $(x+1)^3$.*

The proof of Theorem 1 is delayed until Section 5. We shall prove Theorems 2 and 3 in Section 2. The proof of Theorem 2 is based on Theorem 1 and Laudau's approximate formula, which is in Section 6. Then, it is not difficult to prove Theorem 3 from Theorem 2, as shown in Section 2.

2. Proof of Theorems 2 and 3. From [**25**], one has

$$N(T) \leq \frac{T\log T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + 0.137\log T + 0.443\log\log T + 4.350,$$

for $T \geq 2$. The following proposition follows in a straightforward manner.

Proposition 2.1. *For $T \geq 6$, one has $N(T) \leq (T\log T)/(2\pi)$.*

Proposition 2.2. *Let C_D be defined as in Theorem 1. Assume that the Riemann zeta-function does not vanish for $\sigma > 1 - z(t)$. Suppose that $T_0 \leq T < x^{3/8}$. For any $h > 0$, one has*

$$\left|\sum_{|\Im(\rho)|\leq T} \frac{(x+h)^\rho - x^\rho}{\rho}\right| \leq \frac{2C_D T^{(8/3)z(t)}\log x \log^5 T}{x^{z(t)}(\log x - (8/3)\log T)}\, h.$$

Proof. Note that

$$\left|\frac{(x+h)^\rho - x^\rho}{\rho}\right| = \left|\int_x^{x+h} u^{\rho-1}\,\mathbf{d}u\right| \leq hx^{\beta-1},$$

where $\beta = \Re(\rho)$ is the real part of ρ;

$$x^\beta = 1 + \log x \int_0^\beta x^u\,\mathbf{d}u;$$

and

$$\int_0^\beta x^u\,\mathbf{d}u = \int_0^1 x^u I_\beta(u)\,\mathbf{d}u,$$

where $I_\beta(u)$ is the unit step function or $I_\beta(u) = 1$ for $0 \le u \le \beta$ and $I_\beta(u) = 0$ for $\beta < u \le 1$. After interchanging the summation and integration, one has

$$\left|\sum_{|\Im(\rho)|\le T} \frac{(x+h)^\rho - x^\rho}{\rho}\right| \le \frac{h}{x}\sum_{|\Im(\rho)|\le T} x^\beta \le \frac{h}{x}\left(\sum_{|\Im(\rho)|\le T} 1 + \log x \int_0^1 x^u \left(\sum_{|\Im(\rho)|\le T} I_\beta(u)\right)\mathbf{d}u\right). \tag{2.1}$$

If the Riemann zeta-function does not vanish in the region $\sigma > 1 - z(t)$, then the expression in the outmost parenthesis in (2.1) is bounded by

$$2N(0,T) + 2N(0,T)\log x \int_0^{5/8} x^u\,\mathbf{d}u + 2\log x \int_{5/8}^{1-z(t)} x^u N(u,T)\mathbf{d}u. \tag{2.2}$$

Since $T \ge 6$, one can apply Proposition 2.1. The sum of the first two terms in (2.2) is

$$2\left(1+\log x \int_0^{5/8} x^u\,\mathbf{d}u\right)N(0,T) = 2x^{5/8}N(0,T) \le \frac{x^{5/8}T\log T}{\pi}. \tag{2.3}$$

From Theorem 1, one sees that the last term in (2.2) is bounded by

$$\begin{aligned}
&2C_D \log\, x \log^5 T \int_{5/8}^{1-z(t)} x^u T^{(8/3)(1-\sigma)}\,\mathbf{d}u \\
&= 2C_D T^{8/3}\log x \log^5 T \int_{5/8}^{1-z(t)} \left(\frac{x}{T^{8/3}}\right)^u \mathbf{d}u \\
&= \frac{2C_D T^{8/3}\log x \log^5 T}{\log\, x - (8/3)\log T}\left(\left(\frac{x}{T^{8/3}}\right)^{1-z(t)} - \left(\frac{x}{T^{8/3}}\right)^{5/8}\right) \\
&= \frac{2C_D x T^{(8/3)z(t)}\log x \log^5 T}{x^{z(t)}(\log\, x - (8/3)\log T)} - \frac{2C_D x^{5/8} T \log x \log^5 T}{\log\, x - (8/3)\log T}.
\end{aligned} \tag{2.4}$$

One sees that the sum of the upper bound in (2.3) and the second term on the right side in (2.4) is negative. Finally, one combines (2.1) and the first term in the last expression in (2.4) to finish the proof of Lemma 2.1. □

Proof of Theorem 2. From Lemma 9.1 and Proposition 2.2, one sees that

$$\psi(x+h) - \psi(x) = h + h\varepsilon(x),$$

with

$$\begin{aligned} |\varepsilon(x)| &\leq \frac{1}{h}\left(\left|\sum_{|\Im(\rho)|\leq T_u} \frac{(x+h)^\rho - x^\rho}{\rho}\right| + |E(x+h)| + |E(x)|\right) \\ &\leq \frac{2C_D T^{(8/3)z(t)} \log x \log^5 T}{x^{z(t)}(\log x - (8/3)\log T)} + 10.52\,\frac{(x+h)\log^2(x+h)}{hT} \\ &\quad + 66.976\frac{(x+h)\log^2 T}{hT\log x} + 6\frac{\log^2 T}{hx}. \end{aligned} \tag{2.5}$$

Let $3x^{2/3} \leq h$. Also, let

$$T = T(x) := x^{1/3}\exp\left(\frac{1}{256.59}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right),$$

with some undetermined constant $u > 1$. Then,

$$\log T = \frac{1}{3}\log x + \frac{1}{256.59}\left(\frac{\log x}{\log\log x}\right)^{1/3} \leq 0.34\log x,$$
$$\log\log T \leq \log\log x,$$

and

$$T^{8/3} = x^{8/9}\exp\left(\frac{8}{3\times 256.59}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right).$$

From [**10**], it is known that the Riemann zeta function does not vanish for $T \geq 1 - z(T)$ with

$$z(T) = \frac{1}{58.51\log^{2/3} T(\log\log T)^{1/3}}.$$

Let $Z(x) := z(T(x))$. Then

$$Z(x) \geq \frac{1}{28.51 \log^{2/3} x (\log\log x)^{2/3}},$$

$$\left(\frac{x}{T^{8/3}}\right)^{z(T)} \geq \left(\frac{x^{1/9}}{\exp((8/3 \times 256.59)(\log x)/(\log\log x))^{1/3}}\right)^{Z(x)}$$
$$= \exp\left(\frac{1}{256.59}\left(\frac{\log x}{\log\log x}\right)^{1/3} - \frac{8}{3 \times 256.59 \times 28.51 \log^{1/3} x (\log\log x)^{(1 \text{ or } 2)/3}}\right).$$

It follows that for $x \geq \exp(\exp(45))$, the right side in (2.5) is bounded from above by

$$3192.34 \exp\left(-\frac{1}{273.79}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) + 1.76 \exp\left(-\frac{1}{256.6}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) + 2.59 \exp\left(-\frac{1}{256.6}\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) + \frac{0.24 \log^2 + 4.65 \log x + 260.48}{x}.$$

We conclude that Theorem 2 has been proved. □

Proof of Theorem 3. By the definition of $\pi(x)$ and $\psi(x)$, one has

$$\pi(x+h) - \pi(x) = \sum_{x<p\leq x+h} 1 \geq \sum_{x<p\leq x+h} \frac{\log p}{\log x} = \frac{\psi(x+h) - \psi(x)}{\log x} \geq \frac{h}{\log x}(1 - \varepsilon(x)).$$

This finishes the proof of Theorem 3. □

Proof of Corollary. Let $X = x^3$ and $h = (x+1)^3 - x^3$. Then $h \geq 3x^2 = 3X^{2/3}$. By Theorem 3,

$$\pi\left((x+1)^3\right)-\pi\left(x^3\right)\geq\frac{3x^2}{3\log x}\left(1-\varepsilon(x^3)\right)>1.$$

This proves the corollary. □

3. Three auxiliary functions. Three auxiliary functions U_A, V_A and W_A are introduced in this section. For references, one may see [**4, 17, 26**].

Definitions of three auxiliary functions. Let A be a positive integer. Define

$$U_A(s)=\sum_{n=1}^{A}\frac{\mu(n)}{n^s}.$$

Here μ is the Möbius μ-function. Then,

$$V_A(s)=\zeta(s)U_A(s)-1,\quad W_A(s)=1-V_A^2(s).$$

Lemma 3.1. *Let $\nu(n)=\sum_{m\leq A:m|n}\mu(m)$. Then $|\nu(n)|\leq d(n)$ and*

$$V_A(s)=\sum_{n>A}\frac{\nu(n)}{n^s}.$$

Every nontrivial zero of $\zeta(s)$ is a zero of $W_A(s)$.

Lemma 3.2. *One has*

$$|V_A(2+it)|^2\leq\frac{7.9}{A}.$$

If $A\geq 8$, then both $\Re(W_A(2+it))$ and $W_A(2+it)$ do not vanish; if $A\geq 16$, then $|V_A(2+it)|^2<1/2$ and $|W_A(2+it)|>1/2$.

Lemma 3.3. *Let $b_1=5.134$. For $\sigma\geq 1/4$ and $t\geq 3.297$, one has*

$$|V_A(s)|\ll t^{3/2},$$

and

$$|W_A(s)| \le \left(\frac{16}{9}A^{3/4}t^{3/2} + b_1A^{3/4}t^{1/2}\right)\left(\frac{16}{9}A^{3/4}t^{3/2} + b_1A^{3/4}t^{1/2} + 2\right).$$

Proof of Lemma 3.1. Easy. □

Proof of Lemma 3.2. We may assume $A \ge 5$. Observe

$$\begin{aligned}\sum_{n>A}\frac{d(n)}{n^2} &= \zeta(2)\sum_{m>A}\frac{1}{m^2} + \sum_{m\le A}\frac{1}{m^2}\sum_{n>(A/m)}\frac{1}{n^2}\\ &\le \frac{\zeta(2)}{A} + \sum_{m\le A}\frac{1}{m^2}\frac{1}{\lfloor A/m\rfloor}\\ &\le \frac{\zeta(2)}{A} + \sum_{m\le (A/2)+1}\frac{1}{m(A-m)} + \sum_{(A/2)+1<m\le A}\frac{1}{m^2}\\ &\le \frac{\zeta(2)}{A} + \frac{1}{A-1} + \frac{4}{A^2-4} + \frac{\log(A-1)}{A} + \frac{1}{A}\\ &< \frac{2.8}{A}\end{aligned}$$

for $A \ge 5$. □

One needs the following proposition.

Proposition 3.1. *Let $\sigma \ge 1/4$ and $t \ge 3.297$. Then*

$$\zeta(s) = \sum_{n=1}^{\lfloor t^2\rfloor}\frac{1}{n^s} + B(s),$$

where $|B(s)| \le b_1 t^{1/2}$ with $b_1 := 5.134$.

Proof. Note that in [**4, 17, 26**],

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} - s\int_N^\infty \frac{u-[u]}{u^{s+1}}\,\mathbf{d}u + \frac{1}{(s-1)N^{s-1}}, \quad \sigma > 0,\ s \ne 1.$$

From this, we have

$$\left|\zeta(s) - \sum_{n\le N} \frac{1}{n^s}\right| \le \frac{N^{1-\sigma}}{t} + \sqrt{\frac{1}{\sigma_0^2} + \frac{1}{t_0^2}}\,\frac{t}{N^\sigma}, \quad \sigma > 0,\ s \ne 1.$$

Using this identity, Proposition 3.1 follows. □

Proof of Lemma 3.3. Using its definition, one sees that

$$|U_A(s)| \le \sum_{n=1}^{A} \frac{1}{n^\sigma}.$$

If $0 < \sigma < 1$, one gets

$$|U_A(s)| \le \int_0^A \frac{1}{u^\sigma}\, \mathbf{d}u = \frac{A^{1-\sigma}}{1-\sigma};$$

if $\sigma \ge 1$, one has

$$|U_A(s)| \le \sum_{n=1}^{A} \frac{1}{n} \le \log A + 1.$$

For $\sigma \ge 1/4$, one obtains

$$|U_A(s)| \le \max\left\{\frac{4}{3}A^{3/4}, \log\, A + 1\right\} \le \frac{4}{3}A^{3/4}. \tag{3.2}$$

Similarly, one gets

$$\sum_{n=1}^{\lfloor t^2 \rfloor} \frac{1}{n^\sigma} \le \frac{4}{3} t^{3/2}.$$

Combining this with the result in Proposition 3.2, one has

$$|\zeta(s)| \le \frac{4}{3} t^{3/2} + b_1 t^{1/2}, \tag{3.3}$$

for $\sigma \ge 1/4$ and $t \ge 3.297$.

Recalling the definition of $V_A(s)$, one has

$$|V_A(s)| \leq |\zeta(s)||U_A(s)| + 1;$$

from (3.1), one gets

$$|W_A(s)| \leq |\zeta(s)|\,|U_A(s)|(2 + |\zeta(s)||U_A(s)|).$$

Combining these results with (3.2) and (3.3), Lemma 3.3 is proved. □

4. Representing the number of zeros by an integral.

Notation $N_F(\sigma, T)$. Let $F(s)$ be a complex function and $T > 0$. The notation $N_F(\sigma, T)$ expresses the number of zeros in the form $\beta + i\gamma$ for $F(s)$ with $\sigma \leq \beta$ and $0 \leq \gamma < T$.

It is well known that $\zeta(s)$ does not vanish for $\sigma \geq 1$; so we may restrict our discussion to $\sigma < 1$.

Lemma 4.1. *Let $T_1 = 14$ and $A \geq 16$. Then for $\sigma_0 < \sigma < 1$ and $T \geq T_1$, one has*

$$N_\zeta(\sigma; T) \leq \frac{1}{\sigma - \sigma_0}\left(\frac{1}{2\pi}\int_{T_1}^{T} |V_A(\sigma_0 + it)|^2 \,\mathbf{d}t + \frac{(594)16T}{2\pi\, A} + 1 + c_A(T)\right),$$

with

$$\begin{aligned} c_A(T) := {} & \frac{\log\big(((16)/9)A^{3/4}(T + (7/4))^{3/2} + b_1 A^{3/4}(T + (7/4))^{1/2}\big)}{\log(7/6)} \\ & + \frac{\log\big(((16)/9)A^{3/4}(T + (7/4))^{3/2}}{\log(7/6)} \\ & + \frac{+b_1 A^{3/4}(T + (7/4))^{1/2} + 2\big)\log 2}{\log(7/6)}. \end{aligned}$$

Corollary. *Let $A \leq (595)/(594)T$ and $T \geq \exp(\exp(18))$. Then*

$$c_A(T) \leq 29.193 \log T + 11.978.$$

Notation $N_F(\sigma; T, T_1)$. Let $F(s)$, σ and T be as in the last definition. The notation $N_F(\sigma; T, T_1)$ expresses the number of zeros in the form $\beta + i\gamma$ for $F(s)$ with $\sigma \le \beta$ and $T_1 \le \gamma < T$.

Be definition, one sees that $N_F(\sigma; T, T_1) = N_F(\sigma, T) - N_F(\sigma, T_1)$ for any complex function F. Note here, see [**9, 17**], that there is no zero for the Riemann zeta function $\zeta(\sigma + it)$ for $0 \le t \le 14$. If one takes $T_1 = 14$, then $N_\zeta(\sigma; T, T_1) = N_\zeta(\sigma, T)$.

For an analytic function, a zero is isolated, and the number of zeros in any compact region is finite. Fix σ and T. Let ε_1, ε_2, and ε_3 be sufficiently small positive numbers and $\lambda = \sigma - \varepsilon_1$ and $T_2 = T + \varepsilon_2$. One may assume that λ is not the real part and T_2 is not the imaginary part of any zeros for the function $W_A(s)$. Recalling the second part of Lemma 3.1, one gets the following proposition.

Proposition 4.1. *Let $T_1 = 14$ and ε_1 and ε_2 be small positive numbers such that $\lambda = \sigma - \varepsilon_1$ is not the real part and $T_2 = T + \varepsilon_2$ is not the imaginary part of any zero for the function $W_A(s)$. Then*

$$N_\zeta(\sigma, T) \le N_{W_A}(\lambda; T_2, T_1).$$

Let $\sigma_0 < \lambda$. Since $N_{W_A}(\lambda; T_2, T_1)$ is a nonincreasing function of λ, by the definition one sees that

$$N_{W_A}(\lambda; T_2, T_1) \le \frac{1}{\lambda - \sigma_0} \int_{\sigma_0}^{\lambda} N_{W_A}(\rho; T_2, T_1)\, \mathbf{d}\rho.$$

Noting that

$$\int_{\sigma_0}^{\lambda} N_{W_A}(\rho; T_2, T_1)\, \mathbf{d}\rho \le \int_{\sigma_0}^{2} N_{W_\mathbf{A}}(\rho; T_2, T_1)\, \mathbf{d}\rho,$$

one has the next proposition.

Proposition 4.2. *Let $\sigma_0 < \lambda < 1$ and T_2 $(> T_1)$. Assume that λ is not the real part and T_2 is not the imaginary part of any zero for $W_A(s)$. Then*

$$N_{W_A}(\lambda; T_2, T_1) \leq \frac{1}{\lambda - \sigma_0} \int_{\sigma_0}^{2} N_{W_A}(\rho; T_2, T_1)\, \mathbf{d}\rho.$$

Using the arguments in [**26**, pages 213, 220], one gets the following result.

Proposition 4.3. *Let $(1/2) < \sigma_0 < 2$. Assume that T_2 is not the imaginary part of any zero for $W_A(s)$. Also, let $\mathcal{N}_k$ be the number of zeros for $\Re(W_A(s))$ on the segment between $\sigma_0 + it$ and $2 + it$ on the line $t = T_k$ for $k = 1$ and 2, respectively. Then*

$$\begin{aligned}\int_{\sigma_0}^{2} N_{W_A}(\rho; T_2, T_1)\, \mathbf{d}\rho \leq \frac{1}{2\pi} \int_{T_1}^{T_2} \log \left(\frac{|W_A(\sigma_0 + it)|}{|W_A(2 + it)|} \right) \mathbf{d}t \\ + \frac{\mathcal{N}_1 + \mathcal{N}_2}{2} + 1.\end{aligned} \tag{4.5}$$

The following result can be found in [**4**].

Proposition 4.4. *Suppose that s_0 is a fixed complex number and f is a complex function nonvanishing at s_0 and regular for $|s - s_0| < R$ for a positive number R. Let $0 < r < R$ and $M_f = \max_{|s-s_0|=R} |f(s)|$. Then the number of zeros of f in $|s - s_0| \leq r$, denoted by $\mathcal{N}_f$, multiple zeros being counted according to their order of multiplicity satisfies the following inequality.*

$$\mathcal{N}_f \leq \frac{\log M_f - \log |f(s_0)|}{\log R - \log r}.$$

Proof of Lemma 4.1. From Propositions 4.1, 4.2 and 4.3, one has

$$N(\sigma, T) \leq \frac{1}{\lambda - \sigma_0} \left(\frac{1}{2\pi} \int_{T_1}^{T_2} \log \frac{|W_A(\sigma_0 + it)|}{|W_A(2 + it)|}\, \mathbf{d}t + \frac{\mathcal{N}_1 + \mathcal{N}_2}{2} + 1 \right), \tag{4.6}$$

where $\lambda = \sigma - \varepsilon_1$ as $T_2 = T + \varepsilon_2$ as in Proposition 4.1.

Clearly, we have

$$\Re(W_A(\sigma+it)) = \frac{1}{2}\Big(W_A(\sigma+it) + W_A(\sigma-it)\Big).$$

The number of zeros of $\Re(W_A(s))$ on $\mathcal{S}_k$ are the same for the following regular functions

$$W_0^{(k)}(s) = \frac{1}{2}\left(W_A(s+iT_k) + W_A(s-iT_k)\right)$$

on the real axis between σ_0 and 2.

First, one applies Proposition 4.4 to estimate the number $\mathcal{N}_k'$ of zeros for $W_0^{(k)}(s)$ in $|s-2| \le (3/2)$. It is obvious that $\mathcal{N}_k' \le \mathcal{N}_k$. One takes $s_0 = 2$, $R = (7/4)$, $r = (3/2)$. Recalling (4.7) and Lemma 3.3, one acquires

$$\begin{aligned}\max_{|s-2|=(3/2)} \left|W_0^{(k)}(s)\right| &\le \frac{1}{2}\Bigg(\max_{|s-2|=(3/2)} |W_A(s+iT_k)| \\ &\qquad + \max_{|s-2|=(3/2)} |W_A(s-iT_k)|\Bigg) \\ &\le \max_{|s-2|=(3/2)} |W_A(s+iT_k)| \\ &\le W_A^{(1)}\left(T_k + \frac{7}{4}\right) \\ &\le W_A^{(1)}\left(T_2 + \frac{7}{4}\right),\end{aligned}$$

where $W_A^{(1)}(t)$ is the upper bound of $|W_A(s)|$ in Lemma 3.3. Letting ε_2 tend to zero, one sees that

$$\max_{|s-2|=(3/2)} |W_0^{(k)}(s)| \le W_A^{(1)}\left(T + \frac{7}{4}\right).$$

Also, recall that $|W_0(2+it)| > (1/2)$ from Lemma 3.2. This implies

$$\mathcal{N}_k \le c_A(T) := \frac{\log W_A^{(1)}\left(T + (7/4)\right) - \log(1/2)}{\log(7/6)},$$

for $k = 1$ and 2. Hence,

$$\frac{\mathcal{N}_1 + \mathcal{N}_2}{2} \leq c_A(T). \tag{4.8}$$

Now, transform the integral in (4.6) into one involving the function $V_A(s)$ instead of $W_A(s)$.

Recalling the definition of $W_A(s)$, using the triangular inequality in the form $|x-y| \leq |x| + |y|$ and noting that $\log(1+x) \leq x$ for $x > 0$, one has

$$\begin{aligned} \log|W_A(\sigma_0+it)| &= \log|1 - V^2(\sigma_0+it)| \\ &\leq \log(1 + |V_A(\sigma_0+it)|^2) \leq |V_A(\sigma_0+it)|^2. \end{aligned} \tag{4.9}$$

Also, by the triangular inequality in the form $|x-y| \geq |x| - |y|$, one gets $|1 - V_A^2(1+it)| \geq 1 - |V_A(1+it)|^2$. Using the increasing property of the logarithmic function, one sees that $\log|1 - V_A^2(2+it)| \geq \log(1 - |V_A(2+it)|^2)$. It follows that

$$-\log|W_A(2+it)| = -\log|1 - V_A^2(2+it)| \leq -\log(1 - |V_A(2+it)|^2).$$

From the last part of Lemma 3.2, one sees that $|V_A(2+it)|^2 < (1/2)$ since $A \geq 16$. Applying $-\log(1-x) < 2x$ for $0 < x < (1/2)$, one acquires

$$-\log|W_A(2+it)| \leq 2|V_A(2+it)|^2 < \frac{7.9}{A}. \tag{4.10}$$

Combining (4.9) and (4.10), one obtains

$$\log\left(\frac{|W_A(\sigma_0+it)|}{|W_A(2+it)|}\right) \leq |V_A\,(\sigma_0+it)|^2 + \frac{7.9}{A}.$$

Letting ε_2 tend to zeros in (4.6), one gets

$$N(\sigma, T) \leq \frac{1}{\lambda - \sigma_0}\left(\frac{1}{2\pi}\int_{T_1}^{T} |V_A\,(\sigma_0+it)|^2\,\mathbf{d}t + \frac{7.9T}{2\pi\,A} + 1 + c_A(T)\right).$$

Finally, letting ε_1 tend to zero, one obtains

$$N_\zeta(\sigma, T) \leq \frac{1}{\sigma - \sigma_0}\left(\frac{1}{2\pi}\int_{T_1}^{T} |V_A(\sigma_0 + it)|^2 \, \mathbf{d}t + \frac{7.9T}{2\pi\, A} + 1 + c_A(T)\right).$$

This proves Lemma 4.1. □

5. The proof of Theorem 2. To estimate the integral in Lemma 4.1, we study the following functions.

Definition of $\mathcal{V}_\sigma(t)$. Let $t \geq 0$. Define

$$\mathcal{V}_\sigma(t) = \int_0^t |V_A(\sigma + iy)|^2 \, \mathbf{d}y.$$

One needs an explicit upper bound for the Riemann zeta function on the line $\sigma = 1/2$, for which we summarize the Corollary and Theorems 1, 2 and 3 from [**6**] into the following lemma.

Lemma 5.1. *One has*

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| \leq C\, t^\alpha \log^\beta(t + e) + D$$

for any $t > 0$, where $C = 3$, $\alpha = 1/6$, $\beta = 1$ and $D = 2.657$.

For $\sigma = 1/2$ and $1 + \delta$ with the value of $\delta > 0$ being determined later, one has Lemmas 5.2 and 5.3.

Lemma 5.2. *Let C, α and β be as defined in Lemma 5.1, $A \geq 16$ and $0 \leq t < \infty$. Then,*

$$\mathcal{V}_{1/2}(t) \leq D_1 t^{2\alpha+1} \log^{2\beta}(t + e) + D_2 t^{2\alpha} \log^{2\beta}(t + e) + D_3 t + D_4,$$

where $D_1 := 4C^2(\log A + 1)$, $D_2 := 16C^2 A(\log A + 4)$, $D_3 := 4D^2(\log A + 1)$ and $D_4 := 16D^2 A(\log A + 4)$.

Lemma 5.3. *Let $0 < \delta \leq 1$, $A \geq 16$ and $0 \leq t < \infty$. Then*

$$\mathcal{V}_{1+\delta}(t) \leq D_5 t + D_6,$$

where

$$D_5 := 0.206 \frac{\log^3 A + 3\log^2 A + 6\log A + 6}{A^{1+2\delta}},$$

and

$$\begin{aligned} D_6 := \frac{0.264(1+\delta)}{A^\delta}\left(\frac{\log^3 A}{\delta} + \frac{3\log^2 A}{\delta^2} + \frac{6\log A}{\delta^3} + \frac{6}{\delta^4}\right) \\ + \frac{4.012}{A^{2\delta}}\left(\frac{\log^2 A}{\delta^2} + \frac{2\log A}{\delta^3} + \frac{1}{\delta^4}\right) \\ + \frac{16.020(1+\delta)}{A^\delta}\left(\frac{\log^2 A}{\delta} + \frac{2\log A}{\delta^2} + \frac{2}{\delta^3}\right). \end{aligned}$$

The proofs of Lemma 5.2 and 5.3 will be given in Section 8. One needs another auxiliary function $H(s)$.

Definition of $H(s)$. Let $\sigma > 1/2$ and $t > 0$. Denote

$$H(s) := H_{A,\tau}(s) := \frac{s-1}{s\cos\left(\frac{s}{2\tau}\right)} V_A(s), \tag{5.1}$$

where $V_A(s)$ is defined in Section 3 and τ is a parameter with positive value.

The function $H(s)$ has a close relation to the function $V(s)$, as shown in Lemma 5.4.

Lemma 5.4. *Let $(1/2) \le \sigma \le 2$ and $\tau \ge e$. Then, for $t > 0$,*

$$|H(s)| < 2\, e^{-t/(2\tau)} |V_A(s)|;$$

for $t > 14$,

$$|V_A(s)| < \sqrt{\frac{200}{197}}\, e^{t/(2\tau)} |H(s)|.$$

One then uses Lemmas 5.2 and 5.3 to give estimates as in Lemma 5.5 on $\mathcal{H}(\sigma)$ defined as follows.

Definition of $\mathcal{H}(\sigma)$. Let $\sigma > 1/2$. Define

$$\mathcal{H}(\sigma) = \int_{-\infty}^{\infty} |H(s)|^2 \, \mathbf{d}t. \tag{5.2}$$

Lemma 5.5. *For $T \geq T_0$, one has*

$$\mathcal{H}\left(\frac{1}{2}\right) \leq A_1 T^{4/3} \log^3 T,$$

and

$$\mathcal{H}\left(1 + \frac{\omega}{\log T}\right) \leq A_2 \log^4 T,$$

where

$$A_1 = 685.026\, \kappa^{4/3} + 2061.486\kappa^{1/3} + 0.000001\kappa + 0.001,$$

and

$$\begin{aligned} A_2 = {} & \frac{144.001}{\pi^2 e^{\omega}}\left(\frac{\eta^3}{\omega} + \frac{3\eta^2}{\omega^2} + \frac{6\eta^3}{\omega^3} + \frac{6\eta^3}{\omega^4}\right) \\ & + \frac{4}{e^{2\omega}}\left(\frac{\eta^2}{\omega^2} + \frac{2\eta}{\omega^3} + \frac{1}{\omega^4}\right) \\ & + \frac{4.689\kappa}{\pi^2 e^{2\omega}} + \frac{8.001}{\pi^2 e^{\omega}}\left(\frac{\eta^2}{\omega} + \frac{2\eta}{\omega^2} + \frac{2}{\omega^3}\right), \end{aligned}$$

with $\eta = 1.000001$.

One obtains the following Corollary by taking $\omega = 1.598$ and $\kappa = 1.501$. The justification of these choices of constants will be given in the proof of Theorem 1.

Corollary. *Let $T \geq T_0$. Then*

$$\mathcal{H}\left(\frac{1}{2}\right) \leq A_1 T^{4/3} \log^3 T \quad \textit{and} \quad \mathcal{H}\left(1 + \frac{1.598}{\log T}\right) \leq A_2 \log^4 T,$$

with $A_1 = 3537.613$ and $A_2 = 78.383$.

One may transform the estimates on $\mathcal{H}(\sigma)$ for $\sigma = 1/2$ and $1+\delta$ to any σ between by the following lemma, which is due to Hardy, Ingham and Pólya, see [**4**].

Lemma 5.6. *If $H(s)$ is regular and bounded for $\sigma_1 \le \sigma \le \sigma_2$, and the integral*

$$\mathcal{H}(\sigma) = \int_{-\infty}^{\infty} |H(\sigma + i\,t)|^2 \,\mathbf{d}t$$

exists and converges uniformly for $\sigma_1 \le \sigma \le \sigma_2$, and

$$\lim_{|t|\to\infty} |H(s)| = 0$$

uniformly for $\sigma_1 \le \sigma \le \sigma_2$; then for any positive number T,

$$\mathcal{H}(\sigma) \le \{\mathcal{H}(\sigma_1)\}^{(\sigma_2-\sigma)/(\sigma_2-\sigma_1)} \{\mathcal{H}(\sigma_2)\}^{(\sigma-\sigma_1)/(\sigma_2-\sigma_1)} .$$

The proofs of Lemma 5.5 are given in Section 9.

Proof of Theorem 1. Applying Lemma 5.6 to the function $H(s)$ with $\sigma_1 = 1/2$ and $\sigma_2 = 1+\delta$ with any positive δ, one obtains

$$\mathcal{H}(\sigma) \le A_1^{2(1+\delta-\sigma)}/(1+2\delta) A_2^{(2\sigma-1)}/(1+2\delta) T^{(8(1+\delta-\sigma))/(3(1+2\delta))} \times \log^{3+((2\sigma-1)/(1+2\delta))} T \le A_1 A_2 T^{(8(1+\delta-\sigma))}/3 \log^4 T.$$

Now, from Lemma 5.4, one gets $|V_A(s)|^2 \le (200/197) e^{t/\tau} |H(s)|^2$, or, with $\tau = \kappa T$, for $\kappa \ge (e/T_0)$,

$$|V_A(s)|^2 \le \frac{200}{197} e^{1/\kappa} |H(s)|^2.$$

$$\begin{aligned}\int_{T_1}^{T} |V_A(\sigma_0 + it)|^2 \,\mathbf{d}t &\le \frac{200}{197} e^{1/\kappa} \int_{T_1}^{T} |H(\sigma_0 + it)|^2 \,\mathbf{d}t \\ &\le \frac{200}{197} e^{1/\kappa} \int_{0}^{\infty} |H(\sigma_0 + it)|^2 \,\mathbf{d}t \\ &= \frac{100}{197} e^{1/\kappa} \mathcal{H}(\sigma_0) \\ &\le \frac{100}{197} e^{1/\kappa} A_1 A_2 T^{(8(1+\delta-\sigma_0))/3} \log^4 T.\end{aligned}$$

Recalling Lemma 4.1, one sees that

$$N_\zeta(\sigma,T) \le \frac{100e^{1/\kappa}}{394\pi(\sigma-\sigma_0)} A_1A_2T^{(8(1+\delta-\sigma_0))/3}\log^4 T + \frac{1}{\sigma-\sigma_0}\left(\frac{16T}{2\pi A} + 1 + c_A(T)\right).$$

Note that $A \le (1+(1/T_0))T$ and $\delta = \omega/(\log T)$ as in the proof of Lemma 5.5. Also, let $\sigma_0 = \sigma - (\nu/(\log T))$ for another positive constant ν. It follows that

$$\begin{aligned} N_\zeta(\sigma,T) &\le \frac{100e^{(1/\kappa)+((8(\omega+\nu))/3)}}{394\pi\nu} A_1A_2T^{(8(1-\sigma)/3)}\log^5 T \\ &\quad + \frac{16\log T}{2\pi\nu} + \frac{\log T}{\nu} + \frac{c_A(T)\log T}{\nu} \\ &\le C_D T^{(8(1-\sigma)/3)}\log^5 T, \end{aligned}$$

with

$$C_D := \frac{100e^{(1/\kappa)+((5\omega)/3)+((8\nu)/3)}}{394\pi\nu} A_1A_2 + \frac{1}{\log^4 T_0}\left(\frac{16}{2\pi\nu} + \frac{1}{\nu}\right) + \frac{c_A(T)/\log T}{\nu\log^3 T_0}.$$

The first term in C_D is the major one; one may sub-optimize it in order to sub-optimize C_D. Note that

$$\begin{aligned} C_D \approx \frac{100e^{(1/\kappa)+((5\omega)/3)+((8\nu)/3)}}{394\pi\nu} &(685.026\kappa^{4/3} + 2061.486\kappa^{1/3}) \\ \times \Bigg(&\frac{144}{\pi^2 e^\omega}\left(\frac{1}{\omega} + \frac{3}{\omega^2} + \frac{6}{\omega^3} + \frac{6}{\omega^4}\right) \\ &+ \frac{4}{e^{2\omega}}\left(\frac{1}{\omega^2} + \frac{2}{\omega^3} + \frac{1}{\omega^4}\right)\Bigg). \end{aligned}$$

To optimize the factor $((\exp(8\nu)/3))/(\nu)$, one takes $\nu = (3/8)$, to sub-optimize the factor

$$e^{1/\kappa}(200.593\kappa^{4/3} + 603.656\kappa^{1/3})$$

in C_D, one let $\kappa = 1.501$, and to sub-optimize the factor

$$e^{5\omega/3}\left(\frac{144}{\pi^2 e^\omega}\left(\frac{1}{\omega}+\frac{3}{\omega^2}+\frac{6}{\omega^3}+\frac{6}{\omega^4}\right)+\frac{4}{e^{2\omega}}\left(\frac{1}{\omega^2}+\frac{2}{\omega^3}+\frac{1}{\omega^4}\right)\right),$$

one chooses $\omega = 1.598$. With these choices of constants, one gets the corollary of Lemma 5.5. From the corollary, one justifies the choice of T_0. With computation, the proof of Theorem 1 is finished.

6. Estimates involving the divisor function.

Lemma 6.1. *Let* $\delta > 0$ *and* $\log\log N \geq 18$. *Then*

$$\sum_{N<n} \frac{d^2(n)}{n^{2+2\delta}} \leq \frac{0.206}{N^{1+2\delta}}\left(\log^3 N + 3\log^2 N + 6\log N + 6\right).$$

Lemma 6.2. *Let* $\delta > 0$ *and* $\log\log N \geq 18$. *Then*

$$\sum_{N<n}\sum_{N<m<n} \frac{d(m)d(n)}{(mn)^{1+\delta}} \leq \frac{1.003}{N^{2\delta}}\left(\frac{\log^2 N}{\delta^2}+\frac{2\log N}{\delta^3}+\frac{1}{\delta^4}\right).$$

Lemma 6.3. *Let* $\delta > 0$ *and* $\log\log N \geq 18$. *Then*

$$\sum_{N<m}\frac{1}{m^{1+\delta}}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}$$
$$\leq 0.066\frac{1+\delta}{N^\delta}\left(\frac{\log^3 N}{\delta}+\frac{3\log^2 N}{\delta^2}+\frac{6\log N}{\delta^3}+\frac{6}{\delta^4}\right)$$
$$+ 4.005\frac{1+\delta}{N^\delta}\left(\frac{\log^2 N}{\delta}+\frac{2\log N}{\delta^2}+\frac{2}{\delta^3}\right).$$

Proof of Lemma 6.1. Using the partial summation formula, one gets

$$\begin{aligned}\sum_{N<n<\infty} \frac{d^2(n)}{n^{2+2\delta}} &= \int_N^\infty \frac{1}{y^{2+2\delta}} \,\mathbf{d}\Big(\sum_{N<n\le y} d^2(n)\Big) \\ &= \Big(\frac{1}{y^{2+2\delta}} \sum_{N<n\le y} d^2(n)\Big)\Big|_N^\infty \\ &\qquad + (2+2\delta)\int_N^\infty \Big(\sum_{N<n\le y} d^2(n)\Big)\frac{1}{y^{3+2\delta}}\,\mathbf{d}y \\ &= (2+2\delta)\int_N^\infty \Big(\sum_{N<n\le y} d^2(n)\Big)\frac{1}{y^{3+2\delta}}\,\mathbf{d}y.\end{aligned}$$

By the corollary of Lemma 4.2 in [**5**], one has

$$\begin{aligned}\sum_{n\le x} d^2(x) &\le 0.102x\log^3 x + 1.676x\log^2 x + 8.564x\log x + 23.652x \\ &\quad + 1.334\sqrt{x}\log^3 x - 2.845\sqrt{x}\log^2 x - 4.280\sqrt{x}\log x \\ &\quad - 8.501\sqrt{x} + 1.334\log^3 x - 0.845\log^2 x \\ &\quad + 2.874\log x - 0.111 \\ &\le 0.103x\log^3 x.\end{aligned}$$

It follows that

$$\sum_{N<n<\infty} \frac{d^2(n)}{n^{2+2\delta}} \le 0.103(2+2\delta)\int_N^\infty \frac{\log^3 y}{y^{2+2\delta}}\,\mathbf{d}y. \tag{6.1}$$

From this, Lemma 6.1 follows. □

Proof of Lemma 6.2. We note that

$$\sum_{N<n}\sum_{N<m<n} \frac{d(m)d(n)}{(mn)^{1+\delta}} \le \Big(\sum_{N<n}\frac{d(n)}{n^{1+\delta}}\Big)^2$$

and, by Lemma 5.1 in [**5**],

$$\sum_{n\le x} d(n) \le x\log x + 0.155x + 4\sqrt{x} \le 1.001x\log x.$$

Using these, as before, we similarly prove Lemma 6.2. □

Proposition 6.1. *For* $\log\log x \geq 18$*, one has*

$$\sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)} \leq 0.066x\log^3 x + 4.005x\log^2 x.$$

Proof. Note that $-\log(1-x) > x$ for $0 < x < 1$. Thus, for $n < m$,

$$\begin{aligned}\frac{1}{\log(m/n)} &= \left(-\log\left(1-\frac{m-n}{m}\right)\right)^{-1}\\ &< \left(\frac{m-n}{m}\right)^{-1} = 1+\frac{n}{m-n} < 1+\frac{(mn)^{1/2}}{m-n}.\end{aligned}$$

It follows that

$$\sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)} < \sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{(mn)^{1/2}} + \sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{m-n}. \tag{6.5}$$

For the first sum in (6.5), one sees

$$\sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{(mn)^{1/2}} = \sum_{m\leq x}\frac{d(m)}{m^{1/2}}\sum_{n\leq x}\frac{d(n)}{n^{1/2}} \leq \left(\sum_{n\leq x}\frac{d(n)}{n^{1/2}}\right)^2.$$

Recalling the corollary of Lemma 5.2 in [**5**], one has

$$\sum_{n\leq x}\frac{d(n)}{\sqrt{n}} \leq 2\sqrt{x}\log x - 1.691\sqrt{x} + 2\log x + 5.846 \leq 2.001\sqrt{x}\log x.$$

For the second sum in (6.5), one recalls the corollary of the main theorem in [**5**]. Since $\log\log x \geq 18$, one has

$$\sum_{m\leq x}\sum_{n<m}\frac{d(m)d(n)}{m-n} \leq 0.066x\log^3 x.$$

This concludes the proof of Proposition 6.1. □

Proof of Lemma 6.3. Using the partial summation formula for the sum over n, one gets

$$\begin{aligned}\sum_{N<m}\frac{1}{m^{1+\delta}}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}&\\ =\int_N^\infty \frac{1}{y^{1+\delta}}\,\mathbf{d}\left(\sum_{N<m\le y}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}\right)&\\ =\left(\frac{1}{y^{1+\delta}}\sum_{N<m\le y}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}\right)\Bigg|_N^\infty&\\ +(1+\delta)\int_N^\infty\sum_{N<m\le y}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}\frac{\mathbf{d}y}{y^{2+\delta}}&\\ =\lim_{y\to\infty}\frac{1}{y^{1+\delta}}\sum_{N<m\le y}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}&\\ +(1+\delta)\int_N^\infty\sum_{N<m\le y}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}\frac{\mathbf{d}y}{y^{2+\delta}}.&\end{aligned}$$

Recalling Proposition 6.1, one sees the first term in the last expression is zero; and applying (6.4) and (6.3), one obtains

$$\begin{aligned}\sum_{N<m}\frac{1}{m^{1+\delta}}\sum_{N<n<m}\frac{d(m)d(n)}{(mn)^{1/2}\log(m/n)}&\\ \le 0.066(1+\delta)\int_N^\infty\frac{\log^3 y}{y^{1+\delta}}\,\mathbf{d}y+4.005(1+\delta)\int_N^\infty\frac{\log^2 y}{y^{1+\delta}}\,\mathbf{d}y&\\ \le 0.066\frac{1+\delta}{N^\delta}\left(\frac{\log^3 N}{\delta}+\frac{3\log^2 N}{\delta^2}+\frac{6\log N}{\delta^3}+\frac{6}{\delta^4}\right)&\\ +4.005\frac{1+\delta}{N^\delta}\left(\frac{\log^2 N}{\delta}+\frac{2\log N}{\delta^2}+\frac{2}{\delta^3}\right).&\end{aligned}$$

This proves Lemma 6.3. □

7. Proofs for Lemmas 5.2 and 5.3.

Proof of Lemma 5.2. Recall the definition of $V_A(s)$ from Section 3. Using $(x+y)^2\le 2(x^2+y^2)$ for real numbers x and y, one gets

$$|V_A(s)|^2\le 2(|\zeta(s)|^2|U_A(s)|^2+1). \tag{7.1}$$

Recalling the definition of $\mathcal{V}_\sigma(t)$ from Section 5, applying Lemma 5.1, and using the same inequality for any x and y again, one acquires (7.2)

$$\mathcal{V}_{1/2}(t) \le \left(4C^2 t^{2\alpha} \log^{2\beta}(t+e) + 4D^2\right) \int_0^t |U_A(0.5+i\tau)|^2 \, \mathbf{d}\tau + 2t.$$

The integral in the last expression is

$$\begin{aligned} \int_0^t |U_A(0.5+i\tau)|^2 \, \mathbf{d}\tau &= \int_0^t U_A(0.5+i\tau)\overline{U_A(0.5+i\tau)} \, \mathbf{d}\tau \\ &= \sum_{m=1}^{A} \sum_{n=1}^{A} \frac{\mu(m)\mu(n)}{m^{1/2}n^{1/2}} \int_0^t \left(\frac{m}{n}\right)^{i\tau} \mathbf{d}\tau. \end{aligned}$$

Thus, using the inequality

$$\left| \int_0^t \left(\frac{m}{n}\right)^{i\tau} \mathbf{d}\tau \right| \le \frac{2}{\log(m/n)}, \quad m > n,$$

we immediately get (7.3)

$$\int_0^t |U_A(0.5+i\tau)|^2 \, \mathbf{d}\tau \le t \sum_{n \le A} \frac{1}{n} + 4 \sum_{m \le A} \sum_{n<m} \frac{1}{m^{1/2}n^{1/2} \log(m/n)}.$$

The first term in (7.3) is bounded by $t(\log A + 1)$. For the second term on the right side of the last expression, we note that $x \log x - x + 1 > 0$ for $x > 1$. This implies that

$$\frac{1}{\log x} < \frac{x}{x-1} = 1 + \frac{1}{x-1} < 1 + \frac{x^{1/2}}{x-1}.$$

We use this for $x = m/n$, getting

$$\frac{1}{\log(m/n)} < 1 + \frac{n^{1/2}m^{1/2}}{m-n}.$$

It follows that

$$\sum_{m\le A}\sum_{n<m}\frac{1}{m^{1/2}n^{1/2}\log(m/n)}$$
$$< \sum_{m\le A}\sum_{n<m}\frac{1}{m^{1/2}n^{1/2}} + \sum_{m\le A}\sum_{n<m}\frac{1}{m-n}$$
$$\le \left(\sum_{n\le A}\frac{1}{n^{1/2}}\right)^2 + \sum_{1<n\le A}(1+\log(m-1))$$
$$\le 4A + A\log A \le A(\log A + 4).$$

Thus,

$$\int_0^t |U_A(0.5+i\tau)|^2\,\mathbf{d}\tau \le t(\log A+1) + 4A(\log A+4). \tag{7.4}$$

We conclude that, from (7.2) and (7.4), this proves Lemma 5.2. □

Proof of Lemma 5.3. To estimate $\mathcal{V}_{1+\delta}(t)$, one recalls Lemma 3.1. It follows that

$$\mathcal{V}_{1+\delta}(t) = \int_0^t \left|\sum_{A<n}\frac{\nu(n)}{n^{1+\delta+i\tau}}\right|^2 \mathbf{d}\tau = \sum_{A<m}\frac{\nu(m)}{m^{1+\delta}}\sum_{A<n}\frac{\nu(n)}{n^{1+\delta}}\int_0^t\left(\frac{m}{n}\right)^{i\tau}\mathbf{d}\tau.$$

Similarly to the argument for obtaining (8.3), one deduces

$$\mathcal{V}_{1+\delta}(t) \le t\sum_{A<n}\frac{d^2(n)}{n^{2+2\delta}} + 4\sum_{A<m}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}\log(m/n)}. \tag{7.5}$$

For the second sum in the last expression, we observe that the function $f(x) = \log x + x^{-1/2} - 1 > 0$ for $x > 1$. It follows that

$$\frac{1}{\log x} < 1 + \frac{1}{x^{1/2}\log x}, \quad \text{for } x > 1.$$

With $x = m/n$, one sees that

$$\frac{1}{\log(m/n)} < 1 + \frac{n^{1/2}}{m^{1/2}\log(m/n)}.$$

The second term in (7.5) is less than

$$\begin{aligned}
&\sum_{A<m}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}\log(m/n)}\\
&\le\sum_{A<m}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}}+\sum_{A<m}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1+\delta}n^{\delta}m^{1/2}n^{1/2}\log(m/n)}\\
&\le\sum_{A<m}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}}+\sum_{A<m}\frac{1}{m^{1+\delta}}\sum_{A<n<m}\frac{d(m)d(n)}{m^{1/2}n^{1/2}\log(m/n)}.
\end{aligned}$$

Applying Lemma 6.1 for the first term in (7.5) and Lemma 6.2 for the first term and Lemma 6.3 for the second term in the last expression, Lemma 5.2 is proved. □

8. Proofs for Lemmas 5.4 and 5.5.

Proof of Lemma 5.4. By the definition of the cosine function in the complex variable $s=\sigma+it$, one sees that

$$\begin{aligned}
\cos\left(\frac{s}{2\tau}\right)&=\frac{1}{2}\left(e^{-(i\,s)/(2\tau)}+e^{(i\,s)/(2\tau)}\right)\\
&=\frac{1}{2}\left(e^{(t/2\tau)-i(\sigma/2\tau)}+e^{-(t/2\tau)+i(\sigma/2\tau)}\right)\\
&=\frac{1}{2}e^{(t/2\tau)-i(\sigma/2\tau)}\left(1+e^{-(t/\tau)+i(\sigma/\tau)}\right).
\end{aligned}$$

Since $1/2\le\sigma\le 2$, one has $0<(\sigma/\tau)<(\pi/4)$ for $\tau\ge e$. It follows that $e^{i(\sigma/\tau)}$ is in the first half of the first quadrant so that $1<|1+e^{-(t/\tau)+i(\sigma/\tau)}|=\sqrt{1+e^{-t/\tau}+2e^{-t/\tau}\cos(\sigma/\tau)}<2$. Thus, one sees that

$$\frac{1}{2}e^{t/2\tau}<\left|\cos\left(\frac{s}{2\tau}\right)\right|<e^{t}/2\tau.$$

For $t>0$, it is easy to see that

$$\left|\frac{s-1}{s}\right|=\sqrt{1-\frac{2\sigma-1}{\sigma^2+t^2}}\le 1.$$

For $1/2<\sigma\le 2$ and $t>14$,

$$\left|\frac{s-1}{s}\right| = \sqrt{1-\frac{2\sigma-1}{\sigma^2+t^2}} > \sqrt{1-\frac{2\sigma-1}{\sigma^2+14^2}} \geq \sqrt{\frac{197}{200}}.$$

This concludes the proof of Lemma 5.4. □

Proof of Lemma 5.5. One may first note that H is an analytic function so that

$$\mathcal{H}(\sigma) = 2\int_0^\infty |H(s)|^2 \,\mathbf{d}t.$$

From this equation and the first inequality in Lemma 5.4, one sees

$$\mathcal{H}(\sigma) \leq 8\int_0^\infty e^{-t/\tau}|V_A(s)|^2 \,\mathbf{d}t.$$

One then uses integration by parts, getting

$$\begin{aligned}\int_0^\infty e^{-t/\tau}|V_A(\sigma+it)|^2 \,\mathbf{d}t &= \int_0^\infty e^{-t/\tau}\,\mathbf{d}\left(\int_0^t |V_A(\sigma+iy)|^2 \,\mathbf{d}y\right)\\ &= \int_0^\infty e^{-t/\tau}\,\mathbf{d}\mathcal{V}_\sigma(t) = e^{-t/\tau}\mathcal{V}_\sigma(t)\Big|_0^\infty\\ &\quad + \frac{1}{\tau}\int_0^\infty e^{-t/\tau}\mathcal{V}_\sigma(t)\,\mathbf{d}t.\end{aligned}$$

Note that $\mathcal{V}_\sigma(0) = 0$ by definition. From Lemma 3.3, it is easy to see that $\mathcal{V}_\sigma(t) \ll t^4$; hence, the first term in the last expression is zero. Thus,

$$\mathcal{H}(\sigma) \leq \frac{8}{\tau}\int_0^\infty e^{-t/\tau}\mathcal{V}_\sigma(t)\,\mathbf{d}t.$$

One then substitutes the variable t by τy with the variable y and the parameter τ, obtaining

$$\mathcal{H}(\sigma) \leq 8\int_0^\infty e^{-y}\mathcal{V}_\sigma(\tau y)\,\mathbf{d}y. \tag{8.1}$$

To estimate $\mathcal{H}(1/2)$ and $\mathcal{H}(1+\delta)$, one uses Lemmas 5.2 and 5.3. One needs to calculate the integrals in the forms of

$$\mathcal{J}(a,b) := \int_0^\infty e^{-y}y^a \log^b(y+e)\,\mathbf{d}y,$$

for the ordered sets $\{a, b\} = \{0,0\}$, $\{1,0\}$, $\{(1/3), 0t\}$, $\{(4/3), 0\}$, $\{(1/3), 2\}$, and $\{(4/3), 2\}$.

For the first two sets of values for a and b, it is easy to see

$$\mathcal{J}(0,0) = \int_0^\infty e^{-y}\,\mathbf{d}y = 1, \quad \text{and} \quad \mathcal{J}(1,0) = \int_0^\infty e^{-y} y\,\mathbf{d}y = 1,$$

using the partial integral formula for the second one. One then uses a computation package to obtain

$$\begin{aligned}
\mathcal{J}\left(\frac{1}{3},0\right) &= \int_0^\infty e^{-y} y^{1/3}\,\mathbf{d}y = \Gamma\left(\frac{4}{3}\right) \leq 0.893,\\
\mathcal{J}\left(\frac{4}{3},0\right) &= \int_0^\infty e^{-y} y^{4/3}\,\mathbf{d}y = \Gamma\left(\frac{7}{3}\right) \leq 1.191,\\
\mathcal{J}\left(\frac{1}{3},2\right) &= \int_0^\infty e^{-y} y^{1/3} \log^2(y+e)\,\mathbf{d}y \leq 1.220,
\end{aligned}$$

and

$$\mathcal{J}\left(\frac{4}{3},2\right) = \int_0^\infty e^{-y} y^{4/3} \log^2(y+e)\,\mathbf{d}y \leq 1.881.$$

Note that $\tau y + e \leq \tau(y+e)$ since $\tau \geq e$, so that $\log(\tau y + e) \leq \log\tau + \log(y+e)$. One then has

$$\log^2(\tau y + e) \leq 2\left(\log^2(\tau) + \log^2(y+e)\right),$$

since $(x+y)^2 \leq 2(x^2+y^2)$ is valid for any real numbers x and y. Recalling Lemma 5.2, one obtains that

$$\begin{aligned}
\int_0^\infty e^{-y}\mathcal{V}_{1/2}(\tau y)\,\mathbf{d}y \leq{}& 2D_1\tau^{4/3}\log^2\tau\;\mathcal{J}\left(\frac{4}{3},0\right)\\
&+ 2D_1\tau^{4/3}\;\mathcal{J}\left(\frac{4}{3},2\right)\\
&+ 2D_2\tau^{1/3}\log^2\tau\;\mathcal{J}\left(\frac{1}{3},0\right)\\
&+ 2D_2\tau^{1/3}\mathcal{J}\left(\frac{1}{3},2\right)\\
&+ D_3\tau\mathcal{J}(1,0) + D_4\mathcal{J}(0,0).
\end{aligned}$$

Then, recalling (8.1), one acquires

$$\mathcal{H}\left(\frac{1}{2}\right) \leq a_1\tau^{4/3}\log^2\tau + a_2\tau^{4/3} + a_3\tau^{1/3}\log^2\tau + a_4\tau^{1/3} + a_5\tau + a_6, \tag{8.2}$$

with

$$\begin{aligned}
a_1 &:= 16D_1\mathcal{J}\left(\frac{4}{3}, 0\right) \leq 14.288D_1,\\
a_2 &:= 16D_1\mathcal{J}\left(\frac{4}{3}, 2\right) \leq 19.056D_1,\\
a_3 &:= 16D_2\mathcal{J}\left(\frac{1}{3}, 0\right) \leq 19.520D_2,\\
a_4 &:= 16D_2\mathcal{J}\left(\frac{1}{3}, 2\right) \leq 30.096D_2,\\
a_5 &:= 8D_3\mathcal{J}\left(1, 0\right) = 8D_3,\\
a_6 &:= 8D_4\mathcal{J}\left(0, 0\right) = 8D_4.
\end{aligned}$$

Similarly, but recalling Lemma 5.3 and equation (8.1), one has

$$\mathcal{H}(1+\delta) \leq b_1\tau + b_2, \tag{8.3}$$

with $b_1 = 8D_5\mathcal{J}(1,0) = 8D_5$ and $b_2 = 8D_6\mathcal{J}(1,0) = 8D_6$.

Actually, the "constants" a_j for $j = 1, \ldots, 6$ and b_j for $j = 1$, 2, are not absolute; they depend on the choice of A subject to $A \geq 16$ as well as our choice of the parameter τ. The kink is that we are going to choose suitable A and τ.

Note that $(1 + (1/T_0))T - T = (T/T_0) \geq 1$. One may choose A to be an integer in $T \leq A \leq (1 + (1/T_0))T$. Let κ be a constant such that $\kappa \geq (e/T_0)$ and $\tau = \kappa T$. Then $\tau \geq e$. Also, let $\omega > 0$ and $\delta = (\omega/T)$.

For brevity, denote $A(T) = \log T + \log(1 + (1/T_0))$. We have $A(T) + Z < 1.000,000,001 \log T$ for any $Z = 1$ or 4. Also, we assume that κ is not so large so that $\log T + \kappa \leq 1.000,001 \log T$. It is now straightforward to conclude Lemma 5.5. □

9. Landau's approximate formula. In this section, we give an explicit form of Landau's approximate formula as stated in Lemma 9.1.

Let $T \geq 0$ and $u > 0$. Suppose there are n zeros $\beta_1 + iz_1$, $\beta_2 + iz_2, \ldots, \beta_n + iz_n$ of $\zeta(s)$ in $T - u \leq \Im(s) \leq T + u$ such that

$z_0 = T - u \le z_1 < z_2 < \ldots < z_n \le T + u = z_{n+1}$. Let $1 \le j \le n+1$ be such that $z_j - z_{j-1} \ge z_i - z_{i-1}$ for every other $1 \le i \le n+1$. There may be more than one such a j. Fix one such j, and let $T_u = (z_{j-1} + z_j)/2$. For convenience, T_u is called the associate of T with respect to u.

Lemma 9.1. *Let $x \ge x_0$ and $T \ge \exp(\exp(18))$. Suppose that T_u is the associate of T with respect to $u = 1.155$. Then,*

$$\psi(x) = x - \sum_{|\Im(\rho)| \le T_u} \frac{x^\rho}{\rho} + E(x),$$

where

$$|E(x)| \le 5.26\, \frac{x \log^2 x}{T} + 33.488 \frac{x \log^2 T}{T \log x} + 3 \frac{\log^2 T}{x}.$$

Proposition 9.1. *Let $t \ge 0$ and $\beta_n + i\gamma_n$, $n = 1, 2, \ldots$ be all nontrivial zeros of the Riemann zeta-function. Then*

$$\sum_{n=1}^{\infty} \frac{1}{4 + (t - \gamma_n)^2} \le \frac{1}{4} \log\left(t^2 + 4\right) + 1.483.$$

Proof. Recall the following formula, see [**8**]. That is,

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{n=1}^{\infty} \left(\frac{1}{s - \rho_n} + \frac{1}{\rho_n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{s + 2n} - \frac{1}{2n}\right) + B_0, \tag{9.1}$$

where $\{\rho_n : n = 1, 2, \ldots\}$ is the set of all nontrivial zeros of the Riemann zeta-function and $B_0 = \log(2\pi) - 1$. Using this equation with $s = 2 + it$, Proposition 9.1 follows. □

Proposition 9.2. *Let γ_n be defined as in Lemma 9.1. For $t \ge 0$ and $0 < u$, one has*

(a) *The number of zeros of* $\zeta(s)$ *such that* $|t-\gamma_n| \le u$ *is less than*

$$(4+u^2)\left(\frac{1}{4}\log\left(t^2+4\right)+1.483\right);$$

(b)

$$\sum_{|t-\gamma_n|>u} \frac{1}{(t-\gamma_n)^2} \le \left(1+\frac{4}{u^2}\right)\left(\frac{1}{4}\log\left(t^2+4\right)+1.483\right).$$

Proof. Note that

$$1 \le \frac{4+u^2}{4+(t-\gamma_n)^2}, \quad \text{if } |t-\gamma_n| \le u;$$

therefore,

$$\sum_{|t-\gamma_n|\le u} 1 \le (4+u^2)\sum_{n=1}^{\infty} \frac{1}{4+(t-\gamma_n)^2}.$$

Applying Proposition 9.1, one proves (a) in Proposition 9.2. One shows (b) in the proposition similarly, but note that

$$\frac{1}{(t-\gamma_n)^2} \le \left(1+\frac{4}{u^2}\right)\frac{1}{4+(t-\gamma_n)^2}, \quad \text{if } |t-\gamma_n| > u,$$

so that

$$\sum_{|t-\gamma_n|>u} \frac{1}{(t-\gamma)^2} \le \left(1+\frac{4}{u^2}\right)\sum_{n=1}^{\infty} \frac{1}{4+(t-\gamma_n)^2}. \qquad \square$$

Proposition 9.3. *Let* $-1 \le \sigma \le 2$, $t > 0$, *and* $u > 0$. *Then*

$$\begin{aligned}\left|\frac{\zeta'(\sigma \pm it)}{\zeta(\sigma \pm it)}\right| \le & \sum_{|t-\gamma_n|\le u} \left(\frac{1}{2+it-\rho_n} - \frac{1}{s-\rho_n}\right) \\ & + \frac{3}{2}\left(1+\frac{4}{u^2}\right)\left(\frac{1}{4}\log\left(t^2+4\right)+1.483\right) \\ & + \frac{3}{t^2} + 1.284.\end{aligned}$$

Proof. From (9.1), we have the following equation.

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2+it-1} - \frac{1}{s-1} + \sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n} - \frac{1}{2+it-\rho_n}\right)$$
$$+\sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2+it+2n}\right) - \frac{\zeta'(2+it)}{\zeta(2+it)}.$$

Using this equation and Proposition 9.2 (b), Proposition 9.3 follows. □

Proposition 9.4. *Let* $-1 \le \sigma \le 2$ *and* $T > \exp(\exp(18))$. *Suppose* T_u *is the associate of* T *with respect to* $u = 1.155$. *Then*

(a)

$$\left|\frac{\zeta'(\sigma \pm iT_u)}{\zeta(\sigma \pm iT_u)}\right| \le 6.159\log^2 T + 2.999\log T + 1.285;$$

(b) *For* $12 < t \le T_u$

$$\left|\frac{\zeta'(-1 \pm it)}{\zeta(-1 \pm it)}\right| \le 2.999\log t + 10.241;$$

and

(c) *For* $0 \le t \le 12$

$$\left|\frac{\zeta'(-1 \pm it)}{\zeta(-1 \pm it)}\right| \le 19.172.$$

Proof. Note that

$$\left|\frac{\zeta'(\sigma - it)}{\zeta(\sigma - it)}\right| = \left|\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)}\right|.$$

One only needs to consider the case with the plus sign for each case.

Recalling Proposition 9.3, we only need to estimate the sum

$$\sum_{|t-\gamma_n|\le u}\left(\frac{1}{s-\rho_n}-\frac{1}{2+it-\rho_n}\right). \tag{9.2}$$

Note that

$$\frac{1}{|s-\rho_n|}=\frac{1}{|\sigma-\beta_n+i(t-\gamma_n)|}=\frac{1}{\sqrt{(\sigma-\beta_n)^2+(t-\gamma_n)^2}}\le\frac{1}{|t-\gamma_n|}. \tag{9.3}$$

Similarly,

$$\frac{1}{|2+it-\rho_n|}\le\frac{1}{|t-\gamma_n|}. \tag{9.4}$$

Thus,

$$\left|\sum_{|t-\gamma_n|\le u}\left(\frac{1}{\sigma+it-\rho_n}-\frac{1}{2+it-\rho_n}\right)\right|\le 2\sum_{|t-\gamma_n|\le u}\frac{1}{|t-\gamma_n|}. \tag{9.5}$$

Recalling (a) in Proposition 9.2, one sees that there are at most

$$(4+u^2)(\log(t^2+4)/4+1.483)$$

terms in (9.5). By the setting of T_u, one has

$$|T_u-\gamma_n|\ge\frac{2u}{(4+u^2)(\log(T^2+4)/4+1.483)+1}$$

for every γ_n, $n=1,2,\dots$ (it is T instead of T_u on the right side of the last expression). Or, each summand in the last expression in (9.5) is less than

$$\frac{(4+u^2)(\log(T^2+4)/4+1.483)+1}{2u}.$$

It follows that

$$\begin{aligned}\left|\sum_{|T_u-\gamma_n|\le u}\left(\frac{1}{\sigma+iT_u-\rho_n}-\frac{1}{2+iT_u-\rho_n}\right)\right|&\\ &\le 2(4+u^2)\left(\frac{\log(T_u^2+4)}{4+1.483}\right)\\ &\quad\times\frac{(4+u^2)(\log(T^2+4)/4+1.483)+1}{2u}\\ &\le(4+u^2)\left(\frac{\log((T+u)^2+4)}{4+1.483}\right)\\ &\quad\times\frac{(4+u^2)(\log(T^2+4)/4+1.483)+1}{u}.\end{aligned}$$

To sub-optimize the factor $(4+u^2)^2/u$, we let $u = 1.155$. Also, we remark that

$$\log(T^2+4) = 2\log T + \log\left(1 + 4e^{e^{-36}}\right) \le 2.0000001 \log T,$$

and

$$\begin{aligned}\log((T+u)^2+4) &= 2\log T + \log\left(\left(1+1.155e^{e^{-18}}\right)^2 + 4e^{e^{-36}}\right)\\ &\le 2.0000001 \log T.\end{aligned}$$

Summarizing with the result in Proposition 9.3, (a) is proved.

We prove (b) and (c) similarly. For (b), replacing (9.3) and (9.4) by

$$\frac{1}{|-1+it-\rho_n|} \le \frac{1}{|-1-\beta_n|} \le 1, \text{ and } \frac{1}{|2+it-\rho_n|} \le \frac{1}{|2-\beta_n|} \le 1,$$

and noting that

$$\log(t^2+4) = 2\log t + \log(1+4/12^2) \le 2\log t + 0.028.$$

For (c), we replace the upper bound in (9.8) by 3/2, recalling (13) from [**7**]. Also, note that the terms of the sum in (9.2) are zero and

$$\log(t^2+4) \le \log(12^2+4) \le 4.998. \qquad \square$$

We also need Landau's approximate formula in the following form, see [**7**, Lemma 4].

Proposition 9.5. *Let* $x > 2981$ *and* $T \ge \exp(\exp(18))$. *Suppose that* T_u *is the associate of* T *with respect to* $u = 1.155$. *Then*

$$\psi(x) = \frac{1}{2\pi i}\int_{1+(1/\log x)-iT_u}^{1+(1/\log x)+iT_u} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,\mathbf{d}s + E_0(x),$$

where

$$|E_0(x)| \le 5.25\,\frac{x\log^2 x}{T} + 12.64\,\frac{x\log x}{T} + \log x.$$

Especially, if $x \geq e^{e^{15}}$, *then*

$$|E_0(x)| < 5.26 \, \frac{x \log^2 x}{T} - \log(2\pi x).$$

Proof of Lemma 9.1. We apply the Cauchy residue theorem on the function $-\zeta'(s)/\zeta(s)(x^s/s)$. Utilizing (9.1), we see that the residue of $-\zeta'(s)/\zeta(s)(x^s/s)$ at $s = 1$ is x, those at $s = \rho_n$'s are $-x^\rho/\rho$'s and that at $s = 0$ is $-\zeta'(s)/\zeta(0)$. We let $r = 1 + 1/\log x < 1.01$ as in [**7**, Section 4] and $l = -1$ and apply Cauchy's residue theorem on the rectangle bounded by $s = l$, $s = -iT_u$, $s = r$ and $s = iT_u$, getting

$$\begin{aligned}(9.6)\quad & \frac{1}{2\pi i}\int_{r-iT_u}^{r+iT_u}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right)\mathbf{d}s \\ &= x - \sum_{|\Im(\rho)|\leq T_u}\frac{x^\rho}{\rho} + \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2\pi i}\int_{L_l+L_u+L_b}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right)\mathbf{d}s,\end{aligned}$$

where L_l is the left, L_u is the top and L_b is the bottom side of the rectangle. For the third term on the right side of (9.6), we have

$$\frac{\zeta'(0)}{\zeta(0)} = B_0 + 1 = \log(2\pi).$$

For the integral along with L_l, we use $|-1+it| \geq 1$ for $|t| \leq 12$ and $|-1+it| \geq t$; otherwise, we get

$$\begin{aligned}&\left|\int_{L_l}\left(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\right)\mathbf{d}s\right| \\ &\qquad \leq \int_{-T_u}^{T_u}\left|\left(-\frac{\zeta'(-1+it)}{\zeta(-1+it)}\right)\frac{x^{-1}}{-1+it}\right|\mathbf{d}t \\ &\qquad \leq \frac{2}{x}\int_{12}^{T_u}\frac{2.999\log t + 10.241}{t}\,\mathbf{d}t + \frac{38.344}{x}\int_0^{12}\mathbf{d}t \\ &\qquad = \frac{2.999\log^2 T_u + 20.482\log T_u + 460.128 - 2.999\log^2 12 - 20.482\log 12}{x} \\ &\qquad \leq \frac{2.999\log^2(T+1.155) + 20.482\log(T+1.155) + 390.715}{x} < 3\,\frac{\log^2 T}{x}.\end{aligned}$$

For the integral along with L_u and L_b, we have

$$\begin{aligned}\Bigg|\int_{L_l}\Big(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\Big)\,\mathbf{d}s+\int_{L_b}\Big(-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\Big)\,\mathbf{d}s\Bigg| \\ &\le 2\int_{-1}^{1+(1/\log x)}\Bigg|\Big(-\frac{\zeta'(\sigma+iT_u)}{\zeta(\sigma+iT_u)}\Big)\frac{x^{\sigma+iT_u}}{\sigma+iT_u}\Bigg|\,\mathbf{d}\sigma \\ &\le 2\frac{6.159\log^2 T+2.999\log T+1.285}{T_u}\int_{-1}^{1+(1/\log x)} x^{\sigma}\,\mathbf{d}\sigma \\ &= \frac{(12.318\log^2 T+5.998\log T+2.570)(ex-1/x)}{T_u\log x} \\ &\le \frac{ex(12.318\log^2 T+5.998\log T+2.570)}{(T-u)\log x} \\ &\le \frac{ex(12.319\log^2 T+5.999\log T+2.571)}{T\log x} < 33.488\frac{x\log^2 T}{T\log x}.\end{aligned}$$

This concludes the proof of Lemma 9.1. □

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