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ON WEAK COMPACTNESS IN L_1 SPACES

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ABSTRACT. We will use the concept of strong generating and a simple renorming theorem to give new proofs to slight generalizations of some results of Argyros and Rosenthal on weakly compact sets in $L_1(\mu)$ spaces for finite measures μ.

1. Introduction. The purpose of this note is to show that a simple transfer renorming theorem explains why $L_1(\mu)$-spaces, for finite measures μ, share some properties with superreflexive spaces, though there is no one-to-one bounded linear operator from $L_1(\mu)$ into any reflexive space if $L_1(\mu)$ is nonseparable [**19**, page 232]. The notations used here are standard (see, e.g., [**11**], where we refer, too, for undefined concepts). By a *measure* we always understand a countably additive measure defined on a σ-algebra Σ of subsets of some nonempty set Ω.

Definition 1. We will say that a Banach space X is *strongly generated by a Banach space* Z if there is a bounded linear operator T from Z into X such that, for every weakly compact set $W \subset X$ and every $\varepsilon > 0$, there exists an $m \in \mathbf{N}$ such that $W \subset mT(B_Z) + \varepsilon B_X$. In this case we will say, too, that Z *strongly generates* X.

Remark 2. Definition 1 is motivated by the concept of a *strongly weakly compactly generated Banach space* (SWCG, for short), introduced by Schlüchtermann and Wheeler [**20**]: A Banach space X is SWCG if there exists a weakly compact subset $K \subset X$ such that, for every weakly compact subset $W \subset X$, we can find an $n \in \mathbf{N}$ such

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that $W \subset nK + \varepsilon B_X$ (we say, in this case, that K *strongly generates* X, or that X *is strongly generated by* K, hoping that it does not cause any misunderstanding with Definition 1). Obviously, if X is strongly generated by a reflexive space Z, then it is SWCG. The converse, a straightforward consequence of the factorization theorem of Davis, Figiel, Johnson and Pełczyński [**6**], holds. Precisely, if $K \subset X$ is a weakly compact subset strongly generating X, then there exists a reflexive Banach space Z and a bounded linear mapping $T : Z \to X$ such that $K \subset T(B_Z)$, and so Z strongly generates X.

Note, too, that if X is strongly generated by a Banach space Z via a bounded linear mapping T, then X is strongly generated by the quotient $Z/\ker T$ and now the induced strongly generating mapping $\widehat{T} : Z/\ker T \to X$ is one-to-one.

In [**20**] it is proved that *a Banach space* X *is* SWCG *if and only if the topological space* $(B_{X^*}, \mu(X^*, X))$ *is metrizable*, where $\mu(X^*, X)$ denotes the dual Mackey topology on X^*, i.e., the topology on X^* of the uniform convergence on the family of all absolutely convex and weakly compact subsets of X. It is worth recalling that, according to a result of Grothendieck, see for example, [**16**, subsection 21.6 (4)], for every Banach space X, $(X^*, \mu(X^*, X))$ is complete.

The following result exhibits an important feature of SWCG Banach spaces. We provide here a new, simpler proof of it.

Theorem 3 [**20**]. *Every* SWCG *Banach space is weakly sequentially complete.*

Proof. Let (x_n) be a weakly Cauchy sequence in X. Put $D_n := \overline{\mathrm{aco}}\,\{x_p - x_q; p, q \geq n\}$, $n \in \mathbf{N}$, where $\mathrm{aco}\,(S)$ denotes the absolutely convex hull of a set $S \subset X$. Obviously, $X^* = \cup_{n\in\mathbf{N}} {D_n}^\circ$, where S° denotes the absolute polar in X^* of a set $S \subset X$. In particular, $mB_{X^*} = \cup_{n\in\mathbf{N}} ({D_n}^\circ \cap mB_{X^*})$ for every $m \in \mathbf{N}$. We mentioned above that $(B_{X^*}, \mu(X^*, X))$ is a complete metrizable space. Fix $m \in \mathbf{N}$. The sets $({D_n}^\circ \cap mB_{X^*})$ are $\mu(X^*, X)$-closed; hence, by the Baire category theorem, there exist an $n(m) \in \mathbf{N}$ and an absolutely convex weakly compact subset K_m of X such that

$$({K_m}^\circ \cap mB_{X^*}) \subset ({D_{n(m)}}^\circ \cap mB_{X^*}).$$

By taking polars in X, we get

$$(D_{n(m)} \subset)\ \overline{\text{conv}}\left(D_{n(m)} \cup \frac{1}{m}B_X\right) \subset \overline{\text{conv}}\left(K_m \cup \frac{1}{m}B_X\right)\left(\subset K_m + \frac{1}{m}B_X\right).$$

In particular, $x_p - x_q \in K_m + B_X/m$ for every $p, q \geq n(m)$. Let x^{**} be the weak*-limit of the sequence (x_n) in X^{**}. Then $x^{**} - x_q \in K_m + B_{X^{**}}/m$ for every $q \geq n(m)$, and we obtain $x^{**} \in X + B_{X^{**}}/m$. This happens for every $m \in \mathbf{N}$, so $x^{**} \in X$. □

Throughout the whole note, the following simple consequence of Rosenthal's dichotomy theorem will be frequently used.

Lemma 4. *Let X be a weakly sequentially complete Banach space. Then, the following are equivalent*:

(i) *X contains no isomorphic copy of ℓ_1.*

(ii) *X is reflexive.*

Proof. Obviously, (ii)⇒(i). If (i) holds, every sequence in B_X has, by Rosenthal's dichotomy theorem, a weakly Cauchy (hence weakly convergent because X is weakly sequentially complete) subsequence. Then (ii) follows from the Eberlein-Šmulyan theorem. □

Another useful tool is the following lemma.

Lemma 5. *Let X be a reflexive Banach space strongly generated by a Banach space Z. Then X is isomorphic to a quotient of Z.*

Proof. Let $T : Z \to X$ be a bounded linear mapping witnessing the strong generation. B_X is weakly compact, so for every $\varepsilon > 0$ there exists an $m \in \mathbf{N}$ such that $B_X \subset mTB_Z + \varepsilon B_X$. Then $rB_X \subset \overline{mTB_Z}$ for $0 < r < 1 - \varepsilon$. This follows easily from the separation theorem. A classical argument used in the proof of the open mapping theorem ensures that the sets $\overline{mTB_Z}$ and mTB_Z have the same interior. Then

$\{x \in X; \|x\| < r\} \subset mTB_Z$; hence, the mapping T is open and the factorization $\widehat{T} : Z/\ker T \to X$ of T is an isomorphism onto. □

Proposition 6. *Assume that a Banach space X is strongly generated by a reflexive, respectively superreflexive, space and does not contain an isomorphic copy of ℓ_1. Then X is reflexive, respectively superreflexive.*

Proof. That X is reflexive follows readily from Theorem 3 and Lemma 4. For the superreflexive case, use Lemma 5 and the fact that a quotient of a superreflexive space is superreflexive [**7**, IV.4.6]. □

If $(X, \|\cdot\|)$ is a Banach space, we shall denote again by $\|\cdot\|$ the dual norm on X^* if there is no misunderstanding.

Theorem 7. *Assume that a Banach space X is strongly generated by a superreflexive Banach space. Then X has an equivalent norm $|||\cdot|||$ whose dual norm satisfies the following property: $f_n - g_n \to 0$ uniformly on every weakly compact set in X whenever $f_n, g_n \in S_{(X^*, |||\cdot|||)}$ are such that $|||f_n + g_n||| \to 2$.*

Proof. Assume that $(Z, \|\cdot|_2)$ is a superreflexive space that strongly generates X, via a mapping T. We may assume that $\|\cdot\|_2$ is uniformly rotund (Enflo), cf. e.g., [**7**, Chapter IV]. Then, by a standard argument, cf. e.g., [**7**, Chapter II], the dual norm $|||\cdot|||$ defined on X^* by $|||f|||^2 = \|f\|^2 + \|T^*(f)\|_2^2$ for $f \in X^*$, has the property that $\sup_{T(B_Z)} |f_n - g_n| \to 0$ whenever (f_n) and (g_n) are sequences in $S_{(X^*, |||\cdot|||)}$ such that $|||f_n + g_n||| \to 2$.

We will show that the predual norm to $|||\cdot|||$ is the required norm. Indeed, we need to show that if (f_n) and (g_n) are sequences in $S_{(X^*, |||\cdot|||)}$ such that

$$|||f_n + g_n||| \longrightarrow 2, \tag{1}$$

then $\sup_K |f_n - g_n| \to 0$ for each weakly compact set K in X. For this, let a weakly compact set K in X and $\varepsilon > 0$ be given. From the definition of strong generating, find an $m \in \mathbf{N}$ such that

$K \subset mT(B_Z) + \varepsilon B_X$. Then, from (1), we find an $n_0 \in \mathbf{N}$ such that

$$\sup_{T(B_Z)} |f_n - g_n| \leq \frac{\varepsilon}{m}$$

for each $n > n_0$. So, for each $n > n_0$,

$$\sup_K |f_n - g_n| \leq \sup_{mT(B_Z)} |f_n - g_n| + \sup_{\varepsilon B_X} |f_n - g_n| \leq m\frac{\varepsilon}{m} + 2\varepsilon = 3\varepsilon. \qquad \square$$

The following corollary strengthens Proposition 6.

Corollary 8. *Let X be a Banach space strongly generated by a superreflexive space. Then X admits an equivalent norm the restriction of which to any reflexive subspace Y of X is uniformly Fréchet differentiable. In particular, any such subspace Y is superreflexive.*

Proof. The restriction to Y of the norm on X defined in Theorem 7 is, by Šmulyan's lemma (see, for example, [**7**, Chapter II]), uniformly Fréchet differentiable, and hence X is superreflexive (see, e.g., [**7**, Corollary IV.4.6]). $\square$

Remark 9. In Corollary 8 some condition on the subspace Y is needed in order to ensure that it is superreflexive (here we used reflexivity). In fact, Rosenthal's counterexample to the heredity problem for WCG Banach spaces (a subspace of some $L_1(\mu)$ space which is not WCG) proves that there are subspaces of strongly superreflexive generated Banach spaces, see Proposition 12, which are not WCG, and hence not superreflexive.

Recall that a compact topological space K is *uniform Eberlein* if it is homeomorphic to a compact subset of (H, w), where H is a Hilbert space. A well-known characterization of uniform Eberlein compacta is given by the following result due to Farmaki (here, $\Sigma(\Gamma) := \{s \in \mathbf{R}^\Gamma : \#\{\gamma \in \Gamma; s(\gamma) \neq 0\} \leq \aleph_0\}$, and this set is equipped with the product topology): *Let Γ be an uncountable set, and let $K \subset \Sigma(\Gamma) \cap [-1,1]^\Gamma$ be a compact subset. Then the set K is uniform Eberlein compact if, and* *only if, for every* $\varepsilon > 0$ *there is a decomposition* $\Gamma = \cup_{n=1}^{\infty} \Gamma_n^{\varepsilon}$ *such that, for all* $n \in \mathbf{N}$ *and for all* $k \in K$, $\#\{\gamma \in \Gamma_n^{\varepsilon};\ |k(\gamma)| > \varepsilon\} < n$, see [**12**]; see also [**9**].

We have the following Grothendieck-like stability result:

Proposition 10. *Let* X *be a Banach space. Let* K *be a subset of* X *such that, for every* $\varepsilon > 0$, *there exists a uniform Eberlein compactum* U_ε *in* (X, w) *with* $K \subset U_\varepsilon + \varepsilon B_X$. *Then* (K, w) *is a uniform Eberlein compactum.*

Proof. We may assume that $K \subset B_X$. Let $X_0 := \overline{\operatorname{span}} \cup \{U_\varepsilon;\ \varepsilon \text{ rational}, \varepsilon > 0\}$, a WCG Banach space. Obviously K has the same property stated, now with respect to (X_0, w), so from the very beginning we may also assume that X is WCG. By [**1**] there exists, for some set Γ, a one-to-one linear mapping $T : X \to c_0(\Gamma)$, such that $\|T\| \le 1/2$. Then, $U_\varepsilon \subset 2B_X$ (so $TU_\varepsilon \subset B_{c_0(\Gamma)}$) for $0 < \varepsilon \le 1$. Using Farmaki's characterization mentioned above, for every $0 < \varepsilon \le 1$ there is a decomposition $\Gamma = \cup_{n=1}^{\infty} \Gamma_n^{\varepsilon/2}$ such that, for all $n \in \mathbf{N}$ and for all $u \in U_\varepsilon$,

$$\#\left\{\gamma \in \Gamma_n^{\varepsilon/2};\ |Tu(\gamma)| > \frac{\varepsilon}{2}\right\} < n.$$

Now, if $k \in K$, we can write $k = u + \varepsilon b$, where $u \in U_\varepsilon$ and $b \in B_X$. Hence, $\{\gamma \in \Gamma_n^{\varepsilon/2}; |Tk(\gamma)| > \varepsilon\} \subset \{\gamma \in \Gamma_n^{\varepsilon/2}; |Tu(\gamma)| > \varepsilon/2\}$, and the last set has cardinality $< n$. Thus, this decomposition can be used in Farmaki's theorem, this time for the set TK. This holds for every $1 \ge \varepsilon > 0$, showing that K is a uniform Eberlein compactum. □

Corollary 11. *Assume that* X *is a Banach space strongly generated by a superreflexive space. Then any compact subset* K *of* (X, w) *is uniform Eberlein.*

Proof. Assume that X is strongly generated (via the mapping T) by a superreflexive space Z. In the weak topology, the unit ball of a superreflexive space is a uniform Eberlein compactum [**4**]. Since a quotient of a superreflexive space is superreflexive, see e.g., [**7**, IV.4.6], we may assume that T is one-to-one. It follows that $(mT(B_Z), w)$ is a uniform Eberlein compactum. Now it is enough to use Proposition 10. □

The rest of the paper shows some applications of the former results to the space $L_1(\mu)$.

Proposition 12. *If μ is a finite measure defined on a σ-algebra Σ of subsets of a certain set Ω, then $L_1(\mu)$ is strongly generated by a Hilbert space.*

Proof. We will use [**15**, page 17]. Assume without loss of generality that μ is a probability measure. By using the identity operators, we have $B_{L_\infty(\mu)} \subset B_{L_2(\mu)} \subset B_{L_1(\mu)}$. Let K be a weakly compact set in the unit ball of $L_1(\mu)$. Then K is *uniformly integrable* in $L_1(\mu)$ [**8**, page 292], i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for every $x \in K$, $\int_M |x|\, d\mu < \varepsilon$ whenever $M \in \Sigma$ and $\mu(M) < \delta$.

For $k \in \mathbf{N}$ and for $x \in K$, put $M_k(x) := \{t \in \Omega; |x(t)| \geq k\}$, and write $x = x_1 + x_2$, where $x_1 := x.\chi(\Omega \setminus M_k(x))$ and $x_2 := x.\chi(M_k(x))$ (where $\chi(S)$ denotes the characteristic function of a set $S \subset \Omega$). Let $a_k(K) := \sup\{\|x_2\|_1; x \in K\}$. Then

$$K \subset kB_{L_\infty(\mu)} + a_k(K)B_{L_1(\mu)} \subset kB_{L_2(\mu)} + a_k(K)B_{L_1(\mu)}.$$

We have $k\mu(M_k(x)) \leq \|x_2\|_1 \leq 1$; hence, $\mu(M_k(x)) \leq 1/k$ for all $x \in K$. From the uniform integrability of K, we get that $a_k(K) \to 0$ when $k \to \infty$. This finishes the proof. □

On the other hand, we have the following result.

Corollary 13 [**18**]. *Let X be a subspace of $L_1(\mu)$, for a finite measure μ. Assume that X does not contain an isomorphic copy of ℓ_1. Then X is superreflexive.*

Proof. Combine Proposition 12 and Corollary 8. □

Corollary 14 [**2**]. *Every compact subset of the space $(L_1(\mu), w)$, for a finite measure μ, is uniform Eberlein.*

Proof. Combine Proposition 12 and Corollary 11. □

Remark 15. Note that for the proof of Corollary 14 we do not need to use the full strength of Corollary 11; indeed, the space $L_1(\mu)$ is strongly generated by a Hilbert space, so the appeal to [**4**] is not necessary.

Remark 16. For an uncountable set Γ, the space $\ell_{3/2}(\Gamma)$ is superreflexive and not Hilbert generated. Indeed, it follows from Pitt's theorem that there are no bounded linear mappings with dense images from $\ell_2(\Gamma)$ into $\ell_{3/2}(\Gamma)$, see [**10**].

Remark 17. The research on this paper was motivated by the paper of Giles and Sciffer [**13**], where it is implicitly shown that every reflexive subspace of $L_1(\mu)$ is superreflexive, which is part of a well-known result of Rosenthal in [**18**]. The proof of this result given in this note is different and slightly more general. The proof of Theorem 3 is also different from the original one.

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