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EXISTENCE AND UNIQUENESS RESULTS FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we give some new results concerning solvability of first order singular problems. We study mainly the differential equation $x' = f(t,x)$. We prove that the existence theorem of Caratheodory remains true if f is not defined at the given initial point and satisfies more flexible conditions. This theorem allows us to develop theorems on the existence and uniqueness of the solution of systems of differential equations and high order differential equations. We introduce a more general form of the initial value problems and try to develop this idea.

1. Introduction. In this paper we consider the problem

$$x' = f(t,x), \qquad x(\tau) = \xi, \tag{1}$$

where f is singular in $(t,x) = (\tau,\xi)$. There are different attempts to develop the theory of singular initial value problems. Peano [**9**] and Perron [**10**] have considered the problem (1) when $(\tau,\xi) = (0,0)$, with f continuous, assume the existence of continuous functions $m_1(t)$ and $m_2(t)$ with $m_1(0) = m_2(0)$, $m_1(t) \le m_2(t)$ and

$$D_{\pm} m_1(t) \le f(t, m_1(t)), \qquad D_{\pm} m_2(t) \ge f(t, m_2(t)),$$

and prove the existence of a minimal and a maximal solution of (1) between $m_1(t)$ and $m_2(t)$. In 1931, Dragoni [**6**] considered systems and allowed $m_1(0) \le 0 \le m_2(0)$. Recently, Frigon and O'Regan [**7**] have considered noncontinuous $m_1(t)$ and $m_2(t)$. Marcelli and Rubbioni [**8**] have considered the situation where $m_1(t)$ and $m_2(t)$ can cross. Cherpion and De Coster [**2**] have studied the situation where $m_1(t)$ and $m_2(t)$ are not necessarily continuous nor ordered. In most cases, authors tried to develop so-called lower and upper solution methods.

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In this work we tried to develop theory regardless of the existence of minimal and maximal solutions.

To achieve this goal we introduce a more general concept of the initial value problem. If the point a is a point of accumulation of the domain of function, then we introduce the "set of values" of the function as $t \to a$. For example, the set of values of $\cos(t^{-1})$ as $t \to 0$ is the set $[-1,1]$. The function $\cos(1/t)$ behaves almost like the continuous function about the point 0, and this function can be considered roughly, as a solution of the differential equation $x' = t^{-2}\sin t^{-1}$, with "initial value" (or initial condition) $x(0) = [-1,1]$. That is, $\cos(1/t)$ is, in fact, a solution of the given equation such that for any $c \in [-1,1]$ there exists a sequence $\{t_n\}$, with $t_n \to 0$ and $f(t_n) \to c$.

Remark 1. If the function $f(t)$ is defined on the set I and a is an accumulation (or limit) point of I, we use the symbol $f\langle a\rangle$ for the "set of values" of $f(t)$ at the point a. That is,

$$f\langle a\rangle \stackrel{\text{def}}{=} \left\{A : A = \lim_{t_n \to a} f(t_n), \text{ for some sequence } (t_n) \subset I\right\}.$$

The simplest initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, can be considered as a special form of the (singular) initial condition problems of type $x' = f(t,x)$, $x\langle t_0\rangle = [x_1, x_2]$, where x_1, x_2 are real numbers.

We denote by

$$f\langle a+\rangle \stackrel{\text{def}}{=} \{A : A = \lim f(t_n), \text{ for some sequence } (t_n) \subset [a, a+\epsilon) \cap I\}$$

and

$$f\langle a-\rangle \stackrel{\text{def}}{=} \{A : A = \lim f(t_n), \text{ for some sequence } (t_n) \subset (a-\epsilon, a] \cap I\}.$$

Definition 1. A function f is s-continuous (set-continuous, symmetric continuous, etc.) at the point $t = a$ if $f\langle a+\rangle = f\langle a-\rangle = f\langle a\rangle = [A,B]$ for some real A and B. (If $f(t)$ is continuous at a, then $f\langle a+\rangle = f(a+) = f(a-) = f\langle a-\rangle = f(a)$.)

Similarly, we can define both right s-continuity and left s-continuity. For example, function f is right s-continuous at the point $t = a$ if $f\langle a+\rangle = [A,B]$ for some real A and B.

Definition 2. A function f is an inclusive s-continuous (is-continuous) at the point $x = a$ if f is right s-continuous and left s-continuous and $f\langle a_+\rangle \subseteq f\langle a_-\rangle$ or $f\langle a_-\rangle \subseteq f\langle a+\rangle$.

Definition 3. A function f is weak s-continuous (ws-continuous) at the point $x = a$ if f is right s-continuous and left s-continuous and $f\langle a_+\rangle \cap f\langle a_-\rangle \neq \varnothing$.

Definition 4. The function $f(t)$ is said to be an s-Lebesgue integrable on (a, b), if $f(t)$ is Lebesgue integrable on $[a + \varepsilon, b - \delta]$ for every small enough positive ε, $\delta > 0$ and

$$\left| \int_{a+\varepsilon}^{b-\delta} f(t)\,dt \right| \leq \alpha$$

for some constant α (α does not depend on ε and δ).

Denote by

$$\overset{b}{\underset{a}{S}} f(t)\,dt = \frac{1}{2}\left(\overline{\lim_{\delta,\varepsilon\to 0}} \int_{a-\varepsilon}^{b-\delta} f(t)\,dt + \underline{\lim_{\delta,\varepsilon\to 0}} \int_{a-\varepsilon}^{b-\delta} f(t)\,dt \right).$$

If $f(t)$ is an s-Lebesgue integrable on (a, b), then there exists a sequence $a_n \to a+$, $a_n > a_{n+1}$, $b_n \to b$, $b_n < b_{n+1}$, such that the sequence $\int_{a_n}^{b_n} f(t)\,dt$ converges to $\overset{b}{\underset{a}{S}} f(t)\,dt$. Let's explain this statement in case $a = 0$, and for simplicity suppose that b is a regular point. That is,

$$\begin{aligned}\overset{b}{\underset{a}{S}} f(t)\,dt &= \frac{1}{2}\left(\overline{\lim_{\varepsilon\to 0}} \int_{a+\varepsilon}^{b} f(t)\,dt + \underline{\lim_{\varepsilon\to 0}} \int_{a+\varepsilon}^{b} f(t)\,dt \right) \\ &= \frac{1}{2}\left(\lim_{\alpha_n\to 0} \int_{\alpha_n}^{b} f(t)\,dt + \lim_{\beta_n\to 0} \int_{\beta_n}^{b} f(t)\,dt \right)\end{aligned}$$

for some sequences (α_n) and (β_n). We may take $\alpha_n < \beta_n$ for example. Then we have

$$\overset{b}{\underset{a}{S}} f(t)\,dt = \lim_{\beta_n\to 0} \int_{\beta_n}^{b} f(t)\,dt + \frac{1}{2} \lim_{n\to\infty} \int_{\alpha_n}^{\beta_n} f(t)\,dt.$$

The interval (α_n, β_n) has a point τ_n such that

$$\int_{\alpha_n}^{\beta_n} f(t)\,dt = 2\int_{\tau_n}^{\beta_n} f(t)\,dt$$

(since the values of the integral

$$\int_{\tau}^{\beta_n} f(t)\,dt$$

assume all values between 0 and $\int_{\alpha_n}^{\beta_n} f(t)\,dt$ as $\tau \in [\alpha_n, \beta_n]$). So we have

$$\overset{b}{\underset{a}{S}} f(t)\,dt = \lim_{\tau_n \to 0} \int_{\tau_n}^{b} f(t)\,dt.$$

And let's note that if $am(t) \le f(t) \le bm(t)$ on $(0,1]$ for two positive a and b, and

$$\overset{b}{\underset{0}{S}} m(t)\,dt = \frac{1}{2}\left(\lim_{\alpha_n \to 0} \int_{\alpha_n}^{b} m(t)\,dt + \lim_{\beta_n \to 0} \int_{\beta_n}^{b} m(t)\,dt\right)$$

where the sequences (α_n) and (β_n) correspond to upper and lower limits for the s-Lebesgue integral of $m(t)$; then $f(t)$ is s-Lebesgue integrable, and

$$\overset{b}{\underset{0}{S}} f(t)\,dt = \frac{1}{2}\left(\lim_{\alpha_n \to 0} \int_{\alpha_n}^{b} f(t)\,dt + \lim_{\beta_n \to 0} \int_{\beta_n}^{b} f(t)\,dt\right)$$

for the same sequences (α_n) and (β_n). Indeed, for example,

$$\lim_{\alpha_n \to 0} \int_{\alpha_n}^{b} f(t)\,dt$$

goes to the supremum since

$$\int_{\tau}^{\alpha_n} f(t)\,dt \le b\int_{\tau}^{\alpha_n} m(t)\,dt < \varepsilon$$

for any small enough positive ε as τ changes between 0 and α_n.

Definition 5. The s-Lebesgue integral of the s-Lebesgue integrable on the (a,b) function is the number

$$\underset{a}{\overset{b}{S}} f(t)\,dt.$$

It is clear that, for the integrable on $[a,b]$ functions, we have $\underset{a}{\overset{b}{S}} f(t)\,dt = \int_a^b f(t)\,dt$, and the function

$$F(\tau) = \underset{a}{\overset{\tau}{S}} f(t)\,dt$$

is continuous on $a < \tau < b$. Further, $F(\tau)$ is right s-continuous at a and left s-continuous at b.

For example,

$$\underset{0}{\overset{b}{S}} t^{-2} \sin t^{-1}\,dt = \cos b^{-1}.$$

(It is easily seen that the function $t^{-2}\sin t^{-1}$ is not Lebesgue integrable in the normal sense).

Let's note a few interesting properties of ws-continuous and s-Lebesgue integrable functions. Here we consider the functions having at most a countable and nowhere dense set of points of discontinuity.

1) ws-continuous on the $[a,b]$ function "assumes" (tends to) all values between $f(a)$ and $f(b)$: More exactly, for each real number $A \in f\langle a\rangle$ and $B \in f\langle b\rangle$ and $C \in (A,B)$ (or (B,A)) there exists a $c \in (a,b)$ such that $f\langle c\rangle \ni C$, (that is, there exists the sequence $\{x_n\} \subset (a,b)$, such that $f(x_n) \to C$).

2) If $g(x)$ is ws-continuous and bounded, then $g(x)$ is integrable and $\int_a^b g(x)\,dx = C(b-a)$, where $C \in g\langle c\rangle$ (or $C = g(c)$, if c is not a point of discontinuity).

3) If $g(x)$ is ws-continuous and bounded, then $g(x)$ has a continuous antiderivative defined by the formula $G(x) = \int_a^x g(t)\,dt$.

4) If $f(t)$ is s-Lebesgue integrable on (a,b), then

$$\left(\underset{a}{\overset{x}{S}} f(t)\,dt\right)' = f(x)$$

almost everywhere.

5) If $f(t)$ is s-Lebesgue integrable on (a,b), then the function

$$F(x) = \overset{x}{\underset{a}{S}} f(t)\,dt$$

is right s-continuous at a and left s-continuous at b, and F is bounded and continuous on (a,b).

For the proof of the first property, it is enough to note that if $C \notin \{f\langle x\rangle\}$, then the sets $M_+ = \{x : f \text{ is continuous and } f(x) > C)\}$ and $M_- = \{x : f \text{ is continuous and } f(x) < C\}$ have no common points, but $M_+ \cup M_- \cup S = [a,b]$, where S is a countable and nowhere dense set; that is, we have the sequences $\{x_n\} \subset M_+$ and $\{y_n\} \subset M_-$ such that $\lim x_n = \lim y_n = c$. Then, by the definition of continuity or ws-continuity at the point c, we have that $f(c) = C$ or $f\langle c\rangle \ni C$. For the proof of the second and third properties of the ws-continuous functions, we need to apply Riemann criteria for the integrability of the function (see, for example [**12**, Theorem 8.15 or 8.16]). The fourth property can be obtained immediately, since $f(t)$ is integrable on $[a+\delta, b-\varepsilon]$ for any small enough δ and ε. The fifth property can be received from the definition of s-Lebesgue integrability. (Note that ws-continuous functions and s-Lebesgue integrable functions have a lot of elegant properties and may play an important role in different fields. If the process seems extreme, but "controllable" changes regularly, then the corresponding mathematical model may have the ws-continuous and s-Lebesgue integrable types of functions.)

In a similar manner, we can define an s-Lebesgue integrable on the (a,b) function for the function having "controllable discontinuity" at the finite or countable number points $b_n \in (a,b)$. For simplicity, let's consider just the case $b_1 < b_2 < \cdots < b_n < b_{n+1} < \cdots$, $b_n \to b$. Now we'll define

$$\overset{b}{\underset{a}{S}} f(t)\,dt = \overset{b_1}{\underset{a}{S}} f(t)\,dt + \sum_{k=1}^{\infty} \overset{b_{k+1}}{\underset{b_k}{S}} f(t)\,dt.$$

That is, if $f(t)$ is s-Lebesgue integrable on (a,b_1) and (b_k,b_{k+1}) for every fixed k, and the summation on the righthand side is finite, then we say that $f(t)$ is s-Lebesgue integrable on (a,b). We'll use the term *Lebesgue-defect points of* $f(t)$ or simply L-*defect points of* $f(t)$ for the points of type b_k.

The central problem of the theory of ordinary differential equations may be phrased as follows:

Problem (Initial-value problem). *Let D be a domain in the (t, x) plane, and suppose f is a real-valued function with domain D and (τ, ξ) is a given point in D. To find a function φ defined on a real interval I such that $(t, \varphi(t)) \in D$, and*

$$\varphi(\tau) = \xi$$

and

$$\varphi'(t) = f(t, \varphi(t)).$$

This problem is denoted by

$$x' = f(t, x), \qquad x(\tau) = \xi \tag{2}$$

and is equivalent to the integral equation

$$x(t) = \xi + \int_{\tau}^{t} f(s, x(s))\, ds. \tag{3}$$

Clearly the integral in (3) makes sense for many functions f which are not continuous. Let f be a real valued (not necessarily continuous) function defined in some set D of the (t, x) space. Then one can extend the notion of the differential equation as follows:

Problem. *To find a function φ defined on a real interval I such that $(t, \varphi(t)) \in D$, and*

$$\varphi'(t) = f(t, \varphi(t)) \quad \textit{for all} \quad t \in I,$$
$$\textit{except on a set of Lebesgue measure zero.}$$

If such an interval I and the function φ exist, then φ is said to be a solution of (1) in the extended sense on I.

Caratheodory [**1**] has proved the following quite general theorem under the assumption that f is bounded by a Lebesgue integrable function of t:

Theorem 1 (Caratheodory). *Let f be defined on the rectangle*

$$R : |t-\tau| \leq a,\ |x-\xi| \leq b,$$

and suppose that for all $x \in [\xi - b, \xi + b]$ it is measurable in t, and for $t \in [\tau - a, \tau + a]$ it is continuous in x. If there exists a Lebesgue integrable function m on the interval $|t-\tau| \leq a$ such that

$$|f(t,x)| \leq m(t), \quad ((t,x) \in R), \tag{4}$$

then there exists a solution φ of (2) *in the extended sense on some interval $|t-\tau| \leq \beta$, $\beta > 0$.*

We'll generalize this theorem to the case when f is not defined at the point (τ, ξ). We suppose that f is defined on the set

$$D : 0 < |t-\tau| \leq a,\ 0 \leq |x-\xi| \leq b.$$

At the same time, we'll generalize condition (4) as follows:

Let

$$c_1 m_1(t) \leq f(t,x) \leq c_2 m_2(t), \quad ((t,x) \in D) \tag{5}$$

for some integrable on $[\tau - a, \tau - \delta] \cup [\tau + \delta, \tau + a]$ functions $m_i(t)$, for any small enough positive δ, where c_1 and c_2 are fixed real numbers. It is clear that (5) is a strict extension of (4), for example, $f(t,x) = t^{-2}\cos(t^{-1})$ satisfies (5) but not (4) at some neighborhood of (0,0) and $m_i(t) = (1/t^2)\cos(1/t)$ is integrable on $[-a,-\delta] \cup [\delta, a]$ for any positive δ.

2. The generalization of Caratheodory's theorem.

We begin with the definition of the bounded and continuous set on $(a,b]$ functions.

Let $BC_{(a,b]}$ be a set of all bounded and continuous functions $x(t)$ on the set $(a,b]$, with norm $||x(t)|| = \sup_{t\in(a,b]} |x(t)|$. It is clear that every $x(t) \in BC_{(a,b]}$ is right s-continuous at a.

Lemma. *$BC_{(a,b]}$ is complete.*

Proof. Let $\{x_n(t)\}$ be a Cauchy sequence in $BC_{(a,b]}$. That is, for any $\varepsilon > 0$, there exists an integer N such that

$$||x_n(t) - x_p(t)|| = \sup_{t\in(a,b]} |x_n(t) - x_p(t)| < \varepsilon$$

for all $n, p > N$. For any positive δ, the sequence $\{x_n(t)\}$ considered as a sequence in $C_{[a+\delta,b]}$ converges to a continuous function $x(t)$ on $[a+\delta, b]$ (since the set $C_{[a+\delta,b]}$ is complete). That is, for any fixed $t \in (a,b]$, the limiting function $x(t)$ is continuous at t and therefore is bounded on every interval $[a+\delta, b]$. If $x(t)$ is unbounded at any right neighborhood of a, then there exists a sequence $t_r \in (a,b]$ such that $|x(t_r)| > r$, $t_r \to a+$. Then there exists a positive integer n_r such that $|x_{n_r}(t_r)| > r-1$ and so $\lim_{n_r\to\infty}(\sup_{t\in(a,b]} |x_{n_r}(t)|) \to \infty$. That is, for any fixed n_r it is possible to find a number n_q such that $\sup_{t\in(a,b]} |x_{n_r}(t) - x_{n_q}(t)| > 1$, and therefore the sequence $\{x_{n_r}(t)\}$ is not a Cauchy sequence. But the subsequence of every Cauchy sequence must be a Cauchy sequence, and this contradiction shows that $x(t)$ is bounded function.

In a similar manner, we can prove that the space $BC_{(a,b)}$ of continuous and bounded functions on (a,b) is complete.

Now we generalize the initial value problem:

Problem $\mathbf{A}_+$. *Let $f(t,x)$ be defined on the set*

$$D_+ : 0 < t - \tau \le a, \quad 0 \le |x - \xi| \le b.$$

Find a continuous function φ defined on $(\tau, \tau+\beta]$, with $\beta \le a$, such that $(t, \varphi(t)) \in D_+$ and $\varphi'(t) = f(t, \varphi(t))$ for all $t \in (\tau, \tau+\beta)$ except on a set of Lebesgue measure zero, and $\varphi\langle\tau+\rangle \ni \xi$.

Theorem 2. *Let f be defined on D_+, and suppose for all $x \in [\xi - b, \xi + b]$ it is measurable in t, and for all $t \in (\tau, \tau + a]$, it is continuous in x. Let*

$$c_1 m_1(t) \le f(t,x) \le c_2 m_2(t)$$

for two fixed real numbers c_1 and c_2, where $m_i(t)$ is s-Lebesgue integrable on $(\tau, \tau + a]$ (with only L-defect point τ)and satisfies the next conditions

1) *there exists a positive $\beta \leq a$ such that*

$$\alpha_1 \leq \int_{\tau+\delta}^{\tau+\sigma} m_i(t)\,dt \leq \alpha_2, \quad \textit{for all} \quad \sigma > \delta > 0, \quad \tau + \sigma \leq \beta \tag{6}$$

and $-b < c_1\alpha_1 \leq c_2\alpha_2 < b$,

2)

$$\lim_{\sigma \to 0} \varlimsup_{\delta \to 0} \left(\left| \int_{\tau+\delta}^{\tau+\sigma} m_i(t)\,dt \right| \right) = 0. \tag{7}$$

Then there exists a solution φ of the problem A_+ in the extended sense on some interval $|t - \tau| \leq \beta$ satisfying $\varphi\langle\tau+\rangle \ni \xi$.

Proof. First of all, let's note that conditions 1), 2) are not very strong and any integrable on the $[\tau, \tau + a]$ function in fact satisfies these conditions. The function $t^{-2}\sin(1/t)$ satisfies (6) and (7) but is not integrable on (0,1], since

$$\int_{\delta}^{1} t^{-2} \sin(1/t)\,dt = \cos 1 - \cos\frac{1}{\delta}, \quad \delta > 0.$$

(It is possible to prove another theorem on the existence of the solution without condition 2), if this limit is a number $d \neq 0$. Then the initial condition will be "perturbed" by d units, that is, instead of $\varphi\langle\tau+\rangle \ni \xi$, we'll have to take $\varphi\langle\tau+\rangle \ni \xi + d$.)

For the sake of simplicity, we take $\tau = \xi = 0$. Since

$$\left| \int_{\delta}^{\beta} m_i(t)\,dt \right| \leq \max\{|\alpha_1|, |\alpha_2|\} \equiv \alpha$$

for all $\delta > 0$, there exists a sequence (δ_n), $\delta_n \to 0$ such that the sequence

$$m_{ni} = \int_{\delta_n}^{\beta} m_i(t)\,dt$$

is convergent and tends to $\overset{\beta}{\underset{0}{S}} m_i(t)\,dt$. We denote by

$$M_i(\lambda) = \overset{\lambda}{\underset{0}{S}} m_i(t)\,dt.$$

It is clear that $(t, c_k M_i(t)) \in D_+$, $k = 1, 2$, for $0 < t < \beta < a$. We define φ_j, $j = 1, 2, \dots$, by

$$\begin{aligned} &\varphi_j(t) = 0 && 0 < t \le \frac{\beta}{j} \\ &\varphi_j(t) = \lim_{n\to\infty} \int_{\delta_n}^{t} f(s, \varphi_j(s))\,ds && \frac{\beta}{j} < t \le \beta. \end{aligned} \tag{8}$$

For any fixed $j \ge 1$, the first formula in (8) defines φ_j on $(0, (\beta/j)]$ and, since $(t, 0) \in D_+$ for $0 < t \le \beta/j$, the second formula in (8) defines φ_j as a continuous function on the interval $\beta/j < t \le (2\beta)/j$. (Since $\overset{\delta_n}{\underset{0}{S}} m_i(t)\,dt \to 0$ as $n \to \infty$, we have that the function φ_j is s-right continuous at β/j and $0 \in \varphi_j\langle \beta/j \rangle$.) Further, on this latter interval,

$$c_1 \overset{t-(\beta/j)}{\underset{0}{S}} m_1(s)\,ds \le \varphi_j(t) \le c_2 \overset{t-(\beta/j)}{\underset{0}{S}} m_2(s)\,ds. \tag{9}$$

Then the second formula in (8) defines φ_j for $(2\beta)/j < t \le (3\beta)/j$, and clearly φ_j satisfies (9) on this interval. Therefore, by induction, (8) defines φ_j as a continuous function on $(0, (\beta/j)) \cup ((\beta/j), \beta]$. If t_1 and t_2 are two points in the interval $((\beta/j), \beta]$, then on account of (5) and (8),

$$\begin{aligned} \varphi_j(t_2) - \varphi_j(t_1) &= \int_{t_1-(\beta/j)}^{t_2-(\beta/j)} f(s, \varphi_j(s))\,ds \\ &\le c_2 \int_{t_1-(\beta/j)}^{t_2-(\beta/j)} m_2(s)\,ds \longrightarrow 0, \end{aligned} \tag{10}$$

(regardless of j) for large enough j, where $c = \max\{|c_1|, |c_2|\}$. Since $m_i(t)$ is integrable on every interval $[\delta, a]$, we have that $M_i(t)$ is continuous for all $t \in (0, a]$. Thus, the set of functions $\{\varphi_j(t)\}$ has the following properties: All $\varphi_j(t)$ are continuous on $[\delta,\beta]$ for any fixed, small enough $\delta>0$ and for large enough j, and

$$c\left[M\left(t_2-\frac{\beta}{j}\right)-M_1\left(t_1-\frac{\beta}{j}\right)\right]\le\varphi_j(t_2)-\varphi_j(t_1)\le c_2\left[M_2\left(t_2-\frac{\beta}{j}\right)-M_2\left(t_1-\frac{\beta}{j}\right)\right]. \tag{11}$$

Let $\delta>0$ be a fixed positive number and j_0 a positive integer such that all functions $\{\varphi_{j_0},\varphi_{j_0+1},\dots\}$ are continuous on $[\delta,\beta]$ and $\beta/j_0<\delta$. Then, on account of (11), we have that the set $\{\varphi_j(t)\}$, $j\ge j_0$, is an equicontinuous set on $[\delta,\beta]$. Inequality (8) shows that the sequence $\{\varphi_j(t)\}$ is pointwise bounded. Consequently, it follows by the Ascoli lemma [**5**, Section 1] that there exists a subsequence $\{\varphi_{j_k}(t)\}$ which converges uniformly on $[\delta,\beta]$ to a continuous limit function ϕ_1, as $k\to\infty$. Now the sequence $\{\psi_k\stackrel{\text{def}}{=}\varphi_{j_k}(t)\}$ considered as a continuous function on $[\delta/2,\beta]$ again has a subsequence $\{\psi_{k_l}\}$, which converges uniformly to ϕ_2 on $[\delta/2,\beta]$. It is clear that $\phi_2(t)=\phi_1(t)$ for $t\in[\delta,\beta]$. Continuing this process, we conclude that there exists a subsequence $\{\varphi_{j_r}(t)\}$, which converges to a continuous function $\varphi(t)$ on $(0,\beta]$. Formula (6) implies that $\varphi(t)\in BC_{(0,\beta]}$. Since $f(t,x)$ is continuous in x, we have $f(t,\varphi_{j_r}(t))\to f(t,\varphi(t))$ as $r\to\infty$, for any fixed $t\in(0,\beta]$. The dominated convergence theorem due to Lebesgue gives

$$\lim_{r\to\infty}\int_{\delta_n}^t f(s,\varphi_{j_r}(s))\,ds=\int_{\delta_n}^t f(s,\varphi(s))\,ds$$

for any fixed $\delta_n>0$. Then we have that

$$\lim_{r\to\infty}\overset{t}{\underset{0}{S}} f(s,\varphi_{j_r}(s))\,ds=\overset{t}{\underset{0}{S}} f(s,\varphi(s))\,ds \tag{12}$$

for any $t\in(0,\beta]$. But

$$\varphi_{j_r}(t)=\overset{t}{\underset{0}{S}} f(s,\varphi_{j_r}(s))\,ds-\int_{t-(\beta/j_r)}^t f(s,\varphi_{j_r}(s))\,ds, \tag{13}$$

where it is clear that the latter integral tends to zero as $r\to\infty$. Therefore, using (12) and (13), it follows that

$$\varphi(t)=\overset{t}{\underset{0}{S}} f(s,\varphi(s))\,ds. \tag{14}$$

This relationship implies that $\varphi'(t_0) = f(t_0, \varphi(t_0))$ for almost all $0 < t_0 < \beta$. Indeed, for any positive δ,

$$\varphi(t) - \varphi(\delta) = \int_\delta^t f(s, \varphi(s))\, ds$$

and since $f(t, \varphi(t))$ is measurable, by the Lebesgue theorem [**11**, Theorem 6.2.1], we have that

$$\varphi'(t) = f(t, \varphi(t)),$$

for all t except on a set of Lebesgue measure zero. Let's show that $\varphi\langle 0\rangle$ "assumes" any value between

$$B = \overline{\lim_{\sigma\to 0}} \lim_{\delta_n \to 0} \int_{\delta_n}^{\sigma} f(s, \varphi(s))\, ds$$

and

$$A = \underline{\lim_{\sigma\to 0}} \lim_{\delta_n \to 0} \int_{\delta_n}^{\sigma} f(s, \varphi(s))\, ds.$$

Since the sequence $\int_{\delta_n}^t f(s, \varphi(s))\, ds$ converges as $n \to \infty$, the integrals

$$\int_{\delta_n}^{\delta_m} f(s, \varphi(s))\, ds$$

approach zero as $m \to \infty$ and for $n > m$.

Let C be an arbitrary number from (A, B). For fixed large enough m, we have that

$$\lim_{\delta_n \to 0} \int_{\delta_n}^{\delta_m} f(s, \varphi(s))\, ds \equiv \varphi(\delta_m) \longrightarrow 0 \quad \text{as} \quad m \to \infty,$$

that is, immediately we have $0 \in [A, B]$ (and $0 \in \varphi\langle 0\rangle$). Let's say, for simplicity, $C > 0$ and $C < B$. By the definition of an upper limit for any positive $\varepsilon < (B-C)/2$, there exists a large enough m and $\lambda < \delta_m$ such that

$$\int_{\delta_k}^{\lambda} f(s,\varphi(s))\,ds = B + \varepsilon_k, \quad |\varepsilon_k| < \varepsilon,$$

for some $k > m$. That is, as μ changes from δ_k to λ, values of the integral

$$\int_{\delta_k}^{\mu} f(s,\varphi(s))\,ds$$

assume all values between 0 $(= \int_{\delta_k}^{\mu})$ and $B+\varepsilon_k$, and so we can find λ_1 such that $\int_{\delta_k}^{\lambda_1} f(s,\varphi(s))\,ds = C$. Taking δ_k instead of δ_m and continuing this process, we can find δ_p and λ_2 such that

$$\int_{\delta_p}^{\lambda_2} f(s,\varphi(s))\,ds = C,$$

and so on. That is,

$$\lim_{p\to\infty} \lim_{\delta_n \to 0} \int_{\delta_n}^{\lambda_p} f(s,\varphi(s))\,ds = C = \lim_{p\to\infty} \varphi(\lambda_p)$$

and so $C \in \varphi\langle 0\rangle$. The same can be established for $C < 0$ and $C > A$. Thus, we have $\varphi\langle 0\rangle \supset (A,B)$, and since $\varphi\langle 0\rangle$ is a closed set, we conclude that $\varphi\langle 0\rangle = [A,B]$.

Remark 2. The last note allows us to rewrite the hypothesis of Theorem 2 as follows: there exists a right s-continuous solution φ at τ in the extended sense on some interval $0 < t-\tau \le \beta$ satisfying

$$(15) \quad \varphi\langle \tau + \rangle = \Big[\xi + \underline{\lim}_{\sigma\to 0} \lim_{\delta_n\to 0} \int_{\delta_n}^{\sigma} f(s,\varphi(s))\,ds, \\ \xi + \overline{\lim}_{\sigma\to 0} \lim_{\delta_n\to 0} \int_{\delta_n}^{\sigma} f(s,\varphi(s))\,ds\Big]$$

where the sequence (δ_n) can be taken such that

$$\overset{t}{\underset{0}{S}} f(s,\varphi(s))\,ds = \lim_{\delta_n\to 0} \int_{\delta_n}^{t} f(s,\varphi(s))\,ds.$$

It is clear that Theorem 2 can be established for the lefthand side of τ. Let us just write the similar problem and state the theorem:

Problem A$_-$. *Let $f(t,x)$ be defined on the set*

$$D_- : -a \le t-\tau < 0, \quad 0 \le |x-\xi| \le b.$$

Find a continuous function φ defined on $[\tau-a,\tau)$ such that $(t,\varphi(t)) \in D_-$ and $\varphi'(t) = f(t,\varphi(t))$ for all $t \in [\tau-a,\tau)$ except on a set of Lebesgue measure zero and $\varphi\langle\tau-\rangle \ni \xi$.

Theorem 3. *Let f be defined on D_-, and suppose for all $x \in [\xi-b,\xi+b]$ it is measurable in t and for all $t \in [\tau+a,\tau)$ it is continuous in x. Let*

$$c_1'\widetilde{m}_1(t) \le f(t,x) \le c_2'\widetilde{m}_2(t), \quad ((t,x) \in D_-),$$

for two fixed real numbers c_1' and c_2', where $\widetilde{m}_i(t)$ is s-Lebesgue integrable and

1) *there exists a positive $\gamma \le a$ such that*

$$\alpha_1' \le \int_{\tau-\sigma}^{\tau-\delta} \widetilde{m}_i(t)\,dt \le \alpha_2', \quad \textit{for all} \quad \sigma > \delta > 0, \tau-\sigma \ge -\gamma$$

where $-b < c_1'\alpha_1' \le c_2'\alpha_2' < b$,

2)
$$\lim_{\sigma\to 0}\left(\underline{\lim}_{\delta\to 0}\left|\int_{\tau-\sigma}^{\tau-\delta} \widetilde{m}_i(t)\,dt\right|\right) = 0.$$

Then there exists a solution φ of the problem A_- in the extended sense on some interval $|t-\tau| \le \beta$ satisfying $\varphi\langle\tau-\rangle \ni \xi$.

Now, combining Theorems 2 and 3, we can state a theorem on the existence of s- and ws-continuous solutions of the differential equation.

Theorem 4. *Let f be defined on*

$$D : 0 < |t-\tau| \le a, \ 0 \le |x-\xi| \le b,$$

and suppose that for all $x \in [\xi - b, \xi + b]$ *it is measurable in* t *and for all* $t \in [\tau - a, \tau) \cup (\tau, \tau + a]$ *it is continuous in* x. *Let* f *satisfy the conditions of Theorems* 2 *and* 3 *on* D_+ *and* D_- ($m_i(t)$ *and* $\widetilde{m}_i(t)$ *satisfy the conditions of Theorems* 2 *and* 3*, respectively*)*. Then there exists a ws-continuous solution of the equation* $x' = f(t,x)$ *in the extended sense on some interval* $|t - \tau| \leq \beta$ *satisfying* $\varphi\langle\tau\rangle \ni \xi$.

It is clear that if corresponding upper and lower limits for the lefthand side equal the upper and lower limits for the righthand side, respectively, then φ *is s-continuous and*

$$\varphi\langle\tau\rangle = \Big[\xi + \varliminf_{\sigma\to 0} \lim_{\delta_n \to 0} \int_{\delta_n}^{\sigma} f(s,\varphi(s))\,ds,\ \xi + \varlimsup_{\sigma\to 0} \lim_{\delta_n\to 0} \int_{\delta_n}^{\sigma} f(s,\varphi(s))\,ds\Big].$$

The proof easily follows from (15) and from its analog for D_-.

Example 1. Let us consider the "Cauchy problem"

$$x' = a(t)x + b(t), \quad x\langle 0\rangle = [-1, 1]$$

where the function $a(t)$ is continuous on [0,1] and $b(t)$ is continuous on (0,1], $b\langle 0\rangle = (-\infty, \infty)$, and

$$\varliminf_{t\to 0} \overset{t}{\underset{0}{S}} b(s)\,ds = -1, \qquad \varlimsup_{t\to 0} \overset{t}{\underset{0}{S}} b(s)\,ds = 1.$$

Using Theorem 2 (if we define $b(t)$ on the lefthand side of 0, then using Theorems 3 and 4), we obtain a solution to this problem.

3. Uniqueness results. Now we introduce conditions which ensure at most one Caratheodory type solution for the initial value problem. If the function $f(t,x)$ is continuous on the region

$$D : |t - \tau| \leq a, \qquad |x - \xi| \leq b, \qquad a, b > 0,$$

there are different types of uniqueness theorems of the solution $\varphi(t)$of the equation $x' = f(t,x)$ satisfying $\varphi(\tau) = \xi$. We'll try to find an analog of one of the more general theorems on the uniqueness of the solution [**5**, Chapter 2, Section 2]:

Theorem 5. *Let $\psi(t,r)$ be a nonnegative function defined on*

$$S = \{0 < t < a,\ r \geq 0, a > 0\},$$

which is Lebesgue measurable in t for a fixed r and continuously nondecreasing in r for a fixed t. Further, for every bounded subset B of the set S, let there exist a function κ_B defined on $0 < t < a$ such that

$$\psi(t,r) \leq \kappa_B(t), \qquad (t,r) \in B, \tag{16}$$

and for which κ_B is Lebesgue integrable on $\gamma < t < a$ for every $\gamma > 0$. Suppose that, for each α, $0 < \alpha < a$, the identically zero function is the only absolutely continuous function on $0 \leq t < \alpha$, which satisfies

$$\rho'(t) = \psi(t, \rho(t)) \tag{17}$$

almost everywhere on $0 < t < \alpha$, and such that $\rho'_+(0)$ exists, and

$$\rho(0) = \rho'_+(0) = 0.$$

Let $f(t,x)$ be continuous on $D : |t-\tau| \leq a$, $|x-\xi| \leq b$, and satisfy there, for $t \neq \tau$,

$$|f(t,x) - f(t,y)| \leq \psi(|t-\tau|, |x-y|). \tag{18}$$

Then there exists at most one solution $\varphi(t)$ of the equation $x'(t) = f(t,x(t))$ satisfying $\varphi(\tau) = \xi$.

For the proof, see [**5**, Chapter 2, Section 2, Theorem 2.2].

We'll try to get an analog of this theorem for the case when f is not continuous at the point (τ, ξ) (or undefined at this point-singular case) and when the functions $\kappa_B(t)$ are s-Lebesgue integrable. Conditions (16) and (18) also will be weakened.

Theorem 6. *Let $f(t,x)$ be s-continuous with respect to t for any fixed x on $D_+ : 0 < t - \tau \leq a$, $|x-\xi| \leq b$, with discontinuities just at the points $\{\tau + \delta_n\}_{n=1,2,\dots}$, $\delta_n > 0$, $\delta_n > \delta_{n+1} \to 0$. Let $\psi_l(t,r)$, $l = 1,2$, be a function defined on*

$$S = \{0 < t < a,\ r \geq 0\}, \quad a > 0,$$

which is Lebesgue measurable in t for fixed r and continuous in r for fixed t. Further, for every bounded subset B of the set S, let there exist functions κ_{li} defined on $0 < t < a$ such that

$$c_{l1}\kappa_{l1}(t) \leq \psi_l(t,r) \leq c_{l2}\kappa_{l2}(t), \quad (t,r) \in B,$$

$l = 1, 2$, and for which κ_{li} is s-Lebesgue integrable on $\gamma < t < a$ for every $\gamma > 0$, and has Lebesgue-defect points just at the points $\{\delta_n\}$ and satisfies conditions

$$\lim_{\delta\to 0}\Big(\sup_{\delta}\Big|\overset{\gamma+\delta}{\underset{\gamma}{S}}\kappa_l(t)\,ds\Big|\Big) < b \tag{19}$$

and

$$\lim_{\sigma\to 0}\underline{\lim}_{\delta\to 0}\bigg(\bigg|\int_{\gamma+\delta}^{\gamma+\sigma}\kappa_l(t)\,dt\bigg|\bigg) = 0. \tag{20}$$

Suppose that, for each α, $0 < \alpha < a$, the identically zero function is the only s-continuous function on $0 \leq t < \alpha$ which satisfies

$$\rho'(t) = \psi_l(t, \rho(t))$$

almost everywhere on $0 < t < \alpha$ and $\rho\langle 0\rangle \ni 0$.

Let $f(t,x)$ satisfy $|x-\xi| \leq b$ on $D_+ : 0 < t-\tau \leq a$, for $t \neq \tau$,

$$\psi_1(t-\tau, x-y) \leq f(t,x) - f(t,y) \leq \psi_2(t-\tau, x-y)$$

or

$$\psi_2(t-\tau, x-y) \leq f(t,x) - f(t,y) \leq \psi_1(t-\tau, x-y).$$

Then there exists at most one s-continuous solution $\varphi(t)$ on $(\tau, \tau+a]$ of the equation $x'(t) = f(t, x(t))$ satisfying $\varphi\langle\tau+\rangle = [\xi - \varkappa_1, \xi + \varkappa_2]$, where $\varkappa_1$ and $\varkappa_2$ are fixed positive numbers. (*That is, there exists at most one s-continuous function $\varphi(t)$ such that*

$$\varphi(t) = \overset{t}{\underset{0}{S}} f(s, \varphi(s))\,ds,$$

almost everywhere and $\varphi\langle\tau+\rangle = [\xi - \varkappa_1, \xi + \varkappa_2]$.)

Proof. It will be assumed that $\tau = \xi = 0$ for simplicity.

Suppose that there are two differentiable almost everywhere and s-continuous solutions φ_1 and φ_2 on $(0, a]$ satisfying

$$\varphi_j \langle \tau \rangle = [\xi - \varkappa_1, \xi + \varkappa_2], \quad j = 1, 2. \tag{21}$$

Let $\varphi_1(t) - \varphi_2(t) > 0$ at some fixed σ, $0 < \sigma \le a$, and let $p(t)$ be the function defined by

$$p(t) = \varphi_1(t) - \varphi_2(t), \quad 0 < t \le a.$$

(That is, σ is not a Lebesgue defect point and $p(\sigma) > 0$.) Let $\psi_1(t - \tau, x - y) \le f(t, x) - f(t, y) \le \psi_2(t - \tau, x - y)$ at some small enough left neighborhood $(\sigma - \varepsilon, \sigma)$ of σ. From Theorem 3 it follows that through the point $(\sigma, p(\sigma))$ there exists a function ρ satisfying the equation

$$\rho'(t) = \psi_2(t, \rho(t))$$

on some interval to the left of σ. Let's show that there exists at least one solution, say $\rho(t)$ again, such that

$$\rho(t) \le p(t) \tag{22}$$

almost everywhere, as far to the left of σ as ρ exists. First of all, let's consider the problem of finding a solution to

$$\rho'(t) = \psi_2(t, \rho(t)) + \varepsilon, \quad 0 < \varepsilon < 1.$$

For every such ε, there exists at least one solution ρ_ε of this problem on some interval $\sigma - \alpha \le t \le \sigma$ (through the point $(\sigma, p(\sigma))$). As far to the left of σ as ρ_ε exists, it satisfies the inequality

$$\rho_\varepsilon(t) \le p(t);$$

otherwise, there would exist a point ζ to the left of σ, say ζ, where $\min\{\rho_\varepsilon\langle\zeta_-\rangle\} = \min\{p\langle\zeta_-\rangle\}$ (or simply $\rho_\varepsilon(\zeta) = p(\zeta)$) and $\rho_\varepsilon(t) > p(t)$ for $t < \zeta$ ($\sigma = \zeta$ is not excluded). That is, there exists a sequence (ζ_n) such that $\rho_\varepsilon(\zeta_n) - p(\zeta_n) \to 0$, $\zeta_n < \zeta_{n+1}$, $\zeta_n \to \zeta$ and $\min\{\rho_\varepsilon\langle\zeta_-\rangle\} = \min\{p\langle\zeta_-\rangle\}$. Now, since φ_1 and φ_2 are both solutions

$$\varphi_j(\zeta_n) = \underset{0}{\overset{\zeta_n}{S}} f(t, \varphi_j(t))\, dt, \quad j = 1, 2,$$

and so

$$p(\zeta_n) = \overset{\zeta_n}{\underset{0}{S}}[f(t, \varphi_1(t)) - f(t, \varphi_2(t))]\, dt, \quad j = 1, 2,$$

and for small enough $h > 0$,

$$p(\zeta - h) = \overset{\zeta-h}{\underset{0}{S}}[f(t, \varphi_1(t)) - f(t, \varphi_2(t))]\, dt, \quad j = 1, 2.$$

Now it follows by subtraction that

$$\begin{aligned}(23) \qquad p(\zeta_n) - p(\zeta - h) &= \overset{\zeta_n}{\underset{\zeta-h}{S}}[f(t, \varphi_1(t)) - f(t, \varphi_2(t))]\, dt \\ &\le \overset{\zeta_n}{\underset{\zeta-h}{S}} \psi_2(t, p(t))\, dt \\ &= \int_{\zeta-h}^{\zeta_n} \psi_2(t, p(t))\, dt,\end{aligned}$$

and similarly

$$p(\zeta_n) - p(\zeta - h) \ge \int_{\zeta-h}^{\zeta_n} \psi_1(t, p(t))\, dt,$$

(since κ_l has just a finite number Lebesgue-defect points on $\gamma < t < a$ for every $\gamma > 0$, h, and ζ_n can be taken such that the interval $[\zeta - h, \zeta_n]$ contains no point from the set of Lebesgue-defect points from the sequence (δ_n)). From the definition of ρ_ε, one has, since $\rho_\varepsilon(\zeta_n) - p(\zeta_n) \to 0$, $\rho_\varepsilon(\zeta_n) > p(\zeta_n)$, the lefthand derivative $p'_-(\zeta_n)$ exists, and

$$\begin{aligned}p'_-(\zeta_n) &= \varphi'_1(\zeta_n) - \varphi'_2(\zeta_n) \\ &= f(\zeta_n, \varphi_1(\zeta_n)) - f(\zeta_n, \varphi_2(\zeta_n)) \\ &\le \psi_2(\zeta_n, p(\zeta_n)) < \psi_2(\zeta_n, \rho_\varepsilon(\zeta_n)) + \varepsilon = \rho'_\varepsilon(\zeta_n).\end{aligned}$$

Therefore, for h sufficiently small,

$$\rho_\varepsilon(\zeta - h) < p(\zeta - h),$$

which contradicts the definition of ζ. In the same way (taking ρ instead of p) we have that

$$\rho'_-(\zeta_n) = \psi_2(\zeta_n, \rho(\zeta_n)) < \psi_2(\zeta_n, \rho_\varepsilon(\zeta_n)) + \varepsilon = \rho'_\varepsilon(\zeta_n),$$

and so

$$\lim_{\varepsilon\to 0} (\sup \rho_\varepsilon(t)) \le \rho(t)$$

for every solution of $\rho'(t) = \psi_2(t, \rho(t))$. Since

$$p(\sigma) - \rho_\varepsilon(t) = \underset{t}{\overset{\sigma}{S}}[\psi_2(t, \rho_\varepsilon(t)) + \varepsilon]\, dt = \underset{t}{\overset{\sigma}{S}}\psi_2(t, \rho_\varepsilon(t))\, dt + \varepsilon(\sigma - t)$$

on $\sigma - \alpha \le t \le \sigma$, the set $\{\rho_\varepsilon\}$ is an equicontinuous uniformly bounded set of functions on some small left neighborhood $(\sigma - \alpha_0, \sigma)$ of σ. Therefore, there exists a subsequence $\{\rho_{\varepsilon_k}\}$ that tends to a function ρ, where ρ satisfies $\rho'(t) = \psi_2(t, \rho(t))$. That is, there exists at least one solution $\rho(t)$ to $\rho'(t) = \psi_2(t, \rho(t))$ such that

$$\rho(t) \le p(t).$$

Now $\rho(t) > 0$ on $0 < t \le \sigma$, as far as it exists. Otherwise, $\rho(\tilde\sigma) = 0$ for some $\tilde\sigma$, $0 < \tilde\sigma < \sigma$, and the function defined by

$$\begin{aligned} \widehat{\rho}(t) &= 0 \qquad & 0 \le t \le \tilde\sigma \\ \widehat{\rho}(t) &= \rho(t) \qquad & \tilde\sigma < t \le \sigma \end{aligned}$$

would be a function on $0 \le t \le \sigma$ not identically zero, which satisfies (20). This contradicts the hypothesis of the theorem. Therefore,

$$0 < \rho(t) \le p(t) \tag{24}$$

almost everywhere, as far to the left of σ as ρ exists. (That is, $p(t) > 0$ and $\psi_1(t-\tau, x-y) \le f(t,x) - f(t,y) \le \psi_2(t-\tau, x-y)$ on $(0,\sigma)$.) But, by the application of Theorem 4, it follows that ρ can be continued as a solution, call it ρ again, on the whole interval $0 < t \le \sigma$. (If $\sigma - \alpha_0$ is the Lebesgue defect point, again at the lefthand side of this point there exists a "normal" point σ' such that $|p(\sigma')| > 0 \ldots$.) Let's show that $\rho\langle 0\rangle \ni 0$. Otherwise, $\varphi_1(t) - \varphi_2(t) > d > 0$ in some right neighborhood of 0. Since $\varphi_1(t)$ and $\varphi_2(t)$ are s-continuous, we have that $\varphi_1(t) - \varphi_2(t) > d$ or $\varphi_2(t) - \varphi_1(t) > d$ in some interval $(0, \delta)$, say $\varphi_1(t) - \varphi_2(t) > d$. Then it is clear that

$$\min\{\varphi_1 \langle 0+\rangle\} \geq \min\{\varphi_2 \langle 0+\rangle\} + d,$$

and this contradicts the condition $\varphi_j\langle 0+\rangle = [-\varkappa_1, \varkappa_2]$, $j = 1, 2$. That is, we have $p\langle 0+\rangle \ni 0$, and so $\rho\langle 0+\rangle \ni 0$. This contradicts the hypothesis of the theorem, for ρ is an s-continuous solution of (17) but is not identically zero. Therefore, $p(\sigma) \neq 0$ for any σ, $0 < \delta < a$, almost everywhere, and this proves the theorem.

Example 2. The problem

$$x' = x + \frac{1}{t^2}\sin\frac{1}{t} - \cos\frac{1}{t}, \quad x'(0) \ni 0$$

has a unique solution, $x = \cos 1/t$.

Here we may take ψ_1 and ψ_2 as $\psi_1(t, r) = \psi_2(t, r) = r$, and it is clear that $\psi_1(t, x - y) = f(t, x) - f(t, y) = \psi_2(t, x - y)$. The role of the functions $\kappa_{li}(t)$ may play the constant functions. (The problem $r' = r$, $r(0) = 0$ has a unique solution $r = 0$.)

Note that the ideas developed here can be applied to prove different types of generalization theorems in terms of the reciprocal problem [**3, 4**]. (That is, the roles of t and x sometimes may be interchanged and restriction conditions of $f(t, x)$ may be phrased, for example, as $\psi_1(t - p, x - \eta) \leq f(t, x) - f(p, x) \leq \psi_2(t - p, x - \eta)$.)

4. Systems of differential equations and nth order equations. All the theorems in Section 2 are valid for the system of differential equations

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \dots, x_n) \\ &\cdots\cdots\cdots \\ x_n' &= f_n(t, x_1, x_2, \dots, x_n). \end{aligned} \tag{25}$$

Let's formulate the problem in case of systems:

Problem $S+$. *Let $f_k(t, x_1, \dots, x_n)$ be defined on the set*

$$\mathbf{D}_+ : 0 < t - \tau \leq a, \quad 0 \leq |x_k - \xi_k| \leq b_k,$$

$k = 1, 2, \ldots, n$. *Find the continuous functions* $\varphi_1, \varphi_2, \ldots, \varphi_n$ *defined on the set* $(\tau, \tau + a]$ *such that* $(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)) \in \mathbf{D}_+$, *and*

$$\varphi_k'(t) = f_k(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)) \quad \text{for all} \quad t \in (\tau, \tau + a),$$

except on a set of Lebesgue measure zero, and $\varphi_k \langle \tau + \rangle \ni \xi_k$, $k = 1, 2, \ldots, n$.

The problem $\mathbf{S}-$ for the lefthand side of τ can be formulated similarly.

Theorem 7. *Let* f_k, $k = 1, 2, \ldots, n$, *be defined on* $\mathbf{D}_+$, *and suppose it is measurable in* t *(for fixed* $x_1, x_2, \ldots, x_n$*), continuous in* x_k. *Let*

$$c_{k1} m_{k1}(t) \leq f_k(t, x_1, x_2, \ldots, x_n) \leq c_{k2} m_{k2}(t),$$

for two fixed real numbers c_{k1} *and* c_{k2}, *where* $m_{ki}(t)$ *satisfies the next conditions*

1) *functions* $m_{ki}(t)$ *are integrable on the set* $\delta \leq t - \tau \leq a$ *for any positive* $\delta \leq a$,

2) *there exists a positive* $\gamma \leq a$ *such that*

$$\alpha_{k1} \leq \int_{\tau+\delta}^{\tau+\sigma} m_{ki}(t)\, dt \leq \alpha_{k2}, \quad \text{for all} \quad \sigma > \delta > 0, \quad \tau + \sigma \leq \gamma,$$

and $-b_k < c_{k1}\alpha_{k1} \leq c_{k2}\alpha_{k2} < b_k$, $k = 1, 2, \ldots, n$.

3)
$$\lim_{\sigma \to 0} \left(\underline{\lim_{\delta \to 0}} \left| \int_{\tau+\delta}^{\tau+\sigma} m_{ki}(t)\, dt \right| \right) = 0.$$

Then there exists a solution $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ *of the problem* S_+ *in the extended sense on some interval* $|t - \tau| \leq \beta$ *satisfying* $\varphi_k \langle \tau + \rangle \ni \xi_k$.

We can state a similar theorem for the case $\mathbf{D}_-$ and, combining these theorems, we can state the theorem on the existence of the s- or ws-continuous solution. Let's just write an analog of the theorem

Theorem 8. *Let* $f_k(t, x_1, \ldots, x_n)$ *be defined on*

$$\mathbf{D} : 0 < |t - \tau| \leq a, \quad 0 \leq |x_k - \xi_k| \leq b_k,$$

and suppose it is measurable in t*, continuous in* x_k*,* $k = 1, 2, \dots, n$*. Let* $f_k(t, x_1, \dots, x_n)$ *satisfy the conditions of Theorem 7 on* $\mathbf{D}_+$ *and similar conditions on* $\mathbf{D}_- : -a \le t - \tau < 0$*,* $0 \le |x_k - \xi_k| \le b_k$*:*

$$c'_{k1}\widetilde{m}_{1k}(t) \le f_k(t, x_1, x_2, \dots, x_n) \le c'_{k2}\widetilde{m}_{2k}(t), \quad ((t, x) \in \mathbf{D}_-)$$

$$\alpha'_{k1} \le \int_{\tau-\sigma}^{\tau-\delta} \widetilde{m}_{ki}(t)\,dt \le \alpha'_{k2}, \quad \sigma > \delta > 0, \quad \tau - \sigma \ge -\gamma,$$

where $-b_k < c'_{k1}\alpha'_{k1} \le c'_{k2}\alpha'_{k2} < b_k$*,* $k = 1, 2, \dots, n$*, and*

$$\lim_{\sigma\to 0}\left(\varliminf_{\delta\to 0}\left|\int_{\tau-\sigma}^{\tau-\delta} \widetilde{m}_{ki}(t)\,dt\right|\right) = 0.$$

Then there exists a ws*-continuous function at* $(\tau, \xi_1, \xi_2, \dots, \xi_n)$ *solution* $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ *of the system of equations* (25) *in the extended sense on some interval* $|t-\tau| \le \beta$ *such that* $\varphi_k\langle\tau\rangle \ni \xi_k$*,* $k = 1, 2, \dots, n$*.*

Additionally, we can write conditions for the s-continuity of the functions φ_k.

Now let's consider the nth order equation

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right) \tag{26}$$

where f is a real continuous function defined in a domain D of real $(t, x_1, \dots x_n)$ space. Since equation (26) can be reduced to the theory of a system of n first order differential equations

$$\begin{aligned} &x'_1 = x_2 \\ &\dots\dots\dots \\ &x'_{n-1} = x_n \\ &x'_n = f(t, x_1, \dots, x_n), \end{aligned}$$

the analog of the existence theorem for the nth order equation can be obtained in a similar way:

Theorem 9. *Consider the differential equation*

$$\frac{d^n x}{dt^n} = f\left(t,\ x,\ \frac{dx}{dt}, \dots,\ \frac{d^{n-1}x}{dt^{n-1}}\right)$$

where f satisfies the next conditions on

$$\mathbf{D}_+ = \{\tau < t \le a,\ |x-\xi_1| \le b_1,\ |x'-\xi_2| \le b_2, \dots, |x^{(n-1)} - \xi_n| \le b_n\} \subset R^{n+1},$$

a) *f is measurable in t (for fixed $x_1, x_2, \dots, x_n$), continuous in x_k.*

b) *There exists a function $m_i(t)$ integrable on the set $\tau + \delta \le t \le a$ for any positive small enough δ such that*

$$c_1 m_1(t) \le f(t, x_1, x_2, \dots, x_n) \le c_2 m_2(t), \quad (t, x_1, x_2, \dots, x_n) \in \mathbf{D}_+),$$

for two fixed real numbers c_1 and c_2.

c) *There exists a positive $\gamma \le a$ such that*

$$\alpha_1 \le \int_{\tau+\delta}^{\tau+\sigma} m_i(t)\,dt \le \alpha_2, \quad \text{for all} \quad \sigma > \delta > 0, \quad \tau + \sigma \le \gamma$$

and $-b_n < c_1\alpha_1 \le c_2\alpha_2 < b_n$.

d)

$$\lim_{\sigma \to 0} \Big(\underline{\lim}_{\delta \to 0} \Big| \int_{\tau+\delta}^{\tau+\sigma} m_i(t)\,dt \Big| \Big) = 0.$$

Conclusion. There exists a solution φ of equation (26) defined on some interval $(\tau, \tau+\beta]$, such that

$$\varphi(\tau) = \xi_1, \dots, \varphi^{(n-2)}(\tau) = \xi_{n-1}, \varphi^{(n-1)} \langle \tau \rangle \ni \xi_n.$$

Similarly, we can state theorems on the existence of solutions φ such that $\varphi^{(n-1)}$ is s- or ws-continuous at τ. For example, if f satisfies similar conditions on the set $\mathbf{D}_- = \{a' \le t < \tau,\ |x - \xi_1| \le b_1,\ |x' - \xi_2| \le b_2, \dots,\ |x^{(n-1)} - \xi_n| \le b_n\} \subset R^{n+1}$, and there exists a majoring function $m_i'(t)$ satisfying similar conditions, then φ will be ws-continuous at τ.

Again we can write conditions for the s-continuity of function φ at τ. The corresponding upper and lower limits for the lefthand side must be equal to the upper and lower limits for the righthand side.

Example 3. Let's consider the motion of a mass on a vibrating spring. If there exists an external force $F(t)$ acting proportionally to $(1/t^2)\sin(1/t)$ as t tends to 0, (that is, $F(t)$ is an impulsive type of force and becomes maximum over the given time sequence), then we have a differential equation of the form

$$mx'' + \gamma x' + kx = F(t).$$

Since $F(t)$ is "quasi" integrable about point 0, that is, there exist integrals $\int_\delta^\sigma F(t)\,dt$ for any positive $\delta > 0$ and

$$\left| \int_\delta^\sigma F(t)\,dt \right| \leq c,$$

we have that the problem

$$mx'' + \gamma x' + kx = F(t), \quad x(0) = 0, \quad x'\langle 0 \rangle \ni 0$$

has a right s-continuous at the 0 solution. (More exactly, $x'\langle 0\rangle = [a, b]$, where

$$a = \underline{\lim}_{t\to 0} \overset{t}{\underset{0}{S}} F(t)\,dt, \quad b = \overline{\lim}_{t\to 0} \overset{t}{\underset{0}{S}} F(t)\,dt.)$$

Example 4. This example demonstrates that, even in the solutions of traditional differential equations, the scope of problems can be improved. Consider the "initial value problem" corresponding to the Euler equation

$$t^2 x'' - tx' + 2x = 0, \quad x(0) = 0, \quad x'\langle 0\rangle = [-\sqrt{2}, \sqrt{2}].$$

It is not difficult to show that $x = t\sin(\ln t)$ (and $x = t\cos(\ln t)$) satisfies this initial value problem. But, if we use the substitution $t = e^u$, then we get an equation with constant coefficients

$$x'' - 2x' + 2x = 0, \; x(-\infty) = 0, \quad x'\langle -\infty\rangle = [-\sqrt{2}, \sqrt{2}]$$

which has no solution, since the general solution of the last problem is

$$x = C_1 e^u \cos u + C_2 e^u \sin u$$

and $x' \to 0$ as $u \to -\infty$. That is, $x'\langle -\infty\rangle = 0 \neq [-\sqrt{2}, \sqrt{2}]$.

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