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ON THE WEIGHTED GENERALIZATION OF THE HERMITE-HADAMARD INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. In this paper, we give a weighted generalization of the Hermite-Hadamard inequality. As applications, a refinement of Jensen's inequality is established and some new inequalities of Hermite-Hadamard type are derived.

1. Introduction. Let f be a convex function on $[a,b] \subset \mathbf{R}$. The following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as the Hermite-Hadamard inequality for convex functions [**13**].

It is well known that the Hermite-Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, see [**1–12, 14–18**] and the references therein.

In the present paper, we establish a weighted generalization of the Hermite-Hadamard inequality, where the celebrated Fejér's inequality, see [**7**], is derived as a consequence. Moreover, the results obtained will be applied to establish a refinement of Jensen's inequality and establish two new integral inequalities which are analogous to Dragomir-Agarwal's inequality and Pearce-Pečarić's inequality [**3, 10**]. Finally, an application to special means of positive numbers is given.

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2. Main results.

Theorem 1. *Let* $f : [a,b] \to \mathbf{R}$ *be a convex function on* $[a,b]$, *let* g *be a nonnegative, integrable function on* $[0,1]$, *and let* $\lambda = (\int_0^1 xg(x)\,dx)/(\int_0^1 g(x)\,dx)$. *Then*

(2)
$$f(\lambda a+(1-\lambda)b) \le \frac{\int_a^b f(x)g\left((b-x)/(b-a)\right)\,dx}{\int_a^b g\left((b-x)/(b-a)\right)\,dx} \le \lambda f(a)+(1-\lambda)f(b).$$

Proof. Integration by substitution of $x = ta+(1-t)b$ gives

$$(3)\qquad \frac{\int_a^b f(x)g\left((b-x)/(b-a)\right)\,dx}{\int_a^b g\left((b-x)/(b-a)\right)\,dx} = \frac{\int_0^1 f\left(ta+(1-t)b\right)g(t)\,dt}{\int_0^1 g(t)\,dt}.$$

Note that f is convex on $[a,b]$ and g is nonnegative and integrable on $[0,1]$. It follows from Jensen's inequality [**13**] that

(4)
$$\begin{aligned}\int_0^1 f(ta+(1-t)b)g(t)\,dt &\le \int_0^1 (tg(t)f(a)+(1-t)g(t)f(b))\,dt \\ &= f(a)\int_0^1 tg(t)\,dt + f(b)\int_0^1 (1-t)g(t)\,dt.\end{aligned}$$

Combining inequalities (3) and (4) yields

$$\begin{aligned}&\frac{\int_a^b f(x)g\left((b-x)/(b-a)\right)\,dx}{\int_a^b g\left((b-x)/(b-a)\right)\,dx} \\ &\qquad \le \left(\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)f(a) + \left(1-\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)f(a),\end{aligned}$$

which is the right-hand inequality in (2).

In view of the assumption that f is convex on $[a,b]$, g is nonnegative and integrable on $[0,1]$, we deduce from the Jensen integral inequality

[**13**] that

$$
\begin{aligned}
\frac{\int_0^1 f(ta+(1-t)b)g(t)\,dt}{\int_0^1 g(t)\,dt} & \\
&\geq f\left(\frac{\int_0^1 g(t)(ta+(1-t)b)\,dt}{\int_0^1 g(t)\,dt}\right)\\
&= f\left(\left(\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)a+\left(1-\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)b\right),
\end{aligned}
$$

this is

$$
\frac{\int_a^b f(x)g\left((b-x)/(b-a)\right)dx}{\int_a^b g\left((b-x)/(b-a)\right)\,dx}\geq f\left(\left(\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)a+\left(1-\frac{\int_0^1 tg(t)\,dt}{\int_0^1 g(t)\,dt}\right)b\right);
$$

the left-hand inequality in (2) is proved. This completes the proof of Theorem 1. □

Corollary 1. *Let $f:[a,b]\to\mathbf{R}$ be a convex function on $[a,b]$, let g be a nonnegative, integrable function on $[a,b]$, and let $\lambda=(\int_a^b(b-x)g(a+b-x)\,dx)/(\int_a^b(b-x)g(x)\,dx)$. Then*

$$
f\left(\frac{a+\lambda b}{1+\lambda}\right)\leq\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\leq\frac{f(a)+\lambda f(b)}{1+\lambda}. \tag{5}
$$

Proof. Based on the assumption that g is a nonnegative, integrable function on $[a,b]$, it is easy to verify that $\varphi(x)=g(b-(b-a)x)$ is nonnegative and integrable on $[0,1]$. Hence, we obtain from Theorem 1 that

$$
f(\mu a+(1-\mu)b)\leq\frac{\int_a^b f(x)\varphi((b-x)/(b-a))\,dx}{\int_a^b\varphi((b-x)/(b-a))\,dx}\leq\mu f(a)+(1-\mu)f(b),
$$

i.e.,

$$
f(\mu a+(1-\mu)b)\leq\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\leq\mu f(a)+(1-\mu)f(b),
$$

where

$$\mu = \frac{\int_0^1 xg(b-(b-a)x)\,dx}{\int_0^1 g(b-(b-a)x)\,dx} = \frac{\int_a^b (b-x)g(x)\,dx}{\int_a^b (b-a)g(x)\,dx}$$
$$= \frac{\int_a^b (b-x)g(x)\,dx}{\int_a^b (b-x)g(x)\,dx + \int_a^b (b-x)g(a+b-x)\,dx} = \frac{1}{1+\lambda}.$$

Inequality (5) is proved. □

As a direct consequence of Corollary 1, we have

Corollary 2. *Let $f : [a,b] \to \mathbf{R}$ be a convex function on $[a,b]$, let g be a nonnegative, integrable function on $[a,b]$, satisfying $g(x) = g(a+b-x)$ for any $x \in [a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)d\,x} \le \frac{f(a)+f(b)}{2}. \tag{6}$$

Remark 1. Inequality (6) is the celebrated Fejér's inequality, which was presented in 1906, see [**7**]. It is worth noticing that when $g(x) \equiv 1$, inequalities (2), (5) and (6) would reduce to the Hermite-Hadamard inequality, respectively.

3. Some applications. Let f be a convex function on $[a,b] \subset \mathbf{R}$. The classical Jensen's inequality reads as

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}. \tag{7}$$

We give here a refinement of Jensen's inequality.

Theorem 2. *Let $f : [a,b] \to \mathbf{R}$ be a convex function on $[a,b]$, and let F be a differentiable function such that $F''(x) = f(x)$ on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}. \tag{8}$$

Proof. Define a function $g:\ [a,b]\to\mathbf{R}$ by

$$g(x)=b-a-|2x-a-b|=\begin{cases}2(x-a) & a\le x\le (a+b)/2\\ 2(b-x) & (a+b)/2<x\le b\end{cases}. \tag{9}$$

Using Corollary 1 with a simple calculation:

$$\lambda=\frac{\int_a^b (b-x)g(a+b-x)\,dx}{\int_a^b (b-x)g(x)\,dx}=\frac{\int_a^b (b-x)(b-a-|a+b-2x|)\,dx}{\int_a^b (b-x)(b-a-|2x-a-b|)\,dx}=1,$$

we get

$$f\left(\frac{a+b}{2}\right)\le\frac{\int_a^b f(x)(b-a-|2x-a-b|)\,dx}{\int_a^b (b-a-|2x-a-b|)\,dx}\le\frac{f(a)+f(b)}{2}. \tag{10}$$

On the other hand, direct calculation gives

$$\begin{aligned}
&\frac{\int_a^b f(x)(b-a-|2x-a-b|)\,dx}{\int_a^b (b-a-|2x-a-b|)\,dx}\\
&\quad=\frac{\int_a^{(a+b)/2} 2(x-a)f(x)\,dx+\int_{(a+b)/2}^b 2(b-x)f(x)\,dx}{\int_a^{(a+b)/2} 2(x-a)\,dx+\int_{(a+b)/2}^b 2(b-x)\,dx}\\
&\quad=\frac{4}{(b-a)^2}\left[\int_a^{(a+b)/2}(x-a)\,dF'(x)+\int_{(a+b)/2}^b (b-x)dF'(x)\right]\\
&\quad=\frac{4}{(b-a)^2}\left[\frac{b-a}{2}F'\left(\frac{a+b}{2}\right)-\int_a^{(a+b)/2}F'(x)\,dx\right.\\
&\qquad\qquad\left.-\frac{b-a}{2}F'\left(\frac{a+b}{2}\right)+\int_{(a+b)/2}^b F'(x)\,dx\right]\\
&\quad=\frac{4}{(b-a)^2}\left[-\int_a^{(a+b)/2}F'(x)\,dx+\int_{(a+b)/2}^b F'(x)\,dx\right]\\
&\quad=\frac{4}{(b-a)^2}\left[-\int_a^{(a+b)/2}dF(x)+\int_{(a+b)/2}^b dF(x)\right]\\
&\quad=\frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2}-F\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Combining the above identity and inequality (10) leads to inequality (8). The proof of Theorem 2 is complete. $\square$

Remark 2. Assume that $f : [a,b] \to \mathbf{R}$ is a differentiable function and $|f'|$ is convex on $[a,b]$. Pearce and Pečarić [**10**] and Dragomir and Agarwal [**3**] proved respectively the following inequalities:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{11}$$

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{12}$$

We show here an analogous result relating to the above inequalities.

Theorem 3. *Let $f : [a,b] \to \mathbf{R}$ be a convex function on $[a,b]$, and let F be a differentiable function such that $F''(x) = f(x)$ on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b f(x)\,dx \le \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right], \tag{13}$$

$$\frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(x)\,dx \ge \frac{8}{(b-a)^2}\left[\frac{F(a)+F(b)}{2} - F\left(\frac{a+b}{2}\right)\right]. \tag{14}$$

Proof. Define a function $g : [a,b] \to \mathbf{R}$ by

$$g(x) = |a+b-2x| = \begin{cases} a+b-2x & a \le x \le (a+b)/2 \\ 2x-a-b & (a+b)/2 < x \le b. \end{cases} \tag{15}$$

Using Corollary 1 with a simple calculation:

$$\lambda = \frac{\int_a^b (b-x)g(a+b-x)\,dx}{\int_a^b (b-x)g(x)\,dx} = \frac{\int_a^b (b-x)\,|2x-a-b|\,dx}{\int_a^b (b-x)\,|a+b-2x|\,dx} = 1,$$

we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b |a+b-2x|\, f(x)\, dx}{\int_a^b |a+b-2x|\ dx} \leq \frac{f(a)+f(b)}{2}. \tag{16}$$

In addition, direct calculation gives

$$\begin{aligned}
&\frac{\int_a^b |a+b-2x|\, f(x)\, dx}{\int_a^b |a+b-2x|\ dx} \\
&= \frac{\int_a^{(a+b)/2} (a+b-2x) f(x)\, dx + \int_{(a+b)/2}^b (2x-a-b) f(x)\, dx}{\int_a^{(a+b)/2} (a+b-2x)\, dx + \int_{(a+b)/2}^b (2x-a-b)\, dx} \\
&= \frac{2}{(b-a)^2}\left[\int_a^{(a+b)/2} (a+b-2x) dF'(x) + \int_{(a+b)/2}^b (2x-a-b) dF'(x)\right] \\
&= \frac{2}{(b-a)^2}\left[(b-a)F'(b) - (b-a)F'(a) + 2\int_a^{(a+b)/2} F'(x)\, dx \right. \\
&\qquad\qquad \left. - 2\int_{(a+b)/2}^b F'(x)\, dx\right] \\
&= \frac{2}{(b-a)^2}\left[(b-a)F'(b) - (b-a)F'(a) - 2F(a) \right. \\
&\qquad\qquad \left. - 2F(b) + 4F\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{(b-a)^2}\left[(b-a)\int_a^b f(x)\, dx - 2F(a) - 2F(b) + 4F\left(\frac{a+b}{2}\right)\right] \\
&= \frac{2}{(b-a)}\int_a^b f(x)\, dx + \frac{8}{(b-a)^2}\left[F\left(\frac{a+b}{2}\right) - \frac{F(a)+F(b)}{2}\right].
\end{aligned}$$

Consequently,

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{2}{(b-a)}\int_a^b f(x)\, dx + \frac{8}{(b-a)^2}\left(F\left(\frac{a+b}{2}\right) - \frac{F(a)+F(b)}{2}\right) \\
&\leq \frac{f(a)+f(b)}{2},
\end{aligned}$$

which leads to the desired inequalities (13) and (14). The proof of Theorem 3 is complete. $\square$

Next, we shall consider a class of inequalities concerning arithmetic mean and generalized logarithmic mean for positive numbers α and β, where

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2}, \qquad \text{arithmetic mean,}$$
$$L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{1/p},\ p\neq -1,0, \qquad \text{generalized logarithmic mean.}$$

We have the following results:

Corollary 3. *Let a,b be positive real numbers. Then for all $p\geq 1$ or $p<0$, $p\neq -1,-2$, the following inequalities hold*

(17)
$$\begin{aligned}(A(a,b))^p &\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[A(a^{p+2},b^{p+2})-(A(a,b))^{p+2}\right]\\ &\leq A(a^p,b^p).\end{aligned}$$

(18)
$$\begin{aligned}&(A(a,b))^p-2(L_p(a,b))^p\\ &\quad\leq \frac{8}{(p+1)(p+2)(b-a)^2}\left[A(a^{p+2},b^{p+2})-(A(a,b))^{p+2}\right]\\ &\quad\leq A(a^p,b^p)-2(L_p(a,b))^p.\end{aligned}$$

Inequalities (17) *and* (18) *are reversed for* $0<p<1$.

Proof. It is clear that inequalities (17) and (18) are symmetric with respect to variables a and b. Without loss of generality, we assume that $b>a>0$. Note that the function $f(x)=x^p$ is convex on $(0,+\infty)$ for $p\geq 1$ or $p<0$, and the function $f(x)=-x^p$ is convex on $(0,+\infty)$ for $0<p<1$. We immediately obtain inequalities (17) and (18) by applying these functions to Theorem 2 and Theorem 3.

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