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THE WEYL CORRESPONDENCE AS A FUNCTIONAL CALCULUS FOR NON-COMMUTING OPERATORS

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ABSTRACT. In this expository paper, we describe the Weyl calculus for bounded, self-adjoint operators acting on a Hilbert space as well as the original Weyl correspondence for the position Q and momentum P operators on $S(\mathbf{R}^n)$. We describe some classes of functions for which the calculus is well defined and give a representation for the action of the calculus in these separate cases. In particular, we verify that the Weyl calculus is well defined for polynomials and give results consistent with the natural algebraic definition. The proof of this result for the original Weyl correspondence is obtained via an analysis of the commutator of P and Q on $S(\mathbf{R}^{2n})$. We also discuss the connection of the Weyl calculus with some recent developments in functional calculi.

1. Introduction. The Weyl correspondence was developed in efforts by Hermann Weyl and other mathematical physicists to better understand the correlation between physical observables in classical mechanics and their quantum-mechanical analogues. In classical mechanics, one is concerned with the state space $(p, q) \in \mathbf{R}^{2n}$ that represents the momentum and position information of an object. The observables come in the form of real-valued functions f defined on this space, such as the position operator $(p, q) \mapsto q_j$ and the momentum $(p, q) \mapsto p_j$. In the quantum-mechanical view, this state space is replaced by the set of functions $f \in L^2(\mathbf{R}^n)$ for which $\|f\|_2 = 1$, i.e., the *wavefunctions*, and the observables become self-adjoint linear operators A acting on this space. The fundamental questions to be answered are: how does one connect the state space $\mathbf{R}^{2n}$ to the set of wavefunctions in $L^2(\mathbf{R}^n)$ and, given this, what is the correlation between the real-valued functions $f(p, q)$ and the self-adjoint linear operators A?

The motivation for the Weyl calculus comes from the interpretation of the wavefunctions as probability distribution functions for the posi-

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tion of a particle in space. In this view the spatial coordinate projection $(p, q) \mapsto q_j$ is mapped to the multiplication operator $Q_j f = x_j f$, whereas the momentum projection $(p, q) \mapsto p_j$ corresponds to the differential operator $P_j f = h/(2\pi i)(\partial f/\partial x_j)$. From this, Weyl proposed a calculus based on the following observation: the operator $(p, q) \mapsto e^{2\pi i(pP+qQ)}$ exists as a unitary operator on $L^2(\mathbf{R}^n)$ even though P, Q are themselves unbounded. Then, given a suitable function f and its Fourier transform $\hat{f}$, we consider the formal Bochner integral:

$$f(P, Q) = \int_{\mathbf{R}^{2n}} \hat{f}(p, q) e^{2\pi i(pP+qQ)}\, dp\, dq \tag{1.1}$$

which makes sense as an "evaluation" of the function $f(p, q)$ on the pair of operators P, Q. In particular, if f is in the Schwarz class $S(\mathbf{R}^{2n})$, then its Fourier transform $\hat{f}$ is both Borel-measurable and integrable, and we can make sense of (1.1) as an operator-valued tempered distribution.

This formalism motivates a more general method for defining a map $f(x_1, \dots, x_n) \mapsto f(A_1, \dots, A_n)$ from some suitable class of functions on $\mathbf{R}^n$ to elements of a Banach algebra B as follows: given $\{A_1, \dots, A_n\} \subset B$, we write the formal expression:

$$f(A_1, \dots, A_n) = \int_{\mathbf{R}^n} \hat{f}(\xi_1, \dots, \xi_n) e^{2\pi i(\xi_1 A_1 + \cdots + \xi_n A_n)}\, d\xi_1 \cdots d\xi_n. \tag{1.2}$$

Of course, some restrictions must be placed on the A_i and the classes of functions f in order for this correspondence to be well defined. In particular, the theory is well developed in the case that B is the set $L(X)$ of bounded, linear operators acting on a Hilbert space X and the A_i are self-adjoint, for then the exponential term in (1.2) makes sense as a unitary operator on X, and we may again interpret (1.2) as an operator-valued tempered distribution with values in $L(X)$.

Because of our ability to define the Weyl calculus as a tempered distribution, it makes sense to consider the support of the calculus for a given n-tuple of operators A_i. In particular, if this support is compact, then we may extend the Weyl calculus to polynomials f on $\mathbf{R}^n$ with complex coefficients. A critical result concerning the Weyl calculus is that it gives consistent results with the natural algebraic definition of the polynomial $f(A_1, \ldots, A_n)$. In the case that the A_i are bounded and self-adjoint, this follows as a consequence of the Paley-Wiener theorem for operator-valued distributions. However, the corresponding result for the original Weyl correspondence is a bit more elusive, due to the fact that the exponential $e^{2\pi i(pP+qQ)}$ must be defined more carefully. While there are known results establishing the relationship between the Weyl correspondence for $f(P,Q)$ and the results from the spectral calculus when f satisfies a polynomial bound [**6**], we give here a simple, direct proof for the case when f is a polynomial by considering the commutator of P and Q on $S(\mathbf{R}^{2n})$.

The organization of this paper is as follows. In Section 2, we give a brief background of the von Neumann spectral calculus for bounded, self-adjoint operators. From this context, we introduce the one-dimensional Weyl calculus for bounded self-adjoint operators and illustrate the connection between the two methods. While the correspondence with the spectral calculus serves to give a nice intuitive "feel" for the nature of the Weyl calculus, we note a key failure of the spectral calculus methodology in dealing with several noncommuting operators. This motivates the development of the multi-dimensional Weyl calculus discussed in Section 3. Section 4 considers the development of the original Weyl correspondence for the unbounded P and Q operators on $L^2(\mathbf{R}^n)$. Here, we formally develop the meaning of the operator (1.1), investigate its interpretation for several different types of function spaces, and in particular calculate its action on polynomials in 4.4. Finally, Section 5 discusses some recent results in the development of functional calculi and their relation to the Weyl calculus.

2. A single, bounded, self-adjoint operator.

2.1. The spectral calculus on Hilbert space. Let X be a complex Hilbert space, and let $A \in L(X)$ be self adjoint. We wish to devise a method for mapping a suitable function $f : \mathbf{R} \mapsto \mathbf{C}$ to an operator $f(A) \in L(X)$. Though the question of what constitutes a "suitable" function is yet to be determined, we'd like for this class of functions to at least contain the class of polynomials $P(\mathbf{R})$ and be consistent with the natural definition of $p(A)$ for $p \in P$.

Perhaps the most straightforward means of accomplishing this is through the von Neumann spectral calculus for bounded, self-adjoint operators on a Hilbert space. We give a brief overview of the method here; relevant details can be found in [**3, 7**]. To begin, we note that the self-adjoint operator A can be decomposed as the difference $A^+ - A^-$ of two positive operators, where $A^{\pm} = (\sqrt{(A^2)} \pm A)/2$. Furthermore, we define $m = \inf_{\|x\|=1}(Ax, x)$ and $M = \sup_{\|x\|=1}(Ax, x)$. Now, for each $\lambda \in \mathbf{R}$, we define $E(\lambda) \in L(X)$ to be the projection onto the null space of $(A-\lambda)^+$. A study of the family $E(\lambda)$ reveals that it is the *resolution of identity* associated with the operator A, in particular:

1. $E(\lambda_1) \le E(\lambda_2)$ for $\lambda_1 \le \lambda_2$.
2. The family $E(\lambda)$ is strongly continuous from the right.
3. Each $E(\lambda)$ commutes with A.
4. $E(\lambda) = 0$ for $\lambda < m$, $E(\lambda) = I$ for $\lambda \ge M$.

Now, choose any $x \in X$, and consider the function $(E(\lambda)x, x) : \mathbf{R} \mapsto \mathbf{R}$. We may define the set function m_x on the semi-ring R of half-open intervals of the form $(a, b]$ as $m_x((a, b]) = ([E(b) - E(a)]x, x)$. Since $(E(\lambda)x, x)$ is nondecreasing and continuous from the right, it follows that m_x induces a Lebesgue-Stieltjes measure on $\mathbf{R}$ with the property that $m_x(\mathbf{R}) = m_x((a, b]) = \|x\|^2$ for every $a < m$, $b \ge M$. Thus, any polynomial $p \in P$ is measurable and integrable with respect to this measure. Furthermore, we have for all $\lambda_1 \le \lambda_2$,

$$\lambda_1(E(\lambda_2) - E(\lambda_1)) \le A(E(\lambda_2) - E(\lambda_1)) \le \lambda_2(E(\lambda_2) - E(\lambda_1)). \tag{2.1}$$

From this, it can be shown that, for all polynomials p:

$$\int_{\mathbf{R}} p(\lambda)\, dm_x = (p(A)x, x). \tag{2.2}$$

In particular, $(Ax, x) = \int \lambda\, dm_x$. When coupled with the polarization identity, we find that, for all $x, y \in X$ and $p \in P$, $(p(A)x, y) = \int p(\lambda)\, dm_{xy}$, where m_{xy} is the (complex) measure induced by the function $\lambda \mapsto (E(\lambda)x, y)$. However, it can also be shown that the Riemann partial sums $\sum_{k=1}^{n} p(\lambda_k')(E(\lambda_k) - E(\lambda_{k-1}))$ converge to A uniformly as the partition size $\max_k(\lambda_k - \lambda_{k-1})$ approaches 0. Thus, using a suitable sequence of simple functions to approximate the integral, we may write $p(A)$ as:

$$p(A) = \int_{\mathbf{R}} p(\lambda)\, dE(\lambda), \tag{2.3}$$

where $E(\lambda)$ is the projection-valued measure defined on the sets of R by $E(\lambda)(a,b] = E(b) - E(a)$. Observe that the support of this measure is contained in $[m, M]$. With this in mind, we can extend this definition to a larger class F of functions as follows: let $f : \mathbf{R} \mapsto \mathbf{R}$ be a Borel function that is bounded on $[m, M]$. Then $f(\lambda)$ is measurable with respect to the operator-valued measure $E(\lambda)$, i.e., $f(\lambda)$ is measurable with respect to the complex measures induced by $(E(\lambda)x, y)$ for every $x, y \in X$, and we may define:

$$f(A) = \int_{\mathbf{R}} f(\lambda)\, dE(\lambda). \tag{2.4}$$

Note that this class includes the space of continuous functions $C(\mathbf{R})$.

We observe that there is a natural extension of this correspondence to functions of several variables. Indeed, if $\{A_1, \dots, A_n\} \subset L(X)$ is a collection of *commuting* self-adjoint operators, then the corresponding resolutions of identity $E_i(\lambda)$ are themselves commutative. Hence, we may construct the product measure $E_1(\lambda_1) \times \cdots \times E_n(\lambda_n)$ on $\mathbf{R}^n$. Then, given a suitably well-behaved function $f : \mathbf{R}^n \mapsto \mathbf{R}$, we may define:

$$f(A_1, \dots, A_n) = \int_{\mathbf{R}^n} f(\lambda_1, \dots, \lambda_n)\, dE(\lambda_1) \times \cdots \times E(\lambda_n). \tag{2.5}$$

However, if the A_i, and thus the E_i, are noncommutative, then we cannot define a projection-valued product measure on $\mathbf{R}^n$ as was done in (2.4). It follows that if we wish to define a general correspondence between functions on $\mathbf{R}^n$ and bounded, self-adjoint operators on a Hilbert space, then we must find some means to address the issue of noncommutativity.

2.2. The one-dimensional Weyl calculus. Let f be in the Schwartz space $S(\mathbf{R})$ of smooth, rapidly-decreasing functions with rapidly-decreasing derivatives. It is well known that the Fourier transform $\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi x}\, dx$ belongs to $S(\mathbf{R})$ as well. With X defined as above, let $A \in L(X)$ be self-adjoint. Then, the operator-valued function $\xi \mapsto e^{2\pi i \xi A}$ maps ξ to a unitary operator for every $\xi \in \mathbf{R}$, and it is analytic in ξ (in the sense of the norm topology; cf. Proposition 3.1 below). Let us consider the formal definition [**15**]:

$$f(A) = \int_{\mathbf{R}} \hat{f}(\xi) e^{2\pi i \xi A}\, d\xi. \tag{2.6}$$

We note that this definition makes sense as the Bochner integral of the operator-valued function $\hat{f}(\xi)e^{2\pi i\xi A}$, which is strongly measurable due to the fact that $\hat{f}$ and the function $e^{2\pi i\xi A}$ are both continuous. Since $\|\hat{f}(\xi)e^{2\pi i\xi A}\| = |\hat{f}(\xi)|$, we find that the operator $f(A)$ is well defined. How does this definition compare to that given by the spectral calculus above? To answer this question, we note that the function $x \mapsto e^{2\pi i\xi x}$ is continuous (hence measurable) and bounded on the interval $[m, M]$ defined above, so for each ξ we may apply the spectral calculus to write:

$$e^{2\pi i\xi A} = \int_{\mathbf{R}} e^{2\pi i\xi\lambda}\, dE(\lambda) \tag{2.7}$$

where the integral is again convergent in the sense of the norm. Substituting this into (2.6), we obtain the iterated integral:

$$f(A) = \iint \hat{f}(\xi)e^{2\pi i\xi\lambda}\, dE(\lambda)\, d\xi. \tag{2.8}$$

Observe that $|\hat{f}(\xi)e^{2\pi i\xi\lambda}| = |\hat{f}(\xi)|$ for all $\xi, \lambda \in \mathbf{R} \times \mathbf{R}$. Thus, since $\hat{f} \in L^1(\mathbf{R})$ and the projection-valued measure $E(\lambda)$ is finite, we may apply the Fubini-Tonelli theorem to reverse the order of integration and obtain:

$$\begin{aligned} f(A) &= \iint \hat{f}(\xi)e^{2\pi i\xi\lambda}\, d\xi\, dE(\lambda) \\ &= \int_{\mathbf{R}} f(\lambda)\, dE(\lambda). \end{aligned} \tag{2.9}$$

Since $f \in S(\mathbf{R})$ clearly satisfies the requirements for $\int f(\lambda)\, dE(\lambda)$ to be well defined, we see that these two definitions of $f(A)$ coincide.

An immediate advantage of the latter approach is that it allows us to extend our definition of $f(A)$ to include those $f \in L^1(\mathbf{R})$ such that $\hat{f} \in L^1(\mathbf{R})$ also, as the norm of the integrand in (2.3) will be bounded by $|\hat{f}(\xi)|$. What is not clear, however, is what to do if f is a polynomial p, in which case the Fourier transform $\hat{p}$ may only make sense as a tempered distribution. As we discuss below, however, the natural algebraic correspondence between A and $p(A)$ can still be recovered when we consider $e^{2\pi i\xi A}$ as an operator-valued tempered distribution. This interpretation also allows us to extend (2.6) to the multi-variable case, even if the operators involved are not commutative.

3. The multi-variable Weyl correspondence for bounded, self-adjoint operators.

3.1. The operator $e^{2\pi i\xi A}$. Let X be a Hilbert space, and fix $\{A_1, \dots, A_n\} \subset L(X)$ with each A_i self-adjoint. For $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, we consider the map $\xi \mapsto e^{2\pi i\xi A}$, where we use ξA to denote the sum $\xi_1 A_1 + \cdots + \xi_n A_n$. Observe that the operator $e^{2\pi i\xi A}$ is unitary for each ξ. Now, let $f \in S(\mathbf{R}^n)$, and consider the formal Bochner integral [**15**]:

$$f(A) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i\xi A}\, d\xi. \tag{3.1}$$

Were the A_i commuting, we could apply the results of subsection 2.1 to obtain $f(A)$ in the sense of spectral calculus. However, since this was not assumed, we must consider the nature of the operator-valued tempered distribution T defined as $\langle T, f\rangle = \int \hat{f}(\xi) e^{2\pi i\xi A}\, d\xi$ more carefully. To do this, we note that we may extend $e^{2\pi i\xi A}$ to be defined for all $\xi \in \mathbf{C}^n$. In particular, from the Taylor expansion of the exponential, we can show that this extension is entire. We shall set aside a moment to establish this fact, as well as develop a result that will be quite useful for us later on.

Before we do so, we must first establish some notation. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ be a standard multi-index, and define the monomial $p_\alpha(\xi)$ as $p_\alpha(\xi) = \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We note that if the operators $\{A_1, \cdots, A_n\}$ above are commutative, then this leads to an immediate definition of $p_\alpha(A) = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$. To obtain the more general definition, we make use of the fact that every monomial $p_\alpha(\xi)$ can be expressed as a linear combination of polynomials of the form $p(\xi) = (a_1\xi_1 + \cdots + a_n\xi_n)^k$ [12]. For such a polynomial p, we define the correspondence $p(A) = (a_1 A_1 + \cdots + a_n A_n)^k$, which can be calculated explicitly. From this definition, we find that each $p_\alpha(A)$ is uniquely given by the *symmetrized product* [**6**]:

$$p_\alpha(A) = \frac{\alpha_1! \cdots \alpha_n!}{|\alpha|!} \sum_\sigma A_{\sigma(1)} \cdots A_{\sigma(|\alpha|)} \tag{3.2}$$

where the sum is performed over all maps $\sigma : \{1, \dots, |\alpha|\} \mapsto \{1, \dots n\}$ that assume the value k exactly α_k times. This readily extends to a definition of $p(A)$ for all polynomials that is consistent with the commutative case discussed earlier. Furthermore, to each monomial ξ^α we define the differential operator:

$$D^\alpha = \left(\frac{1}{2\pi i}\right)^{|\alpha|} \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial \xi_n^{\alpha_n}}. \tag{3.3}$$

From this, the definition of $p(D)$ for any polynomial p is obtained naturally.

Proposition 3.1. *The mapping $e^{2\pi i \xi A}$ for $\xi \in \mathbf{C}^n$ is entire. Furthermore, if $p \in P$ is a polynomial, then $[p(D)e^{2\pi i \xi A}]_{\xi=0} = p(A)$.*

Proof. We will show that $\xi \mapsto e^{2\pi i \xi A}$ is smooth, and that all of its derivatives are expressible in terms of a normally convergent Taylor series. To begin, we note that the Taylor expansion $\sum_{k\geq 0}(2\pi i^k/k!)(\xi A)^k$ converges in $L(X)$ uniformly for $\xi \in K$, where $K \subset \mathbf{C}^n$ is any fixed compact set. To show that this series defines a C^∞ function, set $\xi = (\xi_1, \dots, \xi_n)$ and $\xi_h = (\xi_1 + h, \dots, \xi_n)$, where $h \in \mathbf{C}$. Then, using the Taylor expansion, we may write $[e^{2\pi i \xi_h A} - e^{2\pi i \xi A}]/h$ as:

$$\frac{1}{h} \sum_{k\geq 0} \frac{2\pi i^k}{k!} \left[(hA_1 + \xi A)^k - (\xi A)^k\right]. \tag{3.4}$$

Now, define $N = \sup_{\xi \in K}(\|A_1\| + 2\|\xi A\|)$. Observe that the series (3.4) is bounded termwise by $\sum_{k\geq 0}(2\pi N)^k/k! = e^{2\pi N}$. Thus, we may apply the Lebesgue dominated convergence theorem for counting measure on $\mathbf{N}$ to conclude that as $h \to 0$, the series (3.4) converges in $L(X)$ uniformly for $\xi \in K$ to:

$$\sum_{k\geq 0} \frac{2\pi i^k}{k!}[A_1(\xi A)^{k-1} + (\xi A)A_1(\xi A)^{k-2} + \cdots + (\xi A)^{k-1}A_1]. \tag{3.5}$$

Repeated application of the above process reveals that the derivatives of all orders exist for $e^{2\pi i \xi A}$, hence $e^{2\pi i \xi A}$ is smooth. In fact, the dominated convergence theorem assures us that the Taylor series (3.4) for $\partial/\partial x_1$ and all other derivatives converge uniformly on compact subsets of $\mathbf{C}$. Thus, we conclude that $e^{2\pi i \xi A}$ is entire.

Now, let $p_\alpha \in P$ be a monomial, and consider $[p_\alpha(D)e^{2\pi i\xi A}]_{\xi=0}$. From the analyticity of $e^{2\pi i\xi A}$, it follows that we need only consider the term $2\pi i^{|\alpha|}/|\alpha|[p_\alpha(D)(\xi A)^{|\alpha|}]_{\xi=0}$. Indeed, we may readily calculate this derivative to be:

$$\begin{aligned}\left[p_\alpha(D)e^{2\pi i\xi A}\right]_{\xi=0} &= \frac{2\pi i^{|\alpha|}}{|\alpha|}\left[p_\alpha(D)(\xi A)^{|\alpha|}\right]_{\xi=0}\\ &= \frac{2\pi i^{|\alpha|}}{|\alpha|}\left[p_\alpha(D)(\xi)^\alpha \sum_\sigma A_{\sigma(1)}\cdots A_{\sigma(|\alpha|)}\right]_{\xi=0}\\ &= \frac{\alpha_1!\cdots\alpha_n!}{|\alpha|!}\sum_\sigma A_{\sigma(1)}\cdots A_{\sigma(|\alpha|)}\\ &= p_\alpha(A).\end{aligned}\tag{3.6}$$

The result for an arbitrary $p \in P$ follows immediately. □

The next step in establishing the Weyl correspondence requires us to obtain a stronger estimate for $\|e^{2\pi i\xi A}\|$ when ξ is complex. To do this, we can apply the Trotter product formula concerning the product of semi-groups of the form e^{tA_k} where the A_k may not commute. In the case that the A_k are bounded, then $\|e^{tA_k}\| \le e^{|t|\|A_k\|}$ and each has domain Dom$(A_k) = X$. From this, we find that $e^{t(A_1+\cdots+A_n)}$ defines a semi-group for $t \ge 0$ that satisfies:

$$e^{t(A_1+\cdots+A_n)} = \text{strong}\lim_{k\to\infty}\left[e^{tA_1/k}\cdots e^{tA_n/k}\right]^k. \tag{3.7}$$

Now, let A_k be bounded and self-adjoint, and set $\xi_k = x_k + iy_k$ for each k. Applying the Trotter product formula to $e^{2\pi i\xi A}$, we obtain:

$$e^{2\pi i\xi A} = \text{strong}\lim_{k\to\infty}\left[e^{2\pi ix_1A_1/k}e^{-2\pi y_1A_1/k}\cdots e^{2\pi ix_nA_n/k}e^{-2\pi y_nA_n/k}\right]^k. \tag{3.8}$$

Thus, since each $e^{2\pi ix_1A_1/k}$ is unitary, we calculate (after applying the norm to both sides):

$$\left\|e^{2\pi i\xi A}\right\| \le e^{2\pi|y_1|\|A_1\|}\cdots e^{2\pi|y_n|\|A_n\|}. \tag{3.9}$$

Defining $|A| = [\|A_1\|^2+\cdots+\|A_n\|^2]^{1/2}$ and $|y|$ similarly, we may apply the Cauchy-Schwartz inequality to obtain the bound:

$$\left\|e^{2\pi i\xi A}\right\| \le e^{2\pi|y||A|}. \tag{3.10}$$

As a final remark, we note that the derivatives of the operator $e^{2\pi i\xi A}$ are uniformly bounded for real ξ. Indeed, since $e^{2\pi i\xi A}$ is entire, we may apply the Cauchy-Riemann formula to obtain [**15**]:

$$D^\alpha e^{2\pi iA\xi} = \frac{\alpha_1!\cdots\alpha_n!}{(2\pi i)^{n+|\alpha|}}\int_\Gamma \frac{e^{2\pi iA\lambda}}{(\lambda_1-\xi_1)\cdots(\lambda_n-\xi_n)}\,d\lambda_1\cdots d\lambda_n \tag{3.11}$$

where the region of integration Γ consists of all $\lambda=(\lambda_1,\dots,\lambda_n)$ with $|\lambda_k-\xi_k|=1$ for each k. Since $|\mathrm{Im}\,(\lambda_k-\xi_k)|\le 1$, it follows from (3.10) that the integral is bounded uniformly in $\mathbf{R}^n$.

3.2. Operator-valued tempered distributions and the Paley-Wiener theorem. Let T be an operator-valued tempered distribution acting on $S(\mathbf{R}^n)$. Then the inverse Fourier transform $\check{T}$ is well defined as a tempered distribution. The Paley-Wiener theorem states that T has compact support if and only if $\check{T}$ extends to define an entire operator-valued tempered distribution on $\mathbf{C}^n$ that satisfies the bound:

$$\left\|\check{T}(\xi)\right\| \le q(|\xi|)e^{k|\mathrm{Im}\,(\xi)|} \tag{3.12}$$

for some polynomial q and constant k. Returning our attention to formula (3.1) above, we may (formally) propose this integral to be the evaluation of T on the test function f, in which case $\check{T}$ is defined as $\langle\check{T},f\rangle = \int_{\mathbf{R}^n} e^{2\pi i\xi A}f(\xi)\,d\xi$. However, we have established in (3.10) that $e^{2\pi i\xi A}$ extends to define an entire operator-valued function on $\mathbf{C}^n$ that satisfies the bound (3.11). Thus, T is a tempered distribution with compact support. It follows immediately that (3.1) can be continuously extended to include all $f\in C^m(\mathbf{R}^n)$ for some positive $m\le\infty$ (where C^m is equipped with the topology of uniform convergence of f and its derivatives on compact subsets). Since the polynomials are in this class, we are finally in a position to show the correspondence between the Weyl calculus and the natural definition of $p(A)$ given above. The approach used below is due to Taylor [**15**]:

Proposition 3.2. *Let $p\in P(\mathbf{R}^n)$. Then the operator-valued distribution T satisfies $\langle T,p\rangle = p(A)$.*

Proof. To begin, let $\phi\in C_0^\infty(\mathbf{R}^n)$ be such that $|\phi(x)|=1$ in a neighborhood of the unit ball of $\mathbf{R}^n$. For $a\ge 1$, define $\phi_a=\phi(x/a)$.

Note that $\phi \in S \Rightarrow \widehat{\phi} \in S$; hence, there exists an M such that $|\widehat{\phi}(\xi)| \leq M|\xi|^{-n-1}$ for all $|\xi| \geq 1$. It follows that for all such ξ and a, $|\widehat{\phi}_a(\xi)| = a^n|\widehat{\phi}(a\xi)| \leq a^n M|a\xi|^{-n-1} \leq M|\xi|^{-n-1}$. Now, choose a large enough so that the support of T is contained in that of ϕ_a. We then have:

$$\begin{aligned}\langle T, p\rangle &= \int_{\mathbf{R}^n} [p\phi_a]\widehat{\ }(\xi) e^{2\pi i \xi A}\, d\xi \\ &= \int_{\mathbf{R}^n} \left[p(-D)\widehat{\phi}_a\right](\xi) e^{2\pi i \xi A}\, d\xi \\ &= \int_{\mathbf{R}^n} \widehat{\phi}_a(\xi)\left[p(D) e^{2\pi i \xi A}\right]\, d\xi\end{aligned} \tag{3.13}$$

where the differential operator $P(D)$ is defined as in subsection 3.1. Note that the uniform bound for $p(D)e^{2\pi i\xi A}$ on $\mathbf{R}^n$ justifies the derivative in the last step. The boundedness of $p(D)e^{2\pi i\xi A}$, along with the bound given above for $|\widehat{\phi}_a(\xi)|$, allows us to choose r large enough so that:

$$\left\| \langle T, p\rangle - \int_{\mathbf{R}^n} \widehat{\phi_a}\phi_r(\xi)\left[p(D)e^{2\pi i\xi A}\right]\, d\xi \right\| < \varepsilon \tag{3.14}$$

uniformly for $a \geq 1$, where $\varepsilon > 0$ is arbitrary. We now write the latter term as the iterated integral:

$$\iint \phi_a(x)\phi_r(\xi) e^{-2\pi i \xi x}\left[p(D)e^{2\pi i \xi A}\right](\xi)\, dx\, d\xi. \tag{3.15}$$

Again, from the boundedness of $p(D)e^{2\pi i\xi A}$ and the fact that ϕ_a, ϕ_r have compact support, we may apply the Fubini-Tonelli theorem to reverse the order of integration. Writing $h(\xi) = \phi_r(\xi)e^{-2\pi i\xi x}[p(D)e^{2\pi i\xi A}]$, this reads:

$$\int_{\mathbf{R}^n} \phi_a(x)\hat{h}(x)\, dx. \tag{3.16}$$

Note that the operator-valued function h is smooth and of compact support, hence $\|\hat{h}\| \in S(\mathbf{R}^n)$. It follows that we may choose a large enough so that:

$$\left\| \int_{\mathbf{R}^n} \phi_a(x)\hat{h}(x)\, dx - \int_{\mathbf{R}^n} \hat{h}(x)\, dx \right\| < \varepsilon. \tag{3.17}$$

Since the latter term is $h(0) = [p(D)e^{2\pi i\xi A}]_{\xi=0} = p(A)$ by Proposition 3.1, the result follows. $\square$

Thus, the Weyl correspondence defines a continuous map $C^m \mapsto L(X)$ that is consistent with the natural algebraic definition on polynomials. It follows that the restriction of the calculus to holomorphic functions also gives us the results we would expect from the Taylor expansion coupled with the symmetrized product discussed above. In essence, the validity of any proposed functional calculus defined for a collection of noncommutative bounded operators A_i is determined by its action on polynomials. Such a gauge for determining the usefulness of a functional calculus must be used with care in the case that the A_i are unbounded, since the domain of $p(A)$ generally shrinks as the degree of the polynomial increases.

As a final note, we observe that the Weyl calculus for bounded, self-adjoint operators clearly commutes with affine transformations of the type $A'_j = \sum_k c_{jk}A_k + d_jI$ (where all the constants are real). From this, it is readily verifiable [**1**] that if $f \in C^\infty$ depends only on x_k, then the multi-variable Weyl calculus gives the same result as the spectral calculus for $f(A_k)$.

4. The Weyl correspondence for P and Q.

4.1. The operators P and Q. Recall the discussion of the state space $(p,q) \in \mathbf{R}^{2n}$ introduced in Section 1. Our goal here is to determine the map $f(p,q) \mapsto f(P,Q)$ that takes real-valued functions of the state (p,q) to the corresponding operator on $L^2(\mathbf{R}^n)$ Here, we adopt the quantum-mechanical interpretation of state space as the projective Hilbert space consisting of those $f \in L^2(\mathbf{R}^n)$ with unit norm. Note that, for each state f, the Fourier transform $\hat{f}$ is well defined and also of unit norm. In the Weyl correspondence, we interpret f as representing the probability distribution function $|f(x)|^2$ for the position of the particle in space. Furthermore, as the exponential functions $e^{2\pi i\xi x}$ are eigenfunctions of the momentum operator [**6**], we interpret $|\hat{f}(\xi)|^2$ as the probability distribution corresponding to the normalized momentum p/h of the particle, where h is Planck's constant.

We will now attempt to "build" the operators Q_j and P_j on $L^2(\mathbf{R}^n)$ corresponding to q_j and p_j using the spectral calculus. Fix a coordinate index j. We define a projection-valued measure $E_j(\lambda)$ on $\mathbf{R}$ as follows: given a half-open interval of the form $(a,b]$, we define $A_j(a,b] = \{x \in \mathbf{R}^n : x_j \in (a,b]\}$ and set $E_j(\lambda)(a,b]$ to be multiplication by the characteristic function $\chi_{A_j(a,b]}$. Given any such $f \in L^2(\mathbf{R}^n)$ (in particular, one with unit norm), the function $\lambda \mapsto (E_j(\lambda)f, f) = \|\chi_{A_j(-\infty,\lambda]}f\|_2^2$ is nonnegative, increasing, of bounded variation, and strongly continuous from the right. Thus, we may extend $(E_j(\lambda)f, f)$ to be defined as a Lebesgue-Stieltjes measure on $\mathbf{R}$ and consider $E_j(\lambda)$ to be a spectral measure similar to the type discussed in subsection 2.1. The primary difference here is that the measure clearly does not have compact support, whereas the resolution of identity corresponding to a bounded operator A had support contained in the interval $[m, M]$. This has the effect of limiting the domain of the operator $\int \lambda\, dE_j(\lambda)$ to those functions f for which the measurable function λ is integrable with respect to $(E_j(\lambda)f, f)$.

Proposition 4.1. *For $f \in S(\mathbf{R}^n)$, the integral $\int_{\mathbf{R}} \lambda\, d(E_j(\lambda)f, f)$ is well defined and equals $(x_j f, f)$.*

Proof. Consider the (possibly infinite) integral $\int_{\mathbf{R}} |\lambda|\, d(E_j(\lambda)f, f)$. Let $s_i(\lambda) = \sum_k c_{ik}\chi_{B_k}(\lambda)$ be a sequence of nonnegative, measurable simple functions that converge monotonically ($\nearrow$) to $|\lambda|$ pointwise. Note that we may choose each B_k to be a half-open interval of the form $(a,b]$, so that the sets $A_j(B_k) \subset \mathbf{R}^n$ are well defined. Then the monotone convergence theorem implies:

$$\begin{aligned} \int_{\mathbf{R}} |\lambda|\ d(E_j(\lambda)f, f) &= \lim_i \sum_k \int_{\mathbf{R}} c_{ik}\chi_{B_k}\ d(E_j(\lambda)f, f) \\ &= \lim_i \sum_k \int_{\mathbf{R}^n} c_{ik}\chi_{A_j(B_k)} \left|f(x)\right|^2\ dx. \end{aligned} \tag{4.1}$$

Since the simple functions $S_i = \sum_k c_{ik}\chi_{A_j(B_k)} \nearrow |x_j|$ pointwise on $\mathbf{R}^n$, the monotone convergence theorem again yields:

$$\int_{\mathbf{R}} |\lambda|\ d(E_j(\lambda)f, f) = \int_{\mathbf{R}^n} |x_j|\left|f(x)\right|^2\ dx = (|x_j|\, f, f). \tag{4.2}$$

It follows that $\int_{\mathbf{R}^n} \lambda \, d(E_j(\lambda) f, f)$ is well defined, and considering the positive and negative parts of the above analysis, we readily obtain $\int_{\mathbf{R}} \lambda \, d(E_j(\lambda) f, f) = (x_j f, f)$. $\square$

Recall the polarization identity for a complex Hilbert space:

$$(x, y) = \frac{1}{4}(x + y, x + y) - \frac{1}{4}(x - y, x - y) + \frac{i}{4}(x + iy, x + iy) - \frac{i}{4}(x - iy, x - iy). \tag{4.3}$$

Using this, we may conclude that for every $f, g \in S(\mathbf{R}^n)$, $\int_{\mathbf{R}} \lambda \, d(E_j f, g) = (x_j f, g)$. From the fact that $S(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$, it follows that $\int_{\mathbf{R}} \lambda \, d(E_j f, g) = (x_j f, g)$ for all $f \in S(\mathbf{R}^n), g \in L^2(\mathbf{R}^n)$. Thus, we are led to define the quantum-mechanical analogue of the position operator as the multiplication operator $Q_j : f \mapsto x_j f$, defined on the set of all $f \in L^2(\mathbf{R}^n)$ such that $x_j f \in L^2(\mathbf{R}^n)$ as well. Since this domain contains $S(\mathbf{R}^n)$, it is dense in $L^2(\mathbf{R}^n)$. However, the operator is unbounded, since we can demonstrate a sequence $\{f_k\} \subset \mathrm{Dom}\,(Q_j)$ such that $\{f_k\}$ is Cauchy but the $x_j f_k$ are not. As an example, for $j = 1$, we may define $S_k = \{x \in \mathbf{R}^n : x_1 \in [k, k+1], |x_{i \neq 1}| \leq 1\}$. Then, define $f_k \in \mathrm{Dom}\,(Q_1)$ as $f_k(x) = (1/x_1)\chi_{S_k}$.

The determination of the operator corresponding to the momentum in the direction x_j proceeds similarly. The primary difference is that we now define our projections $C_j(\lambda)$ to be the multiplication by characteristic functions in the Fourier domain:

$$C_j(\lambda) = F^{-1} E_j(\lambda) F \tag{4.4}$$

where F represents the Fourier transform. We see from the above that, given $f \in S(\mathbf{R}^n)$, the operator $\int \lambda dC_j(\lambda)$ maps $f \mapsto g$, where $g \in S(\mathbf{R}^n)$ satisfies $\hat{g}(\xi) = \xi_j \hat{f}$. In other words, $g = (1/2\pi i)(\partial f/\partial x_j) = D_j(f)$. Furthermore, since the Fourier transform is an isometry from $S(\mathbf{R}^n)$ to $S(\mathbf{R}^n)$ (equipped with the L^2 norm), it follows that this mapping preserves the interpretation of $C_j(\lambda)(f, f) = E_j(\lambda)(\hat{f}, \hat{f})$ as the probability distribution corresponding to the normalized momentum. Therefore, we are led to define $P_j f \mapsto h[\xi \hat{f} t]\check{\ } \in L^2(\mathbf{R}^n)$ on the set of all $f \in L^2(\mathbf{R}^n)$ such that $\xi \hat{f} \in L^2(\mathbf{R}^n)$ as well (where $\check{f}$ represents the inverse Fourier transform). Again, this is an unbounded operator whose domain contains $S(\mathbf{R}^n)$.

4.2. The unitary operator $e^{2\pi i(pP+qQ)}$. Let $p, q \in \mathbf{R}^n$, and define the operator $T = -2\pi i(pP + qQ)$ on $S(\mathbf{R}^n)$. Our aim in this section is to show that we can define a unitary operator $e^{2\pi i(pP+qQ)}$ on *all* of $L^2(\mathbf{R}^n)$ even though T itself is unbounded. To do this, we appeal to the Hille-Yosida theorem [**1, 10**], which states that a closed operator A defines a strongly continuous contraction semigroup $e^{-tA} = \lim_{n\to\infty}(1 + (t/n)T)^{-n}$ if and only if the resolvent $(\lambda + A)^{-1}$ exists as a bounded operator with norm $\leq 1/\lambda$ for all $\lambda > 0$.

Before attempting to apply this theorem, however, we need to verify that T admits a closure $\overline{T}$. As described in [**3**], it is sufficient to verify that if $\{f_k\} \subset S(\mathbf{R}^n)$ and $\{g_k\} \subset S(\mathbf{R}^n)$ both converge (in $L^2(\mathbf{R}^n)$) to f, and if $Tf_k \to h_1$ while $Tg_k \to h_2$, then $h_1 = h_2$. Note that this is equivalent to having $(h_1, \phi) = (h_2, \phi)$ for all $\phi \in S(\mathbf{R}^n)$. Now, we observe from integration by parts that for all $\phi, \varphi \in S(\mathbf{R}^n)$, the identity $(T\phi, \varphi) = (\phi, T^t\varphi)$ holds, where $T^t = -2\pi i(-pP + qQ)$ on $S(\mathbf{R}^n)$ is the transpose of T. From this, coupled with Hölder's inequality, we may calculate:

$$
\begin{aligned}
(h_1, \phi) &= \lim_{k\to\infty} (Tf_k, \phi) \\
&= \lim_{k\to\infty} (f_k, T^t\phi) \\
&= (f, T^t\phi) \\
&= \lim_{k\to\infty} (g_k, T^t\phi) \\
&= \lim_{k\to\infty} (Tg_k, \phi) \\
&= (h_2, \phi).
\end{aligned}
\tag{4.5}
$$

It follows that we may define the closure $\overline{T}$ of T with domain $\mathrm{Dom}\,(\overline{T})$ that contains $S(\mathbf{R}^n)$.

Proposition 4.2. *The operator $\overline{T}$ satisfies the hypotheses of the Hille-Yosida theorem.*

Proof. Choose $\lambda > 0$. We wish to demonstrate for any $v \in L^2(\mathbf{R}^n)$, there is a unique $u \in \mathrm{Dom}\,(\overline{T})$ with $(\lambda + \overline{T})u = v$. Indeed, this is clear in the case that $p = 0$; in particular, we find that $u = (\lambda - 2\pi iqx)^{-1}v$ satisfies $\|u\|_2 \leq \|v\|_2/\lambda$.

In the case that p is nonzero, it suffices to search for a solution to the differential equation:

$$\lambda u - hp\frac{\partial}{\partial x}u - 2\pi i q x u = v, \tag{4.6}$$

given some $v \in L^2(\mathbf{R}^n)$. Let $\mathbf{M} \subset \mathbf{R}^n$ be the closed hyperplane perpendicular to p, and choose $x_0 \in \mathbf{M}$. Define $u_{x_0}(t) = u(x_0 + pt)$ and $v_{x_0}(t)$ similarly, so that the above equation reads:

$$\frac{du_{x_0}}{dt} + \frac{(2\pi i q(x_0 + pt) - \lambda)}{h}u_{x_0} = -\frac{v_{x_0}}{h}. \tag{4.7}$$

Solving this equation via standard means, we obtain:

$$u_{x_0}(t) = \frac{1}{h}\int_t^\infty e^{\left(\lambda(t-s) - 2\pi i q x_0(t-s) - \pi i p(t^2 - s^2)\right)/h} v_{x_0}(s)\,ds \tag{4.8}$$

where the integral exists from Hölder's inequality. We claim that $u \in L^2(\mathbf{R}^n)$. Indeed, we may find a measure-preserving change of coordinates $(x_1, \ldots, x_n) \to (y_1, \ldots, y_n)$ such that y_1 is colinear with p and $(y_2, \ldots, y_n)$ form an orthonormal basis for $\mathbf{M}$. Then, applying the Fubini-Tonelli theorem to the integral of $|u|^2$, we calculate:

$$\begin{aligned}
\|u\|_2^2 &\le \frac{1}{|p|^2}\left(\frac{1}{h}\right)^2 \int_{\mathbf{M}} \int_{-\infty}^{\infty} \left(\int_t^\infty e^{\lambda(y_1 - s)/h|p|}\,|v|\;ds\right)^2 dy_1\,dy \\
&= \frac{1}{|p|^2}\left(\frac{1}{h}\right)^2 \int_{\mathbf{M}} \int_{-\infty}^{\infty} \\
&\qquad \left(\int_t^\infty e^{\lambda(y_1 - s)/2h|p|} e^{\lambda(y_1 - s)/2h|p|}\,|v|\;ds\right)^2 dy_1\,dy \\
&\le \frac{1}{|p|^2}\left(\frac{1}{h}\right)^2 \int_{\mathbf{M}} \int_{-\infty}^{\infty} \left(\int_t^\infty e^{\lambda(y_1 - s)/h|p|}\,ds\right) \\
&\qquad \times \left(\int_t^\infty e^{\lambda(y_1 - s)/h|p|}\,|v|^2\;ds\right) dy_1\,dy \\
&= \frac{1}{|p|}\left(\frac{1}{\lambda h}\right) \int_{\mathbf{M}} \int_{-\infty}^{\infty} \int_t^\infty e^{\lambda(t-s)/h|p|}\,|v|^2\;ds\,dy_1\,dy
\end{aligned} \tag{4.9}$$

where we have used Hölder's inequality again. Applying the Fubini-Tonelli theorem once more to the inner integrals, we obtain:

$$\begin{aligned}\|u\|_2^2 &\le \frac{1}{|p|}\left(\frac{1}{\lambda h}\right)\int_{\mathbf{M}}\int_{-\infty}^{\infty}\int_{-\infty}^{t} e^{\lambda(y_1-s)/h|p|}\,|v|^2\;dy_1\,ds\,dy\\ &= \left(\frac{1}{\lambda}\right)^2\int_{\mathbf{M}}\int_{-\infty}^{\infty}|v|^2\;ds\,dy\\ &= \frac{\|v\|_2^2}{\lambda^2}.\end{aligned}\tag{4.10}$$

Thus, $u \in L^2(\mathbf{R}^n)$ and satisfies $\|u\|_2 \le \|v\|_2/\lambda$. Since u is clearly in $\mathrm{Dom}\,(\overline{T})$, it follows that $(\overline{T}+\lambda)^{-1}$ defines a bounded linear operator on $L^2(\mathbf{R}^n)$ with norm $\le 1/\lambda$. Thus, $\overline{T}$ satisfies the hypotheses of the Hille-Yosida theorem. □

Thus, we may apply the Hille-Yosida theorem to conclude that for $t \ge 0$, the operator-valued function $t \mapsto e^{2\pi it(pP+qQ)}$ is strongly continuous, a contraction ($\|e^{2\pi it(pP+qQ)}\| \le 1$), and is in fact a semi-group (for $s, t \ge 0$, we have $e^{2\pi is(pP+qQ)}\cdot e^{2\pi it(pP+qQ)} = e^{2\pi i(s+t)(pP+qQ)}$). However, it is clear that Proposition 4.2 holds if $\overline{T}$ is replaced with $-\overline{T}$, so we may consider $e^{2\pi it(pP+qQ)}$ for $t \in \mathbf{R}$ to be a group of unitary operators on $L^2(\mathbf{R}^n)$. It now remains to find the action of this operator when $t = 1$.

To do this, we appeal to the result [**10**] that the vector-valued function $t \mapsto e^{2\pi it(pP+qQ)}u$ is differentiable for $t \ge 0$ and $u \in \mathrm{Dom}\,(\overline{T})$, i.e., $t \mapsto e^{2\pi it(pP+qQ)}$ is strongly differentiable on $\mathrm{Dom}\,(\overline{T})$, with derivative:

$$\frac{d}{dt}e^{-t\overline{T}}u = -\overline{T}e^{-t\overline{T}}u. \tag{4.11}$$

Thus, if we let $u \in S$ and define $g(x,t) = (e^{-t\overline{T}}u)(x)$, it follows that g is a solution to the differential equation:

$$\frac{\partial g}{\partial t} - hp\frac{\partial}{\partial x}g = 2\pi iqxg. \tag{4.12}$$

As before, we fix $x_0 \in \mathbf{R}^n$ and treat this as the directional derivative of $g(x,t)$ in the direction $(-hp_1, \dots, -hp_n, 1)$. Defining $G(t) = g(x(t),t)$ as in [**6**], this reads:

$$\frac{dG}{dt} = 2\pi iq(x_0 - hpt)G(t) \tag{4.13}$$

which has the immediate solution:

$$G(t) = e^{2\pi i q x_0 t - \pi i h q p t^2} G(0). \tag{4.14}$$

Setting $G(0) = u(x_0)$ and $t = 1$, we obtain:

$$\left(e^{-\overline{T}} u\right)(x_0 - hp) = e^{2\pi i q x_0 - \pi i h q p} u(x_0) \tag{4.15}$$

or, in other words:

$$\left(e^{2\pi i (pP + qQ)} u\right)(x) = e^{2\pi i q x + \pi i h q p} u(x + hp). \tag{4.16}$$

The extension of this result to all of $L^2(\mathbf{R}^n)$ is clear.

The fact that we have an explicit formulation (4.16) for the action of $e^{2\pi i(pP+qQ)}$ gives us a certain advantage over the case when we were dealing with the operators A in the abstract. In particular, when it comes time to discuss the Weyl calculus based on this formulation, it allows us to apply standard results concerning integration in $L^2(\mathbf{R}^n)$ and other function spaces without having to incorporate Bochner integrals or operator-valued measures as we did in the previous sections. As we mention below, this adds some deeper insight into the types of functions to which this calculus can be applied, and how the operators thus obtained compare to more well-known operator classes (such as the Hilbert-Schmidt class).

4.3. The Weyl correspondence on $L^2(\mathbf{R}^{2n})$. Let $f \in S(\mathbf{R}^{2n}) \subset L^2(\mathbf{R}^{2n})$, and $g \in L^2(\mathbf{R}^n)$. We formally define the operator $f(P, Q)$ in the same fashion as (3.1), namely:

$$f(P, Q) g = \int_{\mathbf{R}^{2n}} \hat{f}(p, q) \left(e^{2\pi i (pP + qQ)} g\right) d(p \times q). \tag{4.17}$$

We intend to show that this operator is well defined and to classify it as a more recognizable operator on $L^2(\mathbf{R}^n)$. For brevity, we will set $h = 1$ in the following discussion. To begin, we may use the explicit form (4.16) for the action of $e^{2\pi i(pP+qQ)}$ to write:

$$\left(f(P, Q) g\right)(x) = \int_{\mathbf{R}^{2n}} \hat{f}(p, q) e^{2\pi i q x + \pi i q p} g(x + p)\, d(p \times q). \tag{4.18}$$

We now perform the measure-preserving change of variable $(p, q) \mapsto (y - x, q)$ to obtain:

$$(f(P,Q)g)(x) = \int_{\mathbf{R}^{2n}} \hat{f}(y - x, q) e^{\pi i q(x+y)} g(y)\, d(y \times q). \tag{4.19}$$

Next, since $f \in S(\mathbf{R}^{2n})$ implies that $\hat{f} \in S(\mathbf{R}^{2n})$, we see that (4.19) is well defined and can be written as the iterated integral:

$$\begin{aligned} (f(P,Q)g)(x) &= \iint \hat{f}(y - x, q) e^{\pi i q(x+y)} g(y)\, dy\, dq \\ &= \iint \hat{f}(y - x, q) e^{\pi i q(x+y)} g(y)\, dq\, dy \end{aligned} \tag{4.20}$$

where we have used the Fubini-Tonelli theorem. Writing $K_f(x,y) = \int \hat{f}(y - x, q) e^{\pi i q(x+y)}\, dq = (F_1 f)(y - x, (y + x/2))$, where F_1 represents the Fourier transform in the first n coordinates only, it follows that $f(P,Q)$ is given as the integral operator [**6**]:

$$(f(P,Q)g)(x) = \int_{\mathbf{R}^n} K_f(x,y) g(y)\, dy \tag{4.21}$$

with kernel $K_f(x,y) \in S(\mathbf{R}^{2n})$. In particular, we note that the map $f \mapsto K_f$ consists of a measure-preserving change of variable composed with a Fourier transform, both of which are unitary operators. Using the fact that $S(\mathbf{R}^{2n})$ is dense in $L^2(\mathbf{R}^{2n})$, it follows that this map can be extended to define a unitary map from $L^2(\mathbf{R}^{2n})$ to the kernels of the Hilbert-Schmidt operators on $L^2(\mathbf{R}^n)$.

There are several different ways that one can extend this analysis to include other classes of functions f. For instance, let f be such that $\hat{f} \in L^1(\mathbf{R}^{2n})$. By formally writing the integral (4.18) and introducing absolute-values, we may apply the Fubini-Tonelli theorems to obtain the estimate:

$$\iint \left|\hat{f}(y - x, q)\right| |g(y)|\; dy\, dq. \tag{4.22}$$

We may then use Young's inequality to verify that formula (4.19) is well defined for this f and specifies a bounded operator on $L^2(\mathbf{R}^n)$. Indeed, we may continue through steps (4.20)–(4.21) and conclude that

$f(P,Q)$ is again given by an integral operator with kernel $K_f(x,y) = (F_1 f)(y-x,(y+x/2))$.

Similarly, we may extend this analysis even further to the case when $f(p,q)$ defines a tempered distribution on $S(\mathbf{R}^{2n})$, cf. (4.24) below. Since the change of variable and Fourier transform performed above are well-defined for tempered distributions, it follows that the integral operator (4.21) defines a map from $S(\mathbf{R}^n) \mapsto S'(\mathbf{R}^n)$ via the correspondence $g(x) \mapsto (K_f(g))(x) = \int_{\mathbf{R}^n} K_f(x,y)g(y)\,dy$. In particular, for $h \in S(\mathbf{R}^n)$, we have:

$$\langle K_f(g), h\rangle = \iint K_f(x,y)g(y)h(x)\,dy\,dx. \tag{4.23}$$

We remark that the map $f \mapsto K_f$ from $S'(\mathbf{R}^{2n})$ to the set of continuous linear maps from $S(\mathbf{R}^n) \mapsto S'(\mathbf{R}^n)$ as prescribed above is in fact a bijection. Details can be found in [**6**].

4.4. The Weyl correspondence on polynomials. We now wish to examine why it is proper to call the Weyl correspondence a functional calculus. In particular, we wish to prove the analogue of Proposition 3.2 for the collection of unbounded self-adjoint operators (P,Q). Again, we must be careful when discussing the domain of the operator thus defined, since the unitary bound for $e^{2\pi i(pP+qQ)}$ will preclude our ability to substitute a polynomial for f. Thus, our testing ground is the interpretation of $f(p,q)$ as defining an operator-valued tempered distribution on $S(\mathbf{R}^{2n})$, and of (4.17) representing the action of $f(P,Q)$ on the function $g \in S(\mathbf{R}^n)$.

To begin, we wish to give an alternative form for expressions (4.17) and (4.21) when f is a tempered distribution. Let $f(p,q) \in S(\mathbf{R}^{2n})$ define a tempered distribution via the formula:

$$\langle f, h\rangle = \int_{\mathbf{R}^{2n}} f(p,q)h(p,q)\,d(p\times q). \tag{4.24}$$

Then, for $g \in S(\mathbf{R}^n)$, the integral (4.17) is well defined and given as:

$$(f(P,Q)g)(x) = \iint \hat{f}(p,q)e^{2\pi iqx+\pi ipq}g(x+p)\,dp\,dq. \tag{4.25}$$

Expanding the Fourier transform, we obtain the iterated integral:

$$\iint e^{2\pi iqx+\pi ipq}g(x+p)\left(\iint f(\xi,\eta)e^{-2\pi i(p\xi+q\eta)}\,d\xi\,d\eta\right)dp\,dq. \tag{4.26}$$

Switching the order of integration for $d\xi$ and $d\eta$ (which can be done via Fubini-Tonelli since $S(\mathbf{R}^{2n}) \subset L^1(\mathbf{R}^{2n})$), we obtain the integral:

$$\iiint (F_2 f)(\xi,q)e^{-2\pi ip\xi}e^{2\pi iq(x+p/2)}g(x+p)\,d\xi\,dp\,dq \tag{4.27}$$

where F_2 now represents the Fourier transform in the last n variables only. Since $(F_2 f)(\xi,q) \in S(\mathbf{R}^{2n})$ as well, we may again apply Fubini-Tonelli to integrate over q and obtain the expression [**6**]:

$$\begin{aligned}(f(P,Q)g)(x) &= \iint f(\xi,x+p/2)e^{-2\pi ip\xi}g(x+p)\,dp\,d\xi \\ &= \iint f(\xi,(x+y)/2)e^{2\pi i(x-y)\xi}g(y)\,dy\,d\xi\end{aligned} \tag{4.28}$$

where in the last step we performed the change-of-variable $p \mapsto y-x$ on the inner integral. Note that (4.28) makes sense even if $f(p,q)$ is not in $S(\mathbf{R}^{2n})$, so long as the function $f(p,\cdot)$ defines a tempered distribution on $S(\mathbf{R}^n)$ in a manner similar to (4.24) and the action $p \mapsto \langle f(p,\cdot),g(\cdot)\rangle$ is integrable. Thus, for such functions f we may take (4.28) to be the definition of the operator $f(P,Q)$ on $S(\mathbf{R}^n)$.

Next, we consider the action of the differential operators aP and bQ on a function $g \in S(\mathbf{R}^n)$. It is not hard to show that the commutator $[aP,bQ]$ is simply multiplication by $(ab)/2\pi i$, i.e.:

$$[aP(bQ)g]-[bQ(aP)g] = \frac{(ab)}{2\pi i}g. \tag{4.29}$$

From this result, we may establish the following lemma:

Lemma 4.3. *For any integer $k \geq 1$, the commutator $[(aP+bQ)^k,bQ]$ on $S(\mathbf{R}^n)$ is given as $[(aP+bQ)^k,bQ] = (k(ab)/2\pi i)(aP+bQ)^{k-1}$.*

Proof. The proof is by induction. For $k=1$, we calculate directly from (4.29):

$$\begin{aligned}(aP+bQ)bQ = aPbQ+(bQ)^2 &= \frac{(ab)}{2\pi i}+bQaP+(bQ)^2 \\ &= \frac{(ab)}{2\pi i}+bQ(aP+bQ).\end{aligned} \tag{4.30}$$

Now, consider $(aP + bQ)^k bQ = (aP + bQ)(aP + bQ)^{k-1} bQ$. By induction, we have that this is equal to $(aP+bQ)[((k-1)ab/2\pi i)(aP+bQ)^{k-2} + bQ(aP+bQ)^{k-1}]$. From this, and the $k = 1$ case above, we find:

$$(4.31)\quad (aP+bQ)\left[\frac{(k-1)(ab)}{2\pi i}(aP+bQ)^{k-2} + bQ(aP+bQ)^{k-1}\right]$$
$$= \frac{(k-1)(ab)}{2\pi i}(aP+bQ)^{k-1} + (aP+bQ)bQ(aP+bQ)^{k-1}$$
$$= \frac{(k-1)(ab)}{2\pi i}(aP+bQ)^{k-1} + \frac{(ab)}{2\pi i}(aP+bQ)^{k-1} + bQ(aP+bQ)^k$$
$$= \frac{k(ab)}{2\pi i}(aP+bQ)^{k-1} + (ab)(aP+bQ)^k,$$

and the lemma is proved. □

Proposition 4.4. *Let $f(p,q) = (ap+bq)^k$, where $k \geq 0$ is an integer and $a, b \in \mathbf{R}^n$. Then the tempered distribution $f(P,Q)$ defined by (4.28) is the differential operator $(aP+bQ)^k$.*

Proof. Again, we consider a proof by induction. The cases $k = 0$ and $k = 1$ are trivial, so fix an integer $k \geq 2$ and assume the result holds for all $j = 0, 1, \ldots, k-1$. Fix $g \in S(\mathbf{R}^n)$. We calculate:

$$(4.32)\quad \iint (a\xi + b(x+y)/2)^k e^{2\pi i(x-y)\xi} g(y)\, dy\, d\xi$$
$$= \iint (a\xi + b(x+y)/2)^{k-1}(a\xi) e^{2\pi i(x-y)\xi} g(y)\, dy\, d\xi$$
$$+ \frac{1}{2}(bx) \iint (a\xi + b(x+y)/2)^{k-1}(a\xi) e^{2\pi i(x-y)\xi} g(y)\, dy\, d\xi$$
$$+ \frac{1}{2} \iint (a\xi + b(x+y)/2)^{k-1}(a\xi) e^{2\pi i(x-y)\xi}(by) g(y)\, dy\, d\xi.$$

Using the inductive hypothesis, we see that the second term becomes:

$$(4.33)\quad \frac{1}{2}(bx) \iint (a\xi + b(x+y)/2)^{k-1}(a\xi) e^{2\pi i(x-y)\xi} g(y)\, dy\, d\xi$$
$$= \frac{1}{2}(bQ)[(aP+bQ)^{k-1} g](x)$$

whereas the third can be written as:

$$(4.34)\quad \frac{1}{2}\iint (a\xi + b(x+y)/2)^{k-1}(a\xi)e^{2\pi i(x-y)\xi}(by)g(y)\,dy\,d\xi = \frac{1}{2}[(aP+bQ)^{k-1}(bQ)g](x).$$

As for the first term, we may take advantage of the fact that $g \in S(\mathbf{R}^n)$ and integrate by parts in y to obtain the expression:

$$(4.35)\quad \frac{1}{2\pi i}\iint e^{2\pi i(x-y)\xi}(aD_y)[(a\xi + b(x+y)/2)^{k-1}g(y)]\,dy\,d\xi.$$

Calculating the derivative explicitly, we have the integrals:

$$(4.36)\quad \frac{1}{2}\frac{(k-1)(ab)}{2\pi i}\iint e^{2\pi i(x-y)\xi}(a\xi + b(x+y)/2)^{k-2}g(y)\,dy\,d\xi + \frac{1}{2\pi i}\iint e^{2\pi i(x-y)\xi}(a\xi + b(x+y)/2)^{k-1}(aD_y)[g(y)]\,dy\,d\xi.$$

Applying the inductive hypothesis to each term, this becomes:

$$(4.37)\quad \frac{1}{2}\frac{(k-1)(ab)}{2\pi i}[(aP+bQ)^{k-2}g] + [(aP+bQ)^{k-1}(aP)g].$$

Thus, after combining (4.37) with (4.33) and (4.34) above, it follows that the operator $f(P,Q)$ defined by (4.28) coincides on $S(\mathbf{R}^n)$ with the differential operator:

$$(4.38)\quad \frac{1}{2}\frac{(k-1)(ab)}{2\pi i}(aP+bQ)^{k-2} + (aP+bQ)^{k-1}(aP) + \frac{1}{2}(bQ)(aP+bQ)^{k-1} + \frac{1}{2}(aP+bQ)^{k-1}(bQ).$$

Applying Lemma 4.3 to this expression, we obtain the desired result:
(4.39)

$$f(P,Q) = (aP+bQ)^{k-1}(aP) + (aP+bQ)^{k-1}(bQ) = (aP+bQ)^k. \qquad \square$$

From Proposition 4.4 and the discussion at the beginning of subsection 3.1, we find that for all polynomials p on $\mathbf{R}^{2n}$, the operator $p(P,Q)$

as defined by (4.17) coincides with the natural algebraic definition implied by (3.2). Thus, the analogue of Proposition 3.2 holds for the original Weyl correspondence when the functions $f(p,q)$ are considered as tempered distributions.

In closing this section, we remark that a much stronger form of Proposition 4.4 holds. Let $\phi : \mathbf{R} \mapsto \mathbf{R}$ satisfy the bound $|\phi(x)| \leq (1+|x|)^M$ for some M, and let $f(p,q) = \phi(ap+bq)$. Then f defines a tempered distribution on $S(\mathbf{R}^{2n})$ and $f(P,Q)$ defines a linear map $S(\mathbf{R}^n) \mapsto L^2(\mathbf{R}^n)$ that is essentially self-adjoint on $S(\mathbf{R}^n)$. It follows that we may interpret the meaning of $f(P,Q)$ in the context of the spectral calculus. With this in mind, it can be shown from a consideration of the symplectic group $Sp(n)$ that the definition $f(P,Q)$ as defined by (4.28) corresponds to that given by the spectral calculus. In particular, if $\phi = x^k$ for some integer $k \geq 0$, we obtain the result of Proposition 4.4. The details are beyond the scope of this current work; we refer the interested reader to [**6**].

5. Some directions of current research.

5.1. Polynomially bounded semi-groups. Throughout this paper, we have considered the Weyl calculus strictly for self-adjoint operators A acting on a Hilbert space X, i.e., we've focused on the generators of unitary semigroups $e^{2\pi i\xi A}$. However, it is easily seen that the formal Bochner integral (3.1) is well-defined for f in $S(\mathbf{R}^n)$ if the exponential satisfies the polynomial growth estimate:

$$\left\| e^{2\pi i\xi A} \right\| \leq M(1+|\xi|)^k \tag{5.1}$$

for some $k > 0$. Hence, we may define (3.1) on the space $L^1_k(\mathbf{R}^n)$ of Borel-measurable functions f for which $(1+|\xi|)^k \hat{f} \in L^1$ and with norm given by $\|f\|_{1,k} = \|(1+|\xi|)^k \hat{f}\|_1$. Observe that $S(\mathbf{R}^n) \subset L^1_s(\mathbf{R}^n)$ with a continuous embedding. Using estimate (5.1), it follows that the map $f \mapsto f(A)$ defines an operator-valued tempered distribution. Also from (5.1), we see that the holomorphic extension $\xi \mapsto e^{2\pi i\xi A}$ is an entire function that satisfies estimate (3.12), i.e., the operator-valued distribution $f \mapsto f(A)$ has compact support. Using methods similar to those employed in Proposition 3.1, it can be shown [**1, 11**] that (3.1) satisfies the desired property for polynomials on $\mathbf{R}^n$.

Much work continues in the identification and classification of the generators A for polynomially bounded semigroups. Recent work by Eisner [**5**] characterizes such generators based on an integral criterion for the resolvent $(z - A)^{-1}$ along vertical lines $L \subset \mathbf{C}$, but more concrete examples can be found by a consideration of Fourier multiplier operators [**4**]. Let $m \in L^\infty(\mathbf{R}^n)$ and consider the map $S(\mathbf{R}^n) \mapsto S'(\mathbf{R}^n)$ defined as:

$$f \longmapsto (m\hat{f})\check{}\,. \tag{5.2}$$

As $S(\mathbf{R}^n)$ densely embeds into $L^p(\mathbf{R}^n)$ for any $1 < p < \infty$, the question becomes what conditions are necessary on m to ensure that (5.2) extends to define a continuous linear operator T_m on $L^p(\mathbf{R}^n)$. One result in this direction is given by the Hörmander multiplier theorem: given $k = [n/2] + 1$, if $m \in L^\infty(\mathbf{R}^n) \cap C^k(\mathbf{R}^n \backslash \{0\})$ that satisfies:

$$|D^\alpha m| \leq C\,|x|^{-|\alpha|} \tag{5.3}$$

for all $|\alpha| \leq k$, then T_m defines a continuous operator on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$ and satisfies the norm estimate $\|T_m\| \leq M_p C$.

Consider now the semi-group $e^{2\pi i t T_m}$. The identity $T_m^k = T_{m^k}$ holds for all integers $k \geq 0$; hence, for any polynomial p on $\mathbf{R}^n$, we have $p(T_m) = T_{p(m)}$. From this we derive the result:

$$e^{2\pi i t T_m} = T_{e^{2\pi i t m}} \tag{5.4}$$

using the Taylor expansion for $x \mapsto e^{2\pi i t x}$. We now establish that the function $x \mapsto e^{2\pi i t m(x)}$ satisfies the hypotheses of Hörmander's theorem. To begin, let $f \in C^k(\mathbf{R})$ and $g \in C^k(\mathbf{R}^n \backslash \{0\})$. From the chain rule, given any multi-index $|\alpha| \leq k$ we have the formula:

$$D^\alpha f(g(x)) = \Sigma_{j=1}^{|\alpha|} f^j(g(x)) \Sigma_{\sigma_j} M_{\sigma_j} D^{\sigma_{j,1}} g(x) \cdots D^{\sigma_{j,j}} g(x) \tag{5.5}$$

where σ_j is the set of j-tuples of nonzero multi-indices $(\sigma_{j,1}, \dots, \sigma_{j,j})$ that satisfy $\sigma_{j,1} + \cdots + \sigma_{j,j} = \alpha$. Applying (5.5) with $f(x) = e^{2\pi i t x}$ and $g(x) = m(x)$, then applying estimates (5.3), we derive the new estimates: $|D^\alpha e^{2\pi i t m(x)}| \leq B_\alpha t^{|\alpha|} |x|^{-|\alpha|}$ for each $|\alpha| \leq k$, that is:

$$\left| D^\alpha e^{2\pi i t m(x)} \right| \leq B(1 + |t|)^k\,|x|^{-|\alpha|} \tag{5.6}$$

for all $|\alpha| \leq k$. Thus, formula (5.4) makes sense, and from Hörmander's theorem we have the growth estimate:

$$\left\| e^{2\pi i t T_m} \right\| \leq M_p B(1+|t|)^k; \tag{5.7}$$

hence, the semi-group $e^{2\pi i t T_m}$ has polynomial growth.

5.2. Integral representations. It is prudent to consider how the Weyl calculus fits within the greater context of functional calculi, i.e., with the general problem of defining a map $F \mapsto B$ of some suitable function space F to an operator algebra B that satisfies some desired property, such as continuity, algebraic homomorphism, etc. As can be seen from (1.1) and (1.2), the Weyl correspondence itself is motivated from the Fourier transform representation for smooth, rapidly-decaying functions $f \in S(\mathbf{R}^n)$ and its extensions to $L^1(\mathbf{R}^n)$ or $L^2(\mathbf{R}^n)$; the primary formal challenge is that of making sense of the operator-valued kernel $e^{2\pi i \xi A}$. This same line of reasoning has been used to develop other functional calculi, namely the holomorphic calculus [**14**]: given a Hilbert space X and some $A \in L(X)$, let f be holomorphic in a neighborhood U of the spectrum of A, $\sigma(A)$. Then, from the Cauchy integral representation for f:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma f(\xi)(\xi - z)^{-1}\, d\xi \tag{5.8}$$

where $\Gamma \subset \{U \backslash \sigma(A)\}$ is a continuous curve with winding number 1 relative to z, we may define:

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\xi)(\xi - A)^{-1}\, d\xi \tag{5.9}$$

with a similar expression holding for holomorphic functions of several complex variables. While this elegant formula readily defines a calculus for polynomials p and other entire functions, it is limited in its ability to deal with operators that have an unbounded spectrum, such as the P and Q operators from above. Furthermore, the requirement of analyticity makes the application of this calculus to $S(\mathbf{R}^n)$ and other function spaces of interest difficult, as such functions do not always admit useful holomorphic extensions.

More recent work by Andersson and Sjöstrand [**2**] has focused on a variation of this approach that employs the less-restrictive use of *almost*-holomorphic functions; in particular, they consider functions $f(z) = g(z, \bar{z})$ of a complex variable that satisfy the condition:

$$\left| (\operatorname{Im} z)^k \frac{\partial g}{\partial \bar{z}} \right| \leq C_k \tag{5.10}$$

for all integers k. Observe that this class contains the holomorphic functions, which satisfy $\overline{\partial} f = g_{\bar{z}} = 0$. Such almost-holomorphic functions admit the integral representation [**2**]:

$$f(z) = -\frac{1}{\pi} \int_{\mathbf{C}} \overline{\partial} f(\xi)(\xi - z)^{-1} \, d\xi. \tag{5.11}$$

Using this, [**2**] defines an extension of the holomorphic calculus for smooth functions $f \in C^\infty(\mathbf{R}^n)$ with compact support and operators A_j with real spectrum that satisfy the temperate growth estimate:

$$\left\| (z - A_j)^{-1} \right\| \leq C_{k,j} \left| \operatorname{Im} z \right|^{-N_k} \tag{5.12}$$

where $N_k \geq 0$. Note that such operators include self-adjoint operators, such as the operator $\overline{T}$ of Section 4, cf. Proposition 4.2. Their method is to define, for each $f \in C^\infty(\mathbf{R}^n)$ with compact support, an almost-holomorphic extension $\tilde{f}$ with compact support that satisfies the estimates:

$$\left| (\operatorname{Im} z_j)^k \overline{\partial}_j f \right| \leq C_{k,j} \tag{5.13}$$

in each variable $z_1, \dots, z_n$ separately. From this, they use the temperate growth estimates (5.12) to define the Bochner integral:

(5.14)
$$f(A_1, \dots, A_n) = \left(-\frac{1}{\pi} \right)^n \int_{\mathbf{C}^n} \overline{\partial}_1 \cdots \overline{\partial}_n \tilde{f}(\xi_1, \dots, \xi_n)(\xi - A_1)^{-1} \cdots (\xi - A_n)^{-1} \, d\xi_1 \cdots d\xi_n.$$

Using their explicit form for the extension $\tilde{f}$, they show that for any fixed compact set K, the operators $f(A_1, \dots, A_n)$ thus obtained satisfy the norm estimate:

$$\| f(A_1, \dots, A_n) \| \leq A_K \Sigma_{|\alpha| \leq M_K} \left| D^\alpha f \right|. \tag{5.15}$$

Thus, the functional calculus defined by (5.14) extends to define an operator-valued distribution on the space of test functions $D(\mathbf{R}^n) \subset S(\mathbf{R}^n)$. If the support of this distribution is compact, then we may extend it to define an operator-valued tempered distribution with compact support and analyze its action on $S(\mathbf{R}^n)$ and polynomials in a manner similar to that done with the Weyl calculus in subsection 3.2. However, little is known concerning the support of this distribution outside of the case when the A_i commute. Nevertheless, Andersson and Sjöstrand also define an extension of (5.14) to include those smooth functions f that satisfy an asymptotic growth estimate $f(x) = \Sigma_{k=0}^{N} a_k x^{-k} + x^{-N+1} r_{N+1}(x)$ for $|x| > 1$ and $r_{N+1}(x)$ bounded with all of its derivatives. As this class includes $S(\mathbf{R}^n)$, we can consider the smooth calculus as defining an operator-valued tempered distribution, though not necessarily with compact support.

5.3. The Feynman calculus. We also wish to mention briefly the importance of the Feynman calculus [**9**] in contemporary functional calculus research. The setting of this calculus is rather different than that of the Weyl and holomorphic/smooth calculi, in that we consider not fixed operators A_i but operator-valued functions $A_i(t) : [0, T] \mapsto L(X)$ that satisfy some Bochner integrability condition $A_i \in L_{1,\mu_i}$ relative to a collection of Borel measures μ_i. In its simplest form, given n such functions we map a monomial z^α on $\mathbf{R}^n$ to an element $L(X)$ as follows: let $m = |\alpha|$ and for each $\sigma \in S_m$, define the simplexes $\triangle_\sigma^m = \{t_1, \ldots, t_m \in [0, T]^m : 0 < t_{\sigma_1} < \cdots < t_{\sigma_m} < T\}$. Next, define for each $1 \leq i \leq m$:

$$(5.16) \qquad c(i) = \begin{cases} 1 & \text{if } 1 \leq i \leq \alpha_1, \\ 2 & \text{if } \alpha_1 < i \leq \alpha_1 + \alpha_2, \\ \vdots & \\ k & \text{if } \alpha_1 + \cdots + \alpha_{n-1} < i \leq m. \end{cases}$$

We now make the correspondence [**13**]:

$$(5.17) \quad A^\alpha = \Sigma_\sigma \int_{\triangle_\sigma^m} A_{c(\sigma(1))}(t_{\sigma(1)}) \cdots A_{c(\sigma(m))}(t_{\sigma(m)}) \, d\mu_1^{\alpha_1} \cdots d\mu_n^{\alpha_n}.$$

From this, we may extend the calculus to include those functions f that are holomorphic in the complex disk $|z_i| < r_i$ via their Taylor expansion [**13**].

Note the similarity of (5.17) with definition (3.2) for the monomial $p_\alpha(B)$ of a collection of bounded operators $B = \{B_1, \ldots, B_n\}$; in particular, we see that (5.17) is symmetric in the $A_i(t)$ and for each $\sigma \in S_m$, $c \circ \sigma$ defines a map from $\{1, \ldots, |\alpha|\}$ to $\{1, \ldots, n\}$ that assumes the value i exactly α_i times. A key difference is rather than normalize this sum by the factor $\alpha_1! \cdots \alpha_n!/|\alpha|!$, we lift the redundancy by associating to each σ the simplex $\triangle_\sigma^m$ and integrating over the measures. This adds the flexibility of defining new functional calculi by adjusting the measures μ_i associated to the operator-valued functions $A_i(s)$. For example, if $n = 2$ and the support of the measure μ_1 lies "to the left" of the support of μ_2 on $[0, T]$, then only the integrands of the form $A_2(t)A_1(t)$ will contribute to (5.17).

The question of how the Weyl and Feynman calculi are related continues to be an active area of research [**8, 9**]. We will not go into details of the correspondence here, but we wish to relate a result of this work using a straightforward observation of definition (5.17): assume that the $\mu_i = (1/T)\, dx$ are all probability measures on $[0, T]$ and the $A_i(t) = A_i$ are constant, self-adjoint operators on $L(X)$. Then, by noting that the measure of each simplex $\triangle_\sigma^m$ is $1/|\alpha|!$ and that each unique integrand in 5.17 is repeated $\alpha_1! \cdots \alpha_k!$ times, we reproduce the result of Proposition 3.2 concerning the action of the calculus on polynomials. In the case that the A_i are also bounded, so that (3.1) extends to be defined for holomorphic functions, this result implies that definitions (5.17) and (3.1) are the same. Further results along these lines may be found in [**8**].

REFERENCES

1. R.F.V. Anderson, *The Weyl functional calculus*, J. Functional Anal. **4** (1969), 240–267.

2. M. Andersson and J. Sjöstrand, *Functional calculus for non-commuting operators with real spectra via an iterated cauchy formula*, J. Functional Anal. **210** (2004), 341–375.

3. G. Bachman and L. Narici, *Functional analysis*, Dover, New York, 2000.

4. J. Duoandikoetxea, *Fourier analysis*, Grad. Stud. Math. **29**, American Mathematical Society, 2000.

5. T. Eisner, *Polynomially bounded C_0-semigroups*, Semigroup Forum **70** (2005), 118–126.

6. G.B. Folland, *Harmonic analysis in phase space*, Annals Math. Stud. **122**, Princeton University Press, Princeton, 1989.

7. P.R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1957.

8. B. Jefferies and G.W. Johnson, *Feynman's operational calculi for noncommuting operators*: *Spectral theory*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5** (2002), 171–199.

9. G.W. Johnson and M. Lapidus, *The Feynman integral and Feynman's operational calculus*, The Clarendon Press, Oxford University Press, New York, 2000.

10. T. Kato, *Perturbation theory for linear operators*, Encyclopedia Math. Sci. **132**, Springer-Verlag, New York, 1980.

11. A. McIntosh and A. Pryce, *A functional calculus for several commuting operators*, Indiana University J. **36** (1987), 421–439.

12. E. Nelson, *Operants*: *A functional calculus for non-commuting operators*, in *Functional analysis and related topics*, Springer-Verlag, New York, 1970.

13. L. Nielsen, *Time dependent stability for Feynman's operational calculus*, Rocky Mountain J. Math. **35** (2005), 1347–1368.

14. M. Takesaki, *Theory of operator algebras* I, Encyclopedia Math. Sci. **124**, Springer-Verlag, New York, 2002.

15. M.E. Taylor, *Functions of several self-adjoint operators*, Proc. Amer. Math. Soc. **19** (1968), 91–98.

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