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CONSTRUCTION OF BIHOLOMORPHIC CONVEX MAPPINGS ON D_p IN C^n

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ABSTRACT. In this paper, we first prove some sufficient conditions for the biholomorphic convex mappings on the Reinhardt domain D_p $(p_j \geq 2,\ j = 1, \dots, n)$ in C^n. From these, we construct some concrete examples of biholomorphic convex mappings on the Reinhardt domain D_p $(p_j \geq 2,\ j = 1, \dots, n)$. We also introduce a linear operator and a subclass of biholomorphic convex mappings for the purpose of constructing some biholomorphic convex mappings on D_p in C^n.

Let C^n be the vector space of n-complex variables $z = (z_1, z_2, \dots, z_n)$ with the usual inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w}_j$, where $w = (w_1, w_2, \dots, w_n) \in C^n$. Suppose that D is a domain in C^n. If, for every $z \in D$, $\lambda \in C$ and $|\lambda| \leq 1$, we have $\lambda z \in D$, then we call D a balanced domain. The *Minkowski functional* of a balanced domain D is defined by

$$\rho(z) = \inf\Big\{t > 0, \frac{z}{t} \in D\Big\}, \quad z \in C^n.$$

Suppose that D is a bounded convex balanced domain in C^n and $\rho(z)$ is the Minkowski functional of D. Then $\rho(\bullet)$ is a norm of C^n, and $D = \{z \in C^n : \rho(z) < 1\}$, $\rho(\lambda z) = |\lambda|\rho(z)$, where $\lambda \in C$, $z \in C^n$ and $\rho(z) = 0$ if and only if $z = 0$, see [**15**].

Assume $p_j > 1$, $j = 1, 2, \dots, n$. Let $D_p = \{(z_1, z_2, \dots, z_n) \in C^n : \sum_{j=1}^n |z_j|^{p_j} < 1\}$. Then D_p is a bounded convex balanced domain in

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C^n, and the Minkowski functional $\rho(z)$ of D_p satisfies

$$\sum_{j=1}^{n}\left|\frac{z_j}{\rho(z)}\right|^{p_j}=1. \tag{1}$$

When $p_1=p_2=\cdots=p_n=p$, we denote D_p by B_p^n. At this time,

$$\rho(z)=\sqrt[p]{|z_1|^p+|z_2|^p+\cdots+|z_n|^p}.$$

In particular, let $\Delta=B_p^1$ be the unit disc in the complex plane C.

Let $H(D_p)$ be the class of $f:D_p\to C^n$, which are holomorphic mappings on Reinhardt domain D_p in C^n. The first Fréchet derivative and the second Fréchet derivative of a mapping $f\in H(D_p)$ at point z are denoted by $Df(z)(\cdot)$ and $D^2f(z)(b,\cdot)$, respectively. Their matrix representations are

$$Df(z)=\left(\frac{\partial f_j(z)}{\partial z_k}\right)_{1\le j,k\le n},\quad D^2f(z)(b,\cdot)=\left(\sum_{l=1}^{n}\frac{\partial^2 f_j(z)}{\partial z_k\partial z_l}b_l\right)_{1\le j,k\le n},$$

where $f(z)=(f_1(z),\dots,f_n(z))$ and $b=(b_1,\dots,b_n)\in C^n$. A mapping $f\in H(D_p)$ is said to be locally biholomorphic on D_p if f has a local inverse at each point $z\in D_p$ or, equivalently, if $\det Df(z)\neq 0$ at each point on D_p.

Let $N(D_p)$ denote the class of all locally biholomorphic mappings $f:D_p\to C^n$ such that $f(0)=0, Df(0)=I$, where I is the unit matrix of $n\times n$. If $f\in N(D_p)$ is a biholomorphic mapping on D_p and $f(D_p)$ is a convex domain in C^n, then we say that f is a biholomorphic convex mapping on D_p. The class of all biholomorphic convex mappings on D_p with $f(0)=0, Df(0)=I$, is denoted by $K(D_p)$. In particular, we let $K=K(\Delta)$, $N(\Delta)=N(B_p^1)$. Then

$$N(\Delta)=\{f:\Delta\to C \quad \text{is analytic and} \quad f'(z)\neq 0 \text{ for } z\in\Delta \text{ with } f(0)=f'(0)-1=0\},$$

$$f\in K\Longleftrightarrow f\in N(\Delta)\quad\text{and}\quad \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}>0,\quad \text{for all } z\in\Delta.$$

It is not usually easy to construct concrete biholomorphic convex mappings on some domains in C^n, which is an important content in the study about the geometric function theory of several complex variables, even if on the unit ball B_2^n. Until a few years ago, we only knew a few concrete examples about convex mappings on B_2^n. Roper and Suffridge [**13**] proved that: If $f \in K$ and $F(z_1, z_2) = (f(z_1), \sqrt{f'(z_1)}z_2)$, then $F \in K(B_2^2)$. From this, we may construct a lot of concrete examples about biholomorphic convex mappings on B_2^2. However, its proof is very complex; Graham and Kohr and others [**4**–**9**] gave a simplified proof of the theorem of Roper and Suffridge and studied the other properties of the Roper-Suffridge operator. Roper and Suffridge [**14**] also gave two concrete examples of biholomorphic convex mappings on B_2^2. Moreover, Gong and Liu [**2, 3**] generalized the Roper-Suffridge operator to the Reinhardt domain $D_p = \{z = (z_1, z_2, \dots, z_n) \in C^n : \sum_{j=1}^n |z_j|^{p_j} < 1\}$, where $p_1 = 2$, $p_2 \geq 1$, $p_3 \geq 1, \dots, p_n \geq 1$. Recently, we generalized the Roper-Suffridge operator to Banach spaces in [**10, 18, 19**]. But, according to the result in [**1, 11**], none of these concrete examples belongs to $K(D_p)(p_j > 2, j = 1, 2, \dots, n)$. Hence, at present, the concrete examples about biholomorphic convex mappings on $D_p(p_j > 2, j = 1, 2, \dots, n)$ are scarce. It is rather hard to construct concrete examples of biholomorphic convex mappings on D_p. The purpose of this paper is to prove some sufficient conditions for biholomorphic convex mappings on D_p. From these, we construct some concrete biholomorphic convex mappings on the Reinhardt domain D_p, $p_j \geq 2$, $j = 1, \dots, n$. In order to derive our main results, we need the following lemma, which easily follows from [**16**, Theorem 3 and Corollary 2]. Also, please refer to [**6**, Theorem 6.3.9].

Lemma 1. *Suppose that $p_j \geq 2$, $j = 1, 2, \dots, n$, $\rho(z)$ is the Minkowski functional of D_p, and $f \in N(D_p)$. Then $f \in K(D_p)$ if and only if for any $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that*

$$\operatorname{Re}\left\{\sum_{j=1}^n p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j} \frac{b_j}{z_j}\right\} = 0,$$

we have

$$J_f(z, b) = \operatorname{Re}\left\{\sum_{j=1}^n \frac{{p_j}^2}{2} \bullet \frac{|z_j|^{p_j - 2}}{\rho(z)^{p_j}} |b_j|^2 + \sum_{j=1}^n p_j \left(\frac{p_j}{2} - 1\right) \left|\frac{z_j}{\rho(z)}\right|^{p_j} \left(\frac{b_j}{z_j}\right)^2\right.$$

$$- 2\sum_{j=1}^{n}\frac{p_j}{\rho(z)}\left|\frac{z_j}{\rho(z)}\right|^{p_j}\left\langle Df(z)^{-1}D^2f(z)(b,b),\frac{\partial\rho}{\partial\overline{z}}\right\rangle\bigg\}$$
$$\geq 0.$$

Theorem 1. *Suppose that* $n \geq 2$, $p_j \geq 2$, $j = 1,2,\dots,n$, *and* $g_j : \Delta \to C$ *are analytic on* Δ *with* $g_j(0) = g_j'(0) = 0$, $j = 1,2,\dots,n-1$, $f_j(\zeta) \in N(\Delta)$ *satisfy the conditions* $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$, $\zeta \in \Delta$, $j = 1,2,\dots,n$. *Let*

$$f(z) = (f_1(z_1) + g_1(z_n), f_2(z_2) + g_2(z_n), \dots, f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If, for any $z = (z_1,\dots,z_n) \in D_p \setminus \{0\}$, *the inequality*

$$\sum_{j=1}^{n-1} p_j\left[\left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| + \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right|\left|\frac{f_n''(z_n)}{f_n'(z_n)}\right|\right] \leq p_n\left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right)|z_n|^{p_n-2}$$

holds, then $f \in K(D_p)$.

Proof. By direct computation of the Fréchet derivatives of $f(z)$, we have

$$Df(z) = \begin{pmatrix} f_1'(z_1) & 0 & \cdots & 0 & g_1'(z_n) \\ 0 & f_2'(z_2) & \cdots & 0 & g_2'(z_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-1}'(z_{n-1}) & g_{n-1}'(z_n) \\ 0 & 0 & \cdots & 0 & f_n'(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} 1/(f_1'(z_1)) & 0 & \cdots & 0 & -(g_1'(z_n))/(f_1'(z_1)f_n'(z_n)) \\ 0 & 1/(f_2'(z_2)) & \cdots & 0 & -(g_2'(z_n))/(f_2'(z_2)f_n'(z_n)) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/(f_{n-1}'(z_{n-1})) & -(g_{n-1}'(z_n))/(f_{n-1}'(z_{n-1})f_n'(z_n)) \\ 0 & 0 & \cdots & 0 & 1/(f_n'(z_n)) \end{pmatrix};$$

$$D^2f(z)(b,b) = \begin{pmatrix} f_1''(z_1)b_1 & 0 & \cdots & 0 & g_1''(z_n)b_n \\ 0 & f_2''(z_2)b_2 & \cdots & 0 & g_2''(z_n)b_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-1}''(z_{n-1})b_{n-1} & g_{n-1}''(z_n)b_n \\ 0 & 0 & \cdots & 0 & f_n''(z_n)b_n \end{pmatrix}$$

$$\times \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-1} \\ b_n \end{pmatrix} = \begin{pmatrix} f_1''(z_1)b_1^2 + g_1''(z_n)b_n^2 \\ f_2''(z_2)b_2^2 + g_2''(z_n)b_n^2 \\ \cdots \\ f_{n-1}''(z_{n-1})b_{n-1}^2 + g_{n-1}''(z_n)b_n^2 \\ f_n''(z_n)b_n^2 \end{pmatrix}.$$

According to (1) and $|z_j|^{p_j} = z_j^{p_j/2}\,\overline{z_j}^{p_j/2}$, direct computation yields

$$\frac{\partial\rho}{\partial\overline{z_l}} = \frac{p_l|z_l|^{p_l}}{2\overline{z_l}\rho(z)^{p_l-1}\sum_{j=1}^n p_j\big|z_j/\rho(z)\big|^{p_j}}. \tag{2}$$

Fix $z = (z_1, z_2, \ldots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \ldots, b_n) \in C^n \setminus \{0\}$ such that

$$\mathrm{Re}\left\{\sum_{j=1}^n p_j\left|\frac{z_j}{\rho(z)}\right|^{p_j}\frac{b_j}{z_j}\right\} = 0.$$

From (1), we have $|z_j/\rho(z)| \leq 1$, $j = 1, 2, \ldots, n$, for all $z = (z_1, z_2, \ldots, z_n) \in D_p$. Notice that $p_j \geq 2$, $j = 1, 2, \ldots, n$; by the hypotheses of Theorem 1, we obtain

$$\begin{aligned} J_f(z,b) &\geq \sum_{j=1}^n p_j\frac{z_j|^{p_j-2}}{\rho(z)^{p_j}}|b_j|^2 - 2\sum_{j=1}^n\frac{p_j}{\rho(z)}\left|\frac{z_j}{\rho(z)}\right|^{p_j} \\ &\quad\times \mathrm{Re}\left\langle Df(z)^{-1}D^2f(z)(b,b), \frac{\partial\rho}{\partial\overline{z}}\right\rangle \\ &= \sum_{j=1}^n p_j\frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}|b_j|^2 - \mathrm{Re}\left\{\sum_{j=1}^{n-1}\left[\frac{f_j''(z_j)}{f_j'(z_j)}b_j^2 + \frac{g_j''(z_n)}{f_j'(z_j)}b_n^2\right.\right. \\ &\qquad\left.\left. - \frac{g_j'(z_n)f_n''(z_n)b_n^2}{f_j'(z_j)f_n'(z_n)}\right]\frac{p_j|z_j|^{p_j}}{z_j\rho(z)^{p_j}} + \frac{f_n''(z_n)}{f_n'(z_n)}\frac{p_n|z_n|^{p_n}}{z_n\rho(z)^{p_n}}b_n^2\right\} \\ &\geq \sum_{j=1}^{n-1}|b_j|^2 p_j\frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}\left[1 - \left|\frac{z_jf_j''(z_j)}{f_j'(z_j)}\right|\right] \end{aligned}$$

$$
\begin{aligned}
&+ |b_n|^2 \Bigg[\left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right) \frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} \\
&- \sum_{j=1}^{n-1} \left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| \frac{p_j}{|z_j|} \cdot \left|\frac{z_j}{\rho(z)}\right|^{p_j} \\
&- \sum_{j=1}^{n-1} \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right| \left|\frac{f_n''(z_n)}{f_n'(z_n)}\right| \frac{p_j}{|z_j|} \cdot \left|\frac{z_j}{\rho(z)}\right|^{p_j} \Bigg] \\
= &\sum_{j=1}^{n-1} |b_j|^2 p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} \left[1 - \left|\frac{z_j f_j''(z_j)}{f_j'(z_j)}\right|\right] \\
&+ |b_n|^2 \Bigg[\left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right) \frac{p_n |z_n|^{p_n-2}}{\rho(z)^{p_n}} \\
&- \sum_{j=1}^{n-1} \left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| \frac{p_j}{\rho(z)} \cdot \left|\frac{z_j}{\rho(z)}\right|^{p_j-1} \\
&- \sum_{j=1}^{n-1} \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right| \left|\frac{f_n''(z_n)}{f_n'(z_n)}\right| \frac{p_j}{\rho(z)} \cdot \left|\frac{z_j}{\rho(z)}\right|^{p_j-1} \Bigg] \\
\geq &\frac{|b_n|^2}{\rho(z)^{p_n}} \Bigg\{ p_n \left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right) |z_n|^{p_n-2} \\
&\qquad - \sum_{j=1}^{n-1} p_j \left[\left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| + \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right| \left|\frac{f_n''(z_n)}{f_n'(z_n)}\right| \right] \cdot \rho(z)^{p_n-1} \Bigg\}.
\end{aligned}
$$

Since $p_n \geq 2$ and $0 < \rho(z) < 1$, for every $z \in D_p \setminus \{0\}$, then we have that $\rho(z)^{p_n-1} < 1$ for all $z \in D_p$, so

$$
\begin{aligned}
J_f(z,b) \geq \frac{|b_n|^2}{\rho(z)^{p_n}} \Bigg\{ &p_n \left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right) |z_n|^{p_n-2} \\
&- \sum_{j=1}^{n-1} p_j \left[\left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| + \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right| \left|\frac{f_n''(z_n)}{f_n'(z_n)}\right| \right] \Bigg\} \geq 0.
\end{aligned}
$$

Hence, by Lemma 1, we obtain $f \in K(D_p)$, and the proof is complete. □

Setting $g_j(\zeta) \equiv 0$, $j = 1, 2, \ldots, n-1$, in Theorem 1, we have the following corollary.

Corollary 1. *Suppose that* $p_j \geq 2$, $j = 1, 2, \ldots, n$, *and* $f_j(\zeta) \in N(\Delta)$ *with* $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$, $\zeta \in \Delta$, $j = 1, 2, \ldots, n$. *Let*

$$f(z) = (f_1(z_1), f_2(z_2), \ldots, f_{n-1}(z_{n-1}), f_n(z_n)).$$

Then $f \in K(D_p)$.

Remark 1. Setting $n = 2$, $p_j \equiv p$, $j = 1, 2$, in Corollary 1, we get Theorem 4.1 in [**12**]. Setting $p_j \equiv p$, $j = 1, 2, \ldots, n$, in Corollary 1, we get Theorem 4 in [**17**].

Corollary 2. *Suppose that* $n \geq 2$, $p_j \geq 2$, $j = 1, 2, \ldots, n$. *Let*

$$HK = \left\{ f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ is analytic in } \Delta, \text{ and } \sum_{k=2}^{\infty} k^2 |a_k| \leq 1 \right\}.$$

If $f_j \in HK$, $j = 1, 2, \ldots, n$, *then*

$$f(z) = (f_1(z_1), f_2(z_2), \ldots, f_n(z_n)) \in K(D_p).$$

Proof. Assume $g \in HK$; then we have $g(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and

$$\begin{aligned} 0 \leq |zg''(z)| &= \left| \sum_{k=2}^{\infty} k(k-1) a_k z^{k-1} \right| \\ &\leq \sum_{k=2}^{\infty} k(k-1) |a_k| |z|^{k-1} \\ &< \sum_{k=2}^{\infty} k^2 |a_k| - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} \\ &\leq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} \leq |g'(z)| \end{aligned}$$

for all $z \in \Delta$. Hence, by Corollary 1, we obtain $f(z) \in K(D_p)$, and the proof is complete. □

Example 1. Suppose that $p_j \geq 2$, $0 < |\lambda_j| \leq 1$, $f_j(z_j) = (e^{\lambda_j z_j} - 1)/\lambda_j \in N(\Delta)$, $j = 1, \dots, n$; then

$$0 \leq |z_j f_j''(z_j)| = |\lambda_j z_j f_j'(z_j)| < |f_j'(z_j)|, \quad z_j \in \Delta;$$

hence, by Corollary 1, we obtain

$$f(z) = \left(\frac{e^{\lambda_1 z_1} - 1}{\lambda_1}, \frac{e^{\lambda_2 z_2} - 1}{\lambda_2}, \dots, \frac{e^{\lambda_n z_n} - 1}{\lambda_n} \right) \in K(D_p).$$

Example 2. Suppose that $n \geq 2$, $p_j \geq 2$, $j = 1, 2, \dots, n$, $0 < |\lambda| \leq 1$, and k is a positive integer such that $k < p_n \leq k + 1$. Let

$$f(z) = \Big(z_1 + a_1' z_1^2 + a_1 z_n^{k+1} + b_1 z_n^{k+2}, \dots ,$$
$$z_{n-1} + a_{n-1}' z_{n-1}^2 + a_{n-1} z_n^{k+1} + b_{n-1} z_n^{k+2}, \frac{e^{\lambda z_n} - 1}{\lambda} \Big),$$

where $c = \max\{|a_j'| : j = 1, 2, \dots, n-1\}$. If $c \leq 1/4$, and

$$\frac{k+1}{1-2c} \sum_{j=1}^{n-1} p_j [k|a_j| + (k+2)|b_j|]$$
$$+ \frac{|\lambda|}{1-2c} \sum_{j=1}^{n-1} p_j [(k+1)|a_j| + (k+2)|b_j|] \leq p_n(1 - |\lambda|),$$

then $f(z) \in K(D_p)$.

Proof. Set $f_n(z_n) = (e^{\lambda z_n} - 1)/\lambda$, $f_j(\xi) = \xi + a_j' \xi^2$, $g_j(\xi) = a_j \xi^{k+1} + b_j \xi^{k+2}$, $j = 1, 2, \dots, n-1$. Then

$$f_n'(z_n) = e^{\lambda z_n}, \quad f_n''(z_n) = \lambda f_n'(z_n), \quad f_j'(\xi) = 1 + 2a_j' \xi, \quad f_j''(\xi) = 2a_j',$$

$$g_j'(\xi) = (k+1) a_j \xi^k + (k+2) b_j \xi^{k+1},$$
$$g_j''(\xi) = (k+1) k a_j \xi^{k-1} + (k+2)(k+1) b_j \xi^k;$$

thus, it follows from $c = \max\{|a'_j| : j = 1, 2, \dots, n-1\} \le 1/4$ that $|f'_j(z_j)| \ge 1 - 2c > 0$, $j = 1, 2, \dots, n$. Simple computation yields

$$|z_j f''_j(z_j)| = 2|a'_j||z_j| < 1 - 2|a'_j||z_j| \le |f'_j(z_j)|, \quad j = 1, 2, \dots, n-1,$$

and

$$\left|\frac{z_n f''_n(z_n)}{f'_n(z_n)}\right| = |\lambda||z_n| < |\lambda| \le 1.$$

According to the hypotheses, therefore, we have

$$\begin{aligned}
\sum_{j=1}^{n-1} p_j \left[\left|\frac{g''_j(z_n)}{f'_j(z_j)}\right| + \left|\frac{g'_j(z_n)}{f'_j(z_j)}\right|\left|\frac{f''_n(z_n)}{f'_n(z_n)}\right|\right] \\
&\le \frac{k+1}{1-2c} \sum_{j=1}^{n-1} p_j [k|a_j| + (k+2)|b_j|]|z_n|^{k-1} \\
&\quad + \frac{|\lambda|}{1-2c} \sum_{j=1}^{n-1} p_j [(k+1)|a_j| + (k+2)|b_j|]|z_n|^{k} \\
&\le p_n(1-|\lambda|)|z_n|^{k-1} = p_n(1-|\lambda|)|z_n|^{p_n-2}|z_n|^{k+1-p_n} \\
&\le p_n\left(1 - \left|\frac{z_n f''_n(z_n)}{f'_n(z_n)}\right|\right)|z_n|^{p_n-2}.
\end{aligned}$$

Hence, by Theorem 1, we obtain $f \in K(D_p)$, and the proof is complete. □

Example 3. Suppose that $n \ge 2$, $p_j \ge 2$, $j = 1, 2, \dots, n$, and k is a positive integer such that $k < p_n \le k+1$. Let

$$\begin{aligned}
f(z) = (&z_1 + a'_1 z_1^2 + a_1 z_n^{k+1} + b_1 z_n^{k+2}, \dots, \\
&z_{n-1} + a'_{n-1} z_{n-1}^2 + a_{n-1} z_n^{k+1} + b_{n-1} z_n^{k+2}, z_n + a'_n z_n^2),
\end{aligned}$$

where $c = \max\{|a'_j| : j = 1, 2, \dots, n\}$. If $c \le 1/4$, and

$$\frac{k+1}{1-2c} \sum_{j=1}^{n-1} p_j [k|a_j| + (k+2)|b_j|] + \frac{2c}{(1-2c)^2} \sum_{j=1}^{n-1} p_j [(k+1)|a_j| + (k+2)|b_j|] \le \frac{p_n(1-4c)}{1-2c},$$

then $f(z) \in K(D_p)$.

Proof. Set $f_j(\xi) = \xi + a_j'\xi^2$, $j = 1, \dots, n$, $g_j(\xi) = a_j\xi^{k+1} + b_j\xi^{k+2}$, $j = 1, 2, \dots, n-1$. Then

$$f_j'(\xi) = 1 + 2a_j'\xi, \quad f_j''(\xi) = 2a_j',$$
$$g_j'(\xi) = (k+1)a_j\xi^k + (k+2)b_j\xi^{k+1},$$
$$g_j''(\xi) = (k+1)ka_j\xi^{k-1} + (k+2)(k+1)b_j\xi^k;$$

thus, it follows from $c = \max\{|a_j'| : j = 1, 2, \dots, n\} \leq 1/4$ that for any $|z_j| < 1$, $j = 1, 2, \dots, n$, we have

$$|f_j'(z_j)| \geq 1 - 2c > 0.$$

Simple computation yields

$$|z_j f_j''(z_j)| = 2|a_j'||z_j| < 1 - 2|a_j'||z_j| \leq |f_j'(z_j)|,$$

and

$$\left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right| \leq \frac{2c|z_n|}{1-2c} \leq \frac{2c}{1-2c} \leq 1.$$

According to the hypotheses, therefore, we have

$$\begin{aligned}
&\sum_{j=1}^{n-1} p_j\left[\left|\frac{g_j''(z_n)}{f_j'(z_j)}\right| + \left|\frac{g_j'(z_n)}{f_j'(z_j)}\right|\left|\frac{f_n''(z_n)}{f_n'(z_n)}\right|\right] \\
&\qquad \leq \frac{k+1}{1-2c}\sum_{j=1}^{n-1} p_j[k|a_j| + (k+2)|b_j|]|z_n|^{k-1} \\
&\qquad\quad + \frac{2c}{(1-2c)^2}\sum_{j=1}^{n-1} p_j[(k+1)|a_j| + (k+2)|b_j|]|z_n|^k \\
&\qquad \leq \frac{p_n(1-4c)}{1-2c}|z_n|^{k-1} \\
&\qquad = \frac{p_n(1-4c)}{1-2c}|z_n|^{p_n-2}|z_n|^{k+1-p_n} \\
&\qquad \leq p_n\left(1 - \frac{2c}{1-2c}\right)|z_n|^{p_n-2} \\
&\qquad \leq p_n\left(1 - \left|\frac{z_n f_n''(z_n)}{f_n'(z_n)}\right|\right)|z_n|^{p_n-2}.
\end{aligned}$$

Hence, by Theorem 1, we obtain $f \in K(D_p)$, and the proof is complete. □

Example 4. Suppose that $n \geq 2$, $p_j \geq 2$, $0 < |\lambda_j| < 1$, $j = 2, \dots, n$, and k is a positive integer such that $k < p_n \leq k+1$. Let

$$f(z) = \left(z_1 + a z_n^{k+1}, \frac{e^{\lambda_2 z_2} - 1}{\lambda_2}, \dots, \frac{e^{\lambda_n z_n} - 1}{\lambda_n} \right).$$

If $|a| \leq (p_n(1 - |\lambda_n|))/(p_1(k+1)(k + |\lambda_n|))$, then $f(z) \in K(D_p)$.

Setting $p_j \equiv p$, $j = 1, 2, \dots, n$, in Theorem 1, we have the following corollary.

Corollary 3. *Suppose that $n \geq 2$, $p \geq 2$, and $g_j : \Delta \to C$ are analytic on Δ with $g_j(0) = g_j'(0) = 0$, $j = 1, 2, \dots, n-1$, $f_j(\zeta) \in N(\Delta)$ satisfy the conditions $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$, $\zeta \in \Delta$, $j = 1, 2, \dots, n$. Let*

$$f(z) = (f_1(z_1) + g_1(z_n), f_2(z_2) + g_2(z_n), \dots, \\ f_{n-1}(z_{n-1}) + g_{n-1}(z_n), f_n(z_n)).$$

If, for any $z = (z_1, \dots, z_n) \in B_p^n \setminus \{0\}$, we have

$$\sum_{j=1}^{n-1} \left[\left| \frac{g_j''(z_n)}{f_j'(z_j)} \right| + \left| \frac{g_j'(z_n)}{f_j'(z_j)} \right| \left| \frac{f_n''(z_n)}{f_n'(z_n)} \right| \right] \leq \left(1 - \left| \frac{z_n f_n''(z_n)}{f_n'(z_n)} \right| \right) |z_n|^{p-2},$$

then $f \in K(B_p^n)$.

Theorem 2. *Suppose that $n \geq 2$, $p_j \geq 2$, $j = 1, 2, \dots, n$, and $f_j \in N(\Delta)$, $j = 2, 3, \dots, n$, $f_1(z_1, \dots, z_n) : D_p \to C$ is holomorphic with $f_1(0, 0, \dots, 0) = 0, \partial f_1/\partial z_1(0, 0, \dots, 0) = 1$. Let*

$$f(z) = (f_1(z_1, z_2, \dots, z_n), f_2(z_2), \dots, f_n(z_n)),$$

where $z = (z_1, z_2, \dots, z_n)$. If, for any $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, we have

$$\frac{\partial f_1}{\partial z_1} \neq 0, \quad \sum_{j=1}^{n} \left| z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_j} \right| \leq \left| \frac{\partial f_1}{\partial z_1} \right|,$$

$$\left|\frac{[(\partial f_1)/(\partial z_j)]\cdot[(f_j''(z_j))/(f_j'(z_j))]}{(\partial f_1)/(\partial z_1)}\right| + \sum_{l=1}^{n}\left|\frac{[(\partial^2 f_1)/(\partial z_j \partial z_l)]}{(\partial f_1)/(\partial z_1)}\right| \le \frac{p_j}{p_1}\left(1-\left|\frac{z_j f_j''(z_j)}{f_j'(z_j)}\right|\right)\cdot |z_j|^{p_j-2}, \quad j=2,\dots,n;$$

then $f \in K(D_p)$.

Proof. By direct computation of the Fréchet derivatives of $f(z)$, we have

$$Df(z) = \begin{pmatrix} \partial f_1/\partial z_1 & \partial f_1/\partial z_2 & \cdots & \partial f_1/\partial z_{n-1} & \partial f_1/\partial z_n \\ 0 & f_2'(z_2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-1}'(z_{n-1}) & 0 \\ 0 & 0 & \cdots & 0 & f_n'(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} 1/[(\partial f_1)/(\partial z_1)] & -[(\partial f_1/\partial z_2)]/[(\partial f_1/\partial z_1) f_2'(z_2)] & \cdots & -[(\partial f_1/\partial z_{n-1})]/[(\partial f_1/\partial z_1) f_{n-1}'(z_{n-1})] & -[(\partial f_1/\partial z_n)]/[(\partial f_1/\partial z_1) f_n'(z_n)] \\ 0 & 1/(f_2'(z_2)) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/(f_{n-1}'(z_{n-1})) & 0 \\ 0 & 0 & \cdots & 0 & 1/f_n'(z_n) \end{pmatrix},$$

$$D^2 f(z)(b,b) = \begin{pmatrix} \sum_{l=1}^n (\partial^2 f_1/\partial z_1 \partial z_l) b_l & \sum_{l=1}^n (\partial^2 f_1/\partial z_2 \partial z_l) b_l & \cdots & \sum_{l=1}^n (\partial^2 f_1/\partial z_{n-1} \partial z_l) b_l & \sum_{l=1}^n (\partial^2 f_1/\partial z_n \partial z_l) b_l \\ 0 & f_2''(z_2) b_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{n-1}''(z_{n-1}) b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & f_n''(z_n) b_n \end{pmatrix}$$

$$\times \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{n-1} \\ b_n \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n \sum_{l=1}^n (\partial^2 f_1/\partial z_k \partial z_l) b_k b_l \\ f_2''(z_2) b_2^2 \\ \cdots \\ f_{n-1}''(z_{n-1}) b_{n-1}^2 \\ f_n''(z_n) b_n^2 \end{pmatrix}.$$

Fix $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re}\left\{ \sum_{j=1}^n p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0.$$

From (1), we have $|z_1/\rho(z)| \le 1$. Since $p_j \ge 2$, $j = 1, 2, \dots, n$, and $0 < \rho(z) < 1$ for any $z \in D_p \setminus \{0\}$, then we have $\rho(z)^{p_j - 1} < 1$ for all $z \in D_p$. So, by (2) and the hypotheses of Theorem 2, we obtain

$$\begin{aligned} J_f(z,b) &\ge \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 \\ &\quad - 2 \sum_{j=1}^n \frac{p_j}{\rho(z)} \left| \frac{z_j}{\rho(z)} \right|^{p_j} \operatorname{Re} \left\langle Df(z)^{-1} D^2 f(z)(b,b), \frac{\partial \rho}{\partial \overline{z}} \right\rangle \\ &= \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 \\ &\quad - \operatorname{Re}\left\{ \frac{1}{\partial f_1/\partial z_1} \left[\sum_{k=1}^n \sum_{l=1}^n \frac{\partial^2 f_1}{\partial z_k \partial z_l} b_k b_l \right.\right. \\ &\qquad \left. - \sum_{k=2}^n \frac{\partial f_1}{\partial z_k} \frac{f_k''(z_k)}{f_k'(z_k)} b_k^2 \right] p_1 \frac{|z_1|^{p_1}}{z_1 \rho(z)^{p_1}} \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{j=2}^{n} \frac{f_j''(z_j)}{f_j'(z_j)} b_j^2 p_j \frac{|z_j|^{p_j}}{z_j \rho(z)^{p_j}} \Bigg\} \\
&\geq \sum_{j=1}^{n} p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \sum_{j=2}^{n} \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 p_j \frac{|z_j|^{p_j-1}}{\rho(z)^{p_j}} \\
&\quad - \frac{1}{|\partial f_1/\partial z_1|} \Bigg[\sum_{k=1}^{n} \sum_{l=1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k \partial z_l} \right| |b_k||b_l| \\
&\quad + \sum_{k=2}^{n} \left| \frac{\partial f_1}{\partial z_k} \right| \left| \frac{f_k''(z_k)}{f_k'(z_k)} \right| |b_k|^2 \Bigg] p_1 \frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}} \\
&\geq \sum_{j=1}^{n} p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \sum_{j=2}^{n} \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 p_j \frac{|z_j|^{p_j-1}}{\rho(z)^{p_j}} \\
&\quad - \frac{1}{|\partial f_1/\partial z_1|} \Bigg[\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k \partial z_l} \right| \cdot 2|b_k||b_l| \\
&\quad + \sum_{k=1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k^2} \right| |b_k|^2 + \sum_{k=2}^{n} \left| \frac{\partial f_1}{\partial z_k} \right| \left| \frac{f_k''(z_k)}{f_k'(z_k)} \right| |b_k|^2 \Bigg] p_1 \frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}} \\
&\geq \sum_{j=1}^{n} p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} |b_j|^2 - \sum_{j=2}^{n} \left| \frac{f_j''(z_j)}{f_j'(z_j)} \right| |b_j|^2 p_j \frac{|z_j|^{p_j-1}}{\rho(z)^{p_j}} \\
&\quad - \frac{1}{|\partial f_1/\partial z_1|} \Bigg[\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k \partial z_l} \right| (|b_k|^2 + |b_l|^2) \\
&\qquad + \sum_{k=1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k^2} \right| |b_k|^2 \\
&\qquad + \sum_{k=2}^{n} \left| \frac{\partial f_1}{\partial z_k} \right| \left| \frac{f_k''(z_k)}{f_k'(z_k)} \right| |b_k|^2 \Bigg] p_1 \frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}} \\
&= p_1 \frac{|z_1|^{p_1-2}}{\rho(z)^{p_1}} |b_1|^2 + \sum_{j=2}^{n} |b_j|^2 p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}} \left[1 - \left| \frac{z_j f_j''(z_j)}{f_j'(z_j)} \right| \right] \\
&\quad - \frac{1}{|\partial f_1/\partial z_1|} \Bigg[\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \left| \frac{\partial^2 f_1}{\partial z_k \partial z_l} \right| |b_k|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=2}^{n}\sum_{k=1}^{l-1}\left|\frac{\partial^2 f_1}{\partial z_k \partial z_l}\right||b_l|^2\\
&+\sum_{k=1}^{n}\left|\frac{\partial^2 f_1}{\partial z_k^2}\right||b_k|^2+\sum_{k=2}^{n}\left|\frac{\partial f_1}{\partial z_k}\right|\left|\frac{f_k''(z_k)}{f_k'(z_k)}\right||b_k|^2\Bigg]p_1\frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}}\\
\geq{}& p_1\frac{|z_1|^{p_1-2}}{\rho(z)^{p_1}}|b_1|^2\left[1-\sum_{j=1}^{n}\left|\frac{z_1\frac{\partial^2 f_1}{\partial z_1\partial z_j}}{\partial f_1/\partial z_1}\right|\right]\\
&+\sum_{j=2}^{n}|b_j|^2\Bigg\{p_j\frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}\left[1-\left|\frac{z_jf_j''(z_j)}{f_j'(z_j)}\right|\right]\\
&-p_1\frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}}\left|\frac{(\partial f_1/\partial z_j)\cdot(f_j''(z_j)/f_j'(z_j))}{\partial f_1/\partial z_1}\right|\\
&-p_1\frac{|z_1|^{p_1-1}}{\rho(z)^{p_1}}\sum_{k=1}^{n}\left|\frac{(\partial^2 f_1)/(\partial z_j\partial z_k)}{\partial f_1/\partial z_1}\right|\Bigg\}\\
\geq{}&\sum_{j=2}^{n}\frac{p_1|b_j|^2}{\rho(z)^{p_j}}\Bigg\{\frac{p_j}{p_1}\cdot|z_j|^{p_j-2}\left[1-\left|\frac{z_jf_j''(z_j)}{f_j'(z_j)}\right|\right]\\
&\qquad-\left|\frac{z_1}{\rho(z)}\right|^{p_1-1}\rho(z)^{p_j-1}\\
&\qquad\cdot\Bigg[\left|\frac{(\partial f_1/\partial z_j)\cdot(f_j''(z_j)/f_j'(z_j))}{\partial f_1/\partial z_1}\right|\\
&\qquad+\sum_{k=1}^{n}\left|\frac{\partial^2 f_1/(\partial z_j\partial z_k)}{\partial f_1/\partial z_1}\right|\Bigg]\Bigg\}\\
\geq{}&\sum_{j=2}^{n}\frac{p_1|b_j|^2}{\rho(z)^{p_j}}\Bigg\{\frac{p_j}{p_1}\left[1-\left|\frac{z_jf_j''(z_j)}{f_j'(z_j)}\right|\right]\cdot|z_j|^{p_j-2}\\
&\qquad-\Bigg[\left|\frac{(\partial f_1/\partial z_j)\cdot(f_j''(z_j))/(f_j'(z_j))}{\partial f_1/\partial z_1}\right|\\
&\qquad+\sum_{k=1}^{n}\left|\frac{\partial^2 f_1/(\partial z_j\partial z_k)}{\partial f_1/\partial z_1}\right|\Bigg]\Bigg\}\geq 0.
\end{aligned}
$$

Hence, by Lemma 1, we obtain $f\in K(D_p)$, and the proof is complete. □

Corollary 4. *Suppose that* $n \geq 2$, $p_j \geq 2$, $j = 1, 2, \dots, n$, *and let*

$$f(z_1, z_2, \dots, z_n) = (f_1(z_1, z_2, \dots, z_n), z_2, \dots, z_n),$$

where $f_1(z_1, \dots, z_n) : D_p \to C$ *is holomorphic with* $f_1(0, 0, \dots, 0) = 0$, $\partial f_1/\partial z_1(0, 0, \dots, 0) = 1$. *If for any* $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, *we have*

$$\frac{\partial f_1}{\partial z_1} \neq 0, \quad \sum_{l=1}^{n} \left| \frac{\partial^2 f_1}{\partial z_1 \partial z_l} \right| \leq \left| \frac{\partial f_1}{\partial z_1} \right|, \quad \sum_{l=1}^{n} \left| \frac{\partial^2 f_1}{\partial z_j \partial z_l} \right| \leq \frac{p_j}{p_1} \cdot \left| \frac{\partial f_1}{\partial z_1} \right| |z_j|^{p_j - 2},$$
$$j = 2, \cdots, n,$$

then $f \in K(D_p)$.

Corollary 5. *Suppose that* $n \geq 2$, $p_j \geq 2$, $j = 1, 2, \dots, n$, *and* $f_j \in N(\Delta)$ *with* $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$, $j = 2, 3, \dots, n$, $\zeta \in \Delta$, $f_1(z_1, z_2) : \Delta \times \Delta \to C$ *is holomorphic with* $f_1(0, 0) = 0$, $\partial f_1/\partial z_1(0, 0) = 1$. *Let*

$$f(z) = (f_1(z_1, z_2), f_2(z_2), \dots, f_n(z_n)),$$

where $z = (z_1, z_2, \dots, z_n)$. *If, for any* $z = (z_1, \dots, z_n) \in D_p \setminus \{0\}$, *we have*

$$\frac{\partial f_1}{\partial z_1} \neq 0, \quad \sum_{j=1}^{2} |z_1 \frac{\partial^2 f_1}{\partial z_1 \partial z_j}| \leq \left| \frac{\partial f_1}{\partial z_1} \right|,$$
$$\left| \frac{\partial f_1}{\partial z_2} \right| \cdot \left| \frac{f_2''(z_2)}{f_2'(z_2)} \right| + \sum_{l=1}^{2} \left| \frac{\partial^2 f_1}{\partial z_2 \partial z_l} \right| \leq \frac{p_2}{p_1} \left| \frac{\partial f_1}{\partial z_1} \right| \left(1 - \left| \frac{z_2 f_2''(z_2)}{f_2'(z_2)} \right| \right) \cdot |z_2|^{p_2 - 2},$$

then $f \in K(D_p)$.

Example 5. Suppose that $n \geq 2$, $p_j \geq p_1 \geq 2$, $j = 2, \dots, n$, and k_1, k_j are integers such that $k_1 \geq 0$, $k_j < p_j \leq k_j + 1$, $j = 2, 3, \dots, n$. Let

$$f(z) = \Big(z_1 + \frac{a_1}{k_1 + 2} z_1^{k_1 + 2} + a_2 z_1 z_2^{k_2 + 1} + a_3 z_1 z_3^{k_3 + 1} + \cdots$$
$$+ a_n z_1 z_n^{k_n + 1}, \quad z_2, \dots, z_n \Big).$$

If $\sum_{j=1}^{n} [(k_j + 1)^2 + 1] |a_j| \leq 1$, then $f(z) \in K(D_p)$.

Example 6. Suppose that $p_j \geq p_1 \geq 2$, $j = 2, \dots, n$, and k_1, k_j are positive integers such that $k_1 \geq 1$, $k_j < p_j \leq k_j + 1$, $j = 2, 3, \dots, n$, $k = \max_{1 \leq j \leq n}\{k_j\}$. If

$$|a| \leq \left[(k+1)\sum_{j=1}^{n}(k_j+1)\right]^{-1},$$

then

$$f(z) = \left(z_1 + az_1^{k_1+1}z_2^{k_2+1}\cdots z_n^{k_n+1},\ z_2, \dots, z_n\right) \in K(D_p).$$

Proof. Set $f_1(z) = z_1 + az_1^{k_1+1}z_2^{k_2+1}\cdots z_n^{k_n+1}$. Then, for $j = 1, 2, \dots, n$, we have

$$\begin{aligned}\sum_{l=1}^{n}\left|\frac{\partial^2 f_1}{\partial z_1 \partial z_l}\right| &\leq (k_1+1)|a|\left[\sum_{l=2}^{n}(k_l+1)|z_1|^{k_1} + k_1|z_1|^{k_1-1}\right] \\ &\leq (k+1)|a|\left[\sum_{l=1}^{n}(k_l+1) - 1\right]|z_1|^{k_1-1} \\ &\leq [1-(k+1)|a|],\end{aligned}$$

$$\begin{aligned}\sum_{l=1}^{n}\left|\frac{\partial^2 f_1}{\partial z_j \partial z_l}\right| &\leq (k_j+1)|a|\left[\sum_{l\neq j}(k_l+1)|z_j|^{k_j} + k_j|z_j|^{k_j-1}\right] \\ &\leq (k+1)|a|\left[\sum_{l=1}^{n}(k_l+1) - 1\right]|z_j|^{k_j-1} \\ &\leq [1-(k+1)|a|]|z_j|^{p_j-2}|z_j|^{k_j+1-p_j} \\ &\leq [1-(k+1)|a|]|z_j|^{p_j-2}, \quad j = 2, 3, \dots, n.\end{aligned}$$

Since $|\partial f_1/\partial z_1| > 1 - (k_1+1)|a| \geq 1 - (k+1)|a|$ and $p_j \geq p_1 \geq 2$, $j = 2, \dots, n$, by Corollary 4, we obtain that $f(z) = (z_1 + az_1^{k_1+1}z_2^{k_2+1}\cdots z_n^{k_n+1}, z_2, \dots, z_n) \in K(D_p)$, and the proof is complete. □

By applying Corollary 5, we may prove the following two examples.

Example 7. Suppose that $p_j \geq 2$, $j = 1, 2, \dots, n$, $0 < |\lambda_2| < p_2/(p_1 + p_2)$, $0 < |\lambda_j| \leq 1$, $j = 3, \dots, n$, and k_1, k_2 are positive integers such that $k_1 \geq 1$, $k_2 < p_2 \leq k_2 + 1$. If

$$|a| \leq \min \left\{ \frac{1}{(k_1+1)(k_1+k_2+2)}, \frac{p_2 - (p_1+p_2)|\lambda_2|}{p_1(k_2+1)(k_1+k_2+|\lambda_2|+1) + p_2(k_1+1)(1-|\lambda_2|)} \right\},$$

then

$$f(z) = \left(z_1 + a z_1^{k_1+1} z_2^{k_2+1}, \ \frac{e^{\lambda_2 z_2} - 1}{\lambda_2}, \dots, \frac{e^{\lambda_n z_n} - 1}{\lambda_n} \right) \in K(D_p).$$

Example 8. Suppose that $p_j \geq 2$, $j = 1, 2, \dots, n$, and k_1, k_2 are positive integers such that $k_1 \geq 1$, $k_2 < p_2 \leq k_2 + 1$. If $f_j \in N(\Delta)$ with $|\zeta f_j''(\zeta)| \leq |f_j'(\zeta)|$, $j = 3, \dots, n$, $\zeta \in \Delta$, and

$$|a| \leq \min \left\{ \frac{1}{(k_1+1)(k_1+k_2+2)}, \frac{p_2}{p_1(k_2+1)(k_1+k_2+1) + p_2(k_1+1)} \right\},$$

then

$$f(z) = \left(z_1 + a z_1^{k_1+1} z_2^{k_2+1}, z_2, f_3(z_3), \dots, f_n(z_n) \right) \in K(D_p).$$

Remark 2. Suppose that $g_1, g_2, \dots, g_n \in N(\Delta)$. Let

$$\Phi_n(g_1, g_2, \dots, g_n)(z) = (g_1(z_1), g_2(z_2), \dots, g_n(z_n)).$$

In general, if $g_1(z_1), g_2(z_2), \dots, g_n(z_n) \in K(\Delta)$, we cannot conclude that

$$\Phi_n(g_1, g_2, \dots, g_n)(z) = (g_1(z_1), g_2(z_2), \dots, g_n(z_n)) \in K(D_p).$$

For instance, the mapping $f(z) = (z_1/(1-z_1), z_2, \dots, z_n) \notin K(D_p)$.

In fact, suppose that $f(z) = (z_1/(1-z_1), z_2, \dots, z_n)$ is a biholomorphic convex mapping on D_p; taking $z_1(r) = (r, 0, 0, \dots, 0), z_2(r) = (0, r, 0, \dots, 0) \in D_p$ with $0 < r < 1$, we have

$$\frac{1}{2}[f(z_1(r)) + f(z_2(r))] \in f(D_p).$$

This leads to

$$f^{-1}\left(\frac{1}{2}[f(z_1(r)) + f(z_2(r))]\right) \in D_p. \tag{3}$$

On the other hand, direct computation yields

$$f^{-1}\left(\frac{1}{2}[f(z_1(r)) + f(z_2(r))]\right) = \left(\frac{r}{r+2(1-r)}, \frac{r}{2}, 0, \dots, 0\right),$$

and

$$\left(\frac{r}{r+2(1-r)}\right)^{p_1} + \left(\frac{r}{2}\right)^{p_2} \longrightarrow 1 + \frac{1}{2^{p_2}} > 1 \quad (r \to 1^-),$$

that is,

$$f^{-1}\left(\frac{1}{2}[f(z_1(r)) + f(z_2(r))]\right) \notin D_p \quad (r \to 1^-)$$

which contradicts (3). Hence, $f(z) = (z_1/(1-z_1), z_2, \dots, z_n)$ is not a convex mapping on D_p.

Let $SK(D_p)$ be the subclass of $N(D_p)$ consisting of all mappings $f(z)$ which satisfy the following conditions: for any $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re}\left\{\sum_{j=1}^{n} p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j} \frac{b_j}{z_j}\right\} = 0, \tag{4}$$

we have

$$|w_j| \le |b_j|^2, j = 1, 2, \dots, n, \tag{5}$$

where $Df(z)^{-1}D^2f(z)(b,b) = (w_1, \dots, w_n)$, $f \in N(D_p)$.

It is evident that

$$f \in SK(\Delta) \Longleftrightarrow f \in N(\Delta) \quad \text{and} \quad \left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in \Delta,$$

and $SK(\Delta) \subset K(\Delta)$.

Theorem 3. *Suppose that $p_j \geq 2$, $j = 1, 2, \dots, n$. Then $SK(D_p) \subset K(D_p)$.*

Proof. Fix $f \in SK(D_p)$. For any $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that (4) holds, we have

$$\begin{aligned}
J_f(z,b) &\geq \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}|b_j|^2 \\
&\quad - 2\sum_{j=1}^n \frac{p_j}{\rho(z)}\left|\frac{z_j}{\rho(z)}\right|^{p_j} \operatorname{Re}\left\langle Df(z)^{-1}D^2f(z)(b,b), \frac{\partial\rho}{\partial\overline{z}}\right\rangle \\
&= \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}|b_j|^2 - \operatorname{Re}\left\{\sum_{j=1}^n w_j p_j \frac{|z_j|^{p_j}}{z_j\rho(z)^{p_j}}\right\} \\
&\geq \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}|b_j|^2 - \sum_{j=1}^n |w_j| p_j \frac{|z_j|^{p_j-1}}{\rho(z)^{p_j}} \\
&= \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}[|b_j|^2 - |z_j||w_j|] \\
&\geq \sum_{j=1}^n p_j \frac{|z_j|^{p_j-2}}{\rho(z)^{p_j}}[|b_j|^2 - |w_j|] \geq 0,
\end{aligned}$$

so by Lemma 1, we obtain that $f \in K(D_p)$. Hence, $SK(D_p) \subset K(D_p)$, and the proof is complete. □

Remark 3. From Theorem 3, we have $SK(D_p) \subset K(D_p)$, $p_j \geq 2$, $j = 1, 2, \dots, n$, but $SK(D_p) \neq K(D_p)$. In fact, we let k be a positive integer such that $k < p_n \le k+1$ and

$$f(z) = \left(z_1 + \frac{p_n}{k(k+1)p_1} z_n^{k+1}, z_2, \dots, z_n \right).$$

By Example 3, we have $f(z) \in K(D_p)$.

On the other hand, let $b = (0, \dots, 0, 1)$, $z = (r^2, 0, \dots, 0, \sqrt[p_n]{1 - r^{p_1}}\, i)$, $0 < r < 1$. Then $z \in D_p \setminus \{0\}$ satisfies condition (4). Set $Df(z)^{-1} D^2 f(z)(b, b) = (w_1, w_2, \dots, w_n)$. Simple computation yields

$$|w_1| = \frac{p_n}{p_1} |z_n|^{k-1} > 0 = |b_1|^2,$$

where $z_n = \sqrt[p_n]{1 - r^{p_1}}\, i$. Hence, $f(z) \notin SK(D_p)$.

Theorem 4. *Suppose that $p_j \ge 2$, $j = 1, 2, \dots, n$. Then $\Phi_n(g_1, g_2, \dots, g_n) \in SK(D_p)$ if and only if $g_j \in SK(\Delta)$, $j = 1, 2, \dots, n$.*

Proof. Let $f = \Phi_n(g_1, g_2, \dots, g_n) = (g_1(z_1), g_2(z_2), \dots, g_n(z_n))$. By direct computation of the Fréchet derivatives of $f(z)$, we obtain

$$Df(z) = \begin{pmatrix} g_1'(z_1) & 0 & \cdots & 0 \\ 0 & g_2'(z_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_n'(z_n) \end{pmatrix},$$

$$Df(z)^{-1} = \begin{pmatrix} 1/(g_1'(z_1)) & 0 & \cdots & 0 \\ 0 & 1/(g_2'(z_2)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/(g_n'(z_n)) \end{pmatrix},$$

$$D^2 f(z)(b, b) = \begin{pmatrix} g_1''(z_1)b_1 & 0 & \cdots & 0 \\ 0 & g_2''(z_2)b_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g_n''(z_n)b_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$$
$$= \begin{pmatrix} g_1''(z_1)b_1^2 \\ g_2''(z_2)b_2^2 \\ \cdots \\ g_n''(z_n)b_n^2 \end{pmatrix},$$

so

$$Df(z)^{-1}D^2f(z)(b,b) = \begin{pmatrix} (g_1''(z_1)/g_1'(z_1))b_1^2 \\ (g_2''(z_2)/g_2'(z_2))b_2^2 \\ \cdots \\ (g_n''(z_n)/g_n'(z_n))b_n^2 \end{pmatrix}. \tag{6}$$

We first prove that if $g_j \in SK(\Delta)$, $j = 1, 2, \dots, n$, then $f = \Phi_n(g_1, g_2, \dots, g_n) \in SK(D_p)$.

Since $g_j \in SK(\Delta)$, $j = 1, 2, \dots, n$, it is evident that $f = \Phi_n(g_1, g_2, \dots, g_n) \in N(D_p)$.

Set $Df(z)^{-1}D^2f(z)(b,b) = (w_1, \dots, w_n)$. For any $z = (z_1, z_2, \dots, z_n) \in D_p \setminus \{0\}$ and $b = (b_1, b_2, \dots, b_n) \in C^n \setminus \{0\}$ such that

$$\operatorname{Re}\left\{ \sum_{j=1}^{n} p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = 0.$$

From (6), we have

$$|w_j| = \left| \frac{g_j''(z_j)}{g_j'(z_j)} \right| |b_j|^2 \le |b_j|^2, \quad j = 1, 2, \dots, n.$$

Hence, $f = \Phi_n(g_1, g_2, \dots, g_n) \in SK(D_p)$.

Conversely, assume $f = \Phi_n(g_1, g_2, \dots, g_n) \in SK(D_p)$; it is obvious that $g_j \in N(\Delta)$.

Fix k, $1 \le k \le n$. Taking $z_k \in \Delta \setminus \{0\}$, we have $z = (0, \dots, 0, z_k, 0, \dots, 0) \in D_p \setminus \{0\}$, $b = (0, \dots, 0, iz_k, 0, \dots, 0) \in C^n \setminus \{0\}$, and

$$\operatorname{Re}\left\{ \sum_{j=1}^{n} p_j \left| \frac{z_j}{\rho(z)} \right|^{p_j} \frac{b_j}{z_j} \right\} = \operatorname{Re}\left\{ p_k \left| \frac{z_k}{\rho(z)} \right|^{p_k} i \right\} = 0.$$

By (5) and (6), we have

$$|w_k| = \left| \frac{g_k''(z_k)}{g_k'(z_k)} \right| |z_k|^2 \le |z_k|^2;$$

thus,

$$\left|\frac{g_k''(z_k)}{g_k'(z_k)}\right| \le 1, \quad 0 < |z_k| < 1.$$

According to the continuity of $g_k''(z_k)/g_k'(z_k)$, we have

$$\left|\frac{g_k''(z_k)}{g_k'(z_k)}\right| \le 1, \quad |z_k| < 1;$$

hence, $g_k \in SK(\Delta)$, $1 \le k \le n$. This completes the proof. □

Example 9. Suppose that $p_j \ge 2$, $j = 1, 2, \dots, n$. Let

$$f(z) = \left(\frac{z_1}{1 - a_1 z_1}, \ \frac{z_2}{1 - a_2 z_2}, \dots, \frac{z_n}{1 - a_n z_n}\right).$$

Then

$$f(z) \in SK(D_p) \subset K(D_p) \Longleftrightarrow |a_j| \le \frac{1}{3}, \quad j = 1, 2, \dots, n.$$

Proof. We first suppose that $|a_j| \le 1/3$, $j = 1, 2, \dots, n$. Set $f_j(z_j) = z_j/(1 - a_j z_j)$, $z_j \in \Delta$, $j = 1, 2, \dots, n$; then $f_j(z_j)$ are analytic in Δ, and

$$f_j'(z_j) = \frac{1}{(1 - a_j z_j)^2} \ne 0, \quad f_j''(z_j) = \frac{2a_j}{(1 - a_j z_j)^3}.$$

This leads to $f_j \in N(\Delta)$, and

$$\left|\frac{z_j f_j''(z_j)}{f_j'(z_j)}\right| = \left|\frac{2a_j z_j}{1 - a_j z_j}\right| \le \frac{2|a_j||z_j|}{1 - |a_j||z_j|} \le \frac{2|a_j|}{1 - |a_j|} \le 1,$$

for $j = 1, 2, \dots, n$. So $f_j \in SK(\Delta)$, $j = 1, 2, \dots, n$. Hence, by Theorem 4 and Theorem 3, we obtain that $f(z) \in SK(D_p) \subset K(D_p)$.

Conversely, we suppose that $f(z) \in SK(D_p) \subset K(D_p)$. Then by Theorem 4, we have $f_j \in SK(\Delta)$, $j = 1, 2, \dots, n$.

Fix j, $1 \le j \le n$; the function $f_j(z_j)$ is analytic on Δ, and

$$\left|\frac{z_j f_j''(z_j)}{f_j'(z_j)}\right| = \left|\frac{2a_j z_j}{1 - a_j z_j}\right| \le 1,$$

for all $z_j \in \Delta$. So $|a_j| \le 1$. If $a_j = 0$, it is obvious that $|a_j| < 1/3$. In the following, we assume that $a_j \ne 0$.

Let $a_j = r_j e^{i\alpha_j}$ with $r_j = |a_j|$ and α_j a real number. Taking $z_j = R_j e^{-i\alpha_j} \in \Delta$, we have

$$\left|\frac{z_j f_j''(z_j)}{f_j'(z_j)}\right| = \left|\frac{2a_j z_j}{1 - a_j z_j}\right| = \frac{2r_j R_j}{1 - r_j R_j} \le 1;$$

this leads to $r_j R_j \le 1/3$. Setting $R_j \to 1^-$, we obtain $|a_j| = r_j \le 1/3$, and the proof is complete. □

Example 10. Suppose that $p_j \ge 2$, $j = 1, 2, \dots, n$, $|\lambda_j| > 0$, $j = 2, \dots, n$, and let

$$f(z) = \left(\frac{z_1}{1 - a_1 z_1}, \frac{e^{\lambda_2 z_2} - 1}{\lambda_2}, \dots, \frac{e^{\lambda_n z_n} - 1}{\lambda_n}\right).$$

Then

$$f(z) \in SK(D_p) \subset K(D_p) \Longleftrightarrow |a_1| \le \frac{1}{3}, \text{ and } |\lambda_j| \le 1, \quad j = 2, \dots, n.$$

Example 11. Suppose that $p_j \ge 2$, $|a_j| > 0$, $j = 1, 2, \dots, n$, and let

$$f(z) = \left(\frac{1}{2a_1}\log\frac{1 + a_1 z_1}{1 - a_1 z_1}, \frac{1}{2a_2}\log\frac{1 + a_2 z_2}{1 - a_2 z_2}, \dots, \frac{1}{2a_n}\log\frac{1 + a_n z_n}{1 - a_n z_n}\right),$$

with $\log 1 = 0$. Then

$$f(z) \in SK(D_p) \subset K(D_p) \Longleftrightarrow |a_j| \le \frac{1}{\sqrt{3}}, \quad j = 1, 2, \dots, n.$$

Example 12. Suppose that $p_j \ge 2$, $|a_j| > 0$, $j = 2, \dots, n$, and let

$$f(z) = \left(\frac{z_1}{1 - a_1 z_1}, \frac{1}{2a_2}\log\frac{1 + a_2 z_2}{1 - a_2 z_2}, \dots, \frac{1}{2a_n}\log\frac{1 + a_n z_n}{1 - a_n z_n}\right),$$

with $\log 1 = 0$. Then

$$f(z) \in SK(D_p) \subset K(D_p) \Longleftrightarrow |a_1| \leq \frac{1}{3}$$

and

$$|a_j| \leq \frac{1}{\sqrt{3}}, \quad j = 2, 3, \dots, n.$$

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