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# HYPOELLIPTIC CONVOLUTION EQUATIONS IN THE SPACE OF DISTRIBUTIONS ON NONCOMPACT SEMI-SIMPLE LIE GROUPS

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ABSTRACT. We characterize by simple proofs the hypoelliptic convolution equations in the space of distributions on noncompact semi-simple Lie groups, in terms of their spherical transform.

**1. Introduction.** In this paper we recall first the main results of harmonic analysis on noncompact semi-simple Lie groups $G$ of real rank $\ell$, and next we define the generalized translation operators on $G$ and we give their properties. With the aid of these operators we define and study in this work the convolution product on spaces of distributions. We present also the spherical transform of distributions. The results obtained have permitted to characterize by simple proofs the hypoelliptic convolution equations in the space of distributions in terms of their spherical transform. This characterization was first given by Ehrenpreis [**3**] and next by Hörmander [**7**] in the case of the classical Fourier transform on $\mathbf{R}^\ell$. In [**1, 2**] the authors have studied this characterization for the Hankel, Jacobi and Chébli-Trimèche transforms. We remark that their proofs are complicated and our proofs can be applied in the case of these transforms (see [**9**]).

**2. Preliminaries**. In this section we recall some basic results on real semi-simple Lie groups. (See [**4, 5, 6**]).

**2.1. Structure of real semi-simple Lie groups.** Let $G$ be a noncompact connected real semi-simple Lie group with finite center, $\mathcal{G}$ the Lie algebra of $G$. Let $\theta$ be a Cartan involution of $\mathcal{G}, \mathcal{G} = \mathcal{K} + \mathcal{P}$ the

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corresponding Cartan decomposition and $K$ the analytic subgroup of $G$ with Lie algebra $\mathcal{K}$. Let $\boldsymbol{\alpha} \subset \mathcal{P}$ be a maximal abelian subspace, $\boldsymbol{\alpha}^*$ its (real) dual, $\boldsymbol{\alpha}^*_{\mathbf{C}}$ the complexification.

The Killing form of $\mathcal{G}$ induces a scalar product on $\boldsymbol{\alpha}$ and hence on $\boldsymbol{\alpha}^*$. We denote by $\langle .,. \rangle$ its $\mathbf{C}$-bilinear extension to $\boldsymbol{\alpha}^*_{\mathbf{C}}$.

The $\ell = \dim \boldsymbol{\alpha}$ is called the real rank of $G$. Let $e_1, e_2, \dots, e_\ell$ be an orthonormal basis of $\boldsymbol{\alpha}$ and $e_1^*, e_2^*, \dots, e_\ell^*$ the dual basis of $\boldsymbol{\alpha}^*_{\mathbf{C}}$. Then every $\lambda$ in $\boldsymbol{\alpha}^*_{\mathbf{C}}$ is uniquely written in the form

$$\lambda = z_1 e_1^* + z_2 e_2^* + \cdots + z_\ell e_\ell^*, \quad z_j \in \mathbf{C}, \quad j = 1, 2, \dots, \ell.$$

Using the basis $e_1, e_2, \dots, e_\ell$ we can identify $\boldsymbol{\alpha}$ with $\mathbf{R}^\ell$.

For $\lambda \in \alpha^*$, put $\mathcal{G}_\lambda = \{X \in \mathcal{G} / [H, X] = \lambda(H)X, \text{ for all } H \in \boldsymbol{\alpha}\}$. If $\lambda \neq 0$ and $\dim \mathcal{G}_\lambda \neq 0$, then $\lambda$ is called a (restricted) root and $m_\lambda = \dim \mathcal{G}_\lambda$ is called its multiplicity. The set of restricted roots will be denoted by $\sum$. If $\lambda, \mu$ are in $\boldsymbol{\alpha}^*$, let $H_\lambda$ in $\boldsymbol{\alpha}^*$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for $H \in \boldsymbol{\alpha}$, and put $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$. Let $W$ be the Weyl group associated with $\sum$ and $|W|$ its cardinality.

Fix a Weyl chamber $\boldsymbol{\alpha}^+$ in $\boldsymbol{\alpha}$, and let $\overline{\boldsymbol{\alpha}^+}$ be its closure. We call a root positive if it is positive on $\boldsymbol{\alpha}^+$. The corresponding Weyl chamber in $\boldsymbol{\alpha}^*$ will be denoted by $\boldsymbol{\alpha}^*_+$, and let $\overline{\boldsymbol{\alpha}^*_+}$ be its closure. Let $\sum^+$ be the set of positive roots. Put $\rho = 1/2 \sum_{\alpha \in \sum^+} m_\alpha \alpha$. Let $\sum_0 = \{\alpha \in \sum, 1/2\alpha \notin \sum\}$, and put $\sum_0^+ = \sum^+ \cap \sum_0$. Let $\mathcal{N} = \oplus_{\alpha \in \sum^+} \mathcal{G}_\alpha$ and $\overline{\mathcal{N}} = \theta \mathcal{N}$.

Let $A$ be the analytic subgroup of $G$ with Lie algebra $\boldsymbol{\alpha}$. The exponential map is an isomorphism from $\boldsymbol{\alpha}$ (considered as an abelian Lie group) onto $A$. We put $A^+ = \exp \boldsymbol{\alpha}^+$. Its closure in $G$ is $\overline{A^+} = \exp \overline{\boldsymbol{\alpha}^+}$. Let $N$, respectively $\overline{N}$, be the analytic subgroup of $G$ with Lie algebra $\mathcal{N}$, respectively $\overline{\mathcal{N}}$.

Let $x^+$ be in $\overline{\boldsymbol{\alpha}^+}$-component of $x \in G$ in the Cartan decomposition $G = K(\exp \overline{\boldsymbol{\alpha}^+})K$, and let $|x| = \|x^+\|$. Viewed on $G/K$ (or $K \backslash G$), $|.|$ is the distance to the origin $0 = \{K\}$.

Let $H : G \to \boldsymbol{\alpha}$ be the Iwasawan projection according to the Iwasawa decomposition $G = KAN$, i.e., if $x \in G$ then $H(x)$ is the unique element in $\boldsymbol{\alpha}$ such that $x \in K(\exp H(x))N$.

We normalize the Lebesgue measures $dH$ and $d\lambda$ on $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ such that for the classical Fourier transform

$$\mathcal{F}_0(f)(\lambda) = \int_A f(a)e^{-i\lambda(\log a)}\,da, \quad \lambda \in \boldsymbol{\alpha}^*, \tag{2.1}$$

of a regular function on $A$, we have the inversion formula

$$\mathcal{F}_0^{-1}(h)(a) = \int_{\boldsymbol{\alpha}^*} h(\lambda)e^{i\lambda(\log a)}\,d\lambda, \quad a \in A; \tag{2.2}$$

here $h$ is a regular function on $\boldsymbol{\alpha}^*$.

On the compact group $K$ the Haar measure $dk$ is normalized such that its total mass is 1. The Haar measure of nilpotent groups $N$ and $\overline{N}$ are normalized such that $\theta(dn) = d\overline{n}$ and

$$\int_{\overline{N}} e^{-2\rho(H(\overline{n}))}\,d\overline{n} = 1.$$

In the Iwasawa decomposition, the Haar measure $dx$ of $G$ is given by

$$\int_G f(x)\,dx = \int_K \int_A \int_N f(kan)\,dk\,da\,dn. \tag{2.3}$$

This relation can also be written in the form

$$\int_G f(x)\,dx = \int_K \int_{\boldsymbol{\alpha}} \int_N f(k(\exp H)n)e^{2\rho(H)}\,dk\,dH\,dn; \tag{2.4}$$

here $f$ is a $C^\infty$-function on $G$ with compact support.

In the Cartan decomposition the Haar measure $dx$ of $G$ is given by

$$\int_G f(x)\,dx = \int_K \int_{A^+} \int_K f(k_1\,ak_2)\,dk_1\,da\,dk_2\,. \tag{2.5}$$

We can also write this relation in the form

$$\int_G f(x)\,dx = \int_K \int_{\boldsymbol{\alpha}^+} \int_K f(k_1(\exp H)k_2)w(H)\,dk_1\,dH\,dk_2, \tag{2.6}$$

where

$$w(H) = \prod_{\alpha \in \sum^+} [2sh\alpha(H)]^{m_\alpha}. \tag{2.7}$$

We have the following estimate for the density $w$ :

$$0 \leq w(H) \leq e^{2\rho(H)}, \quad H \in \overline{\boldsymbol{\alpha}^+}. \tag{2.8}$$

*Remark* 2.1. If $G$ has rank one, then for some $\alpha$ in $\boldsymbol{\alpha}^*$, $\sum$ is equal to $\{\alpha, -\alpha\}$ or $\{\alpha, -\alpha, 2\alpha, -2\alpha\}$. Let $H_1$ be in $\boldsymbol{\alpha}$ such that $\alpha(H_1) = 1$, and write $\mathcal{G}_{\pm 1}, \mathcal{G}_{\pm 2}$ instead of $\mathcal{G}_{\pm\alpha}, \mathcal{G}_{\pm 2\alpha}$ with dimension $m_{\pm 1}, m_{\pm 2}$, respectively. Choose the ordering on $\boldsymbol{\alpha}^*$ such that $\alpha$ is positive. Then $\rho = (m_1 + 2m_2)/2$.

For $f$ a $C^\infty$-function on $G$ bi-invariant under $K$ and with compact support the Haar measure on $G$ is given by

$$\int_G f(x)\, dx = \int_0^\infty f(\exp tH_1) A_{p,q}(t)\, dt, \tag{2.9}$$

with

$$A_{p,q}(t) = 2^{2\rho} (sht)^{2p+1} (cht)^{2q+1}, \tag{2.10}$$

and

$$p = \frac{1}{2}(m_1 + m_2 - 1), \quad q = \frac{1}{2}(m_2 - 1), \quad \rho = p + q + 1. \tag{2.11}$$

(See [**8**, pages 14–16, 27]).

**2.2. The Harish-Chandra's $c$-function.** The Harish-Chandra's $c$-function is given by the formula

(2.12)

$$c(\lambda) = c_0 \prod_{\alpha \in \sum_0^+} \frac{\Gamma(\langle i\lambda, \alpha_0\rangle/2)\Gamma(\langle i\lambda, \alpha_0\rangle/2 + 1/2)}{2\sqrt{\pi}\Gamma(m_\alpha/4 + 1/2 + \langle i\lambda, \alpha_0\rangle/2)\Gamma(m_\alpha/4 + m_{2\alpha}/2 + \langle i\lambda, \alpha_0\rangle/2)},$$

where $\alpha_0 = \alpha/\langle \alpha, \alpha\rangle$, the constant $c_0$ is defined by $c(i\rho) = 1$ and $\Gamma$ is the gamma function.

The function $c$ satisfies the following properties

i) $c(-\lambda) = \overline{c(\lambda)}$, $\lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$.

ii) The function $|c(\lambda)|^{-2}$ is analytic and $W$-invariant on $\boldsymbol{\alpha}^*$.

iii) The function $|c(\lambda)|^{-2}$ possesses the estimate

$$|c(\lambda)|^{-2} \leq \text{const}\,(1 + \|\lambda\|)^b, \quad \lambda \in \boldsymbol{\alpha}^*, \tag{2.13}$$

for some positive constant $b$.

**2.3. Spherical functions.** The spherical functions on $G$ are defined by

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}\,dk, \quad x \in G, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}. \tag{2.14}$$

We collect some properties of these functions.

i) The function $\varphi_\lambda(x)$ is bi-invariant under $K$ in $x \in G$ and $W$-invariant in $\lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$.

ii) The function $\varphi_\lambda(x)$ is a $C^\infty$-function in $x$ and a holomorphic function in $\lambda$.

iii) We have

(2.15)
- $\varphi_\lambda(e) = 1 \quad \text{and} \quad \varphi_{-i\rho}(x) = 1, \quad x \in G, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$
- $\varphi_\lambda(x) = \varphi_{-\lambda}(x^{-1}), \varphi_{-\bar\lambda}(x) = \overline{\varphi_\lambda(x)}, \quad x \in G, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$
- $\varphi_\lambda \equiv \varphi_{\lambda'}, \quad$ if and only if $\quad \lambda' = w\lambda \quad$ for some $\quad w \in W$.

iv) We have

$$e^{-\rho(H)} \leq \varphi_0(\exp H) \leq \text{const}\,(1 + \|H\|)^a e^{-\rho(H)}, \quad H \in \overline{\boldsymbol{\alpha}^+}, \tag{2.16}$$

for some positive constant $a$.

v) We have

(2.17)
- $0 < \varphi_{-i\lambda}(\exp H) \leq e^{\lambda(H)}\varphi_0(\exp H), \quad H \in \overline{\boldsymbol{\alpha}^+}, \quad \lambda \in \overline{\boldsymbol{\alpha}^*_+}$.

(2.18)
- $|\varphi_\lambda(x)| \leq \varphi_{i\mathrm{Im}\,\lambda}(x), \quad x \in G, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$.

(2.19)
- $|\varphi_\lambda(x)| \leq 1, \quad x \in G, \quad \lambda \in \boldsymbol{\alpha}^*$.

vi) For all $\lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$, the function $\varphi_\lambda$ is an eigenfunction of each operator $D$ in $\mathbf{D}(G/K)$ the algebra of $G$-invariant differential operators on $G/K$. More precisely, we have

$$D\varphi_\lambda = \gamma(D)(i\lambda)\varphi_\lambda, \quad D \in \mathbf{D}(G/K), \tag{2.20}$$

where $\gamma : D \to \gamma(D)$ is the natural isomorphism between $\mathbf{D}(G/K)$ and $\mathcal{S}(\boldsymbol{\alpha}^*)^W$ the algebra of complex valued polynomial functions on $\boldsymbol{\alpha}^*$ which are $W$-invariant. It is called the Harish-Chandra isomorphism. In particular, when $D$ is the Laplace-Beltrami operator $\Delta$, we have for all $\lambda \in \boldsymbol{\alpha}^*$,

$$\Delta\varphi_\lambda = -(\|\lambda\|^2 + \|\rho\|^2)\varphi_\lambda. \tag{2.21}$$

vii) The function $\varphi_\lambda(x)$ admits the following product formula with respect to the variable $x$:

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_K \varphi_\lambda(xky)\,dk, \quad x, y \in G, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}. \tag{2.22}$$

This relation can also be written in the form

$$\varphi_\lambda(b)\varphi_\lambda(c) = \int_{A^+} \varphi_\lambda(a)\,d\mu_{b,c}(a), \quad b, c \in A^+, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}, \tag{2.23}$$

where $\mu_{b,c}$ is a positive $W$-invariant measure on $A$ with compact support.

viii) The function $\varphi_\lambda(x)$ also satisfies the following product formula with respect to the variable $\lambda$:

$$\varphi_\mu(x)\varphi_\nu(x) = \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} \varphi_\lambda(x)K(\mu,\nu,\lambda)\frac{d\lambda}{|c(\lambda)|^2}, \quad \mu,\nu \in \boldsymbol{\alpha}^*, \quad x \in G, \tag{2.24}$$

where $K(\mu,\nu,\lambda)$ is a positive function on $\boldsymbol{\alpha}^* \times \boldsymbol{\alpha}^* \times \boldsymbol{\alpha}^*$, which is analytic and $W$-invariant in all three variables. It is given by the relation

$$K(\mu,\nu,\lambda) = \int_G \varphi_\mu(x)\varphi_\nu(x)\varphi_{-\lambda}(x)\,dx. \tag{2.25}$$

and satisfies

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} K(\mu,\nu,\lambda)\,\frac{d\lambda}{|c(\lambda)|^2} = 1. \tag{2.26}$$

*Remark* 2.2. When $G$ has rank one, it follows from [**8**, page 27], that the set of all spherical functions for $(G, K)$

$$\varphi_\lambda(a_t) = \varphi_\lambda^{(p,q)}(t), \quad a_t \in A, \quad t \in \mathbf{R}, \tag{2.27}$$

where

$$p = \frac{1}{2}(m_1 + m_2 - 1), \qquad q = \frac{1}{2}(m_2 - 1), \tag{2.28}$$

and $\varphi_\lambda^{(p,q)}$ are the Jacobi functions defined by

$$\varphi_\lambda^{(p,q)}(t) = {}_2F_1((p+q+1-i\lambda)/2, (p+q+1+i\lambda)/2; \\ p+1; -sh^2 t), \tag{2.29}$$

with ${}_2F_1$ the Gaussian hypergeometric function.

**3. The Abel transform and its dual.** In this section we consider the Abel transform $\mathcal{A}$ and its dual $\mathcal{A}^*$ on spaces of functions, and we give their properties (see [**4, 5, 6**]). Next we define and study the Abel transform on spaces of distributions.

**Notations.** We denote by $C^\natural(G)$, respectively $C(A)^W$, the space of continuous functions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant. $C_c^\natural(G)$, respectively $C_c(A)^W$, is the space of continuous functions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant, and with compact support. $\mathcal{E}^\natural(G)$, respectively $\mathcal{E}(A)^W$, is the space of $C^\infty$-functions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant. $\mathcal{D}^\natural(G)$, respectively $\mathcal{D}(A)^W$, is the space of $C^\infty$-functions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant, and with compact support. We provide these spaces with their classical topology.

We consider also the following spaces of distributions. $\mathcal{D}'_\natural(G)$, respectively $\mathcal{D}'_W(A)$, is the space of distributions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant. It is the topological dual of $\mathcal{D}^\natural(G)$, respectively $\mathcal{D}(A)^W$. $\mathcal{E}'_\natural(G)$, respectively $\mathcal{E}'_W(A)$, is the space of distributions on $G$, respectively $A$, which are $K$ bi-invariant, respectively $W$-invariant. It is the topological dual of $\mathcal{E}^\natural(G)$, respectively $\mathcal{E}(A)^W$.

**Definition 3.1.** The Abel transform $\mathcal{A}$ is defined on $C_c^\natural(G)$ by

$$\mathcal{A}(f)(a) = e^{\rho(\log a)} \int_N f(an)\, dn, \quad \text{for all } a \in A. \tag{3.1}$$

The function $\mathcal{A}(f)$ belongs to $C_c(A)^W$ and satisfies the properties given by the following theorem.

**Theorem 3.1.** i) *The transform $\mathcal{A}$ is a topological isomorphism from $\mathcal{D}^\natural(G)$ onto $\mathcal{D}(A)^W$. More precisely, $f$ has support in the closed ball $\{x \in G/|x| \leq R\}$ if and only if $\mathcal{A}(f)$ has support in the closed ball $\{a \in A/|a| \leq R\}$.*

ii) *The transform $\mathcal{A}$ has the transmutation property*

$$\mathcal{A}(Df) = \gamma(D)\mathcal{A}(f), \quad f \in \mathcal{D}^\natural(G), \tag{3.2}$$

*for each $D \in \mathbf{D}(G/K)$.*

**Definition 3.2.** The dual Abel transform $\mathcal{A}^*$ is defined on $C(A)^W$ by

$$\mathcal{A}^*(g)(x) = \int_K g(\exp H(xk)) e^{-\rho(H(xk))}\, dk, \quad \text{for all } x \in G. \tag{3.3}$$

The function $\mathcal{A}^*(g)$ belongs to $C^\natural(G)$ and satisfies the following properties.

i) We have

$$\mathcal{A}^*(e^{i\lambda(\cdot)}) = \varphi_\lambda, \quad \lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}. \tag{3.4}$$

ii) For all $x \in G$, we have

$$\mathcal{A}^*(1)(x) = \int_K e^{-\rho(H(xk))}\,dk \leq 1. \tag{3.5}$$

**Theorem 3.2.** i) *The transform $\mathcal{A}^*$ is a topological isomorphism from $\mathcal{E}(A)^W$ onto $\mathcal{E}^\natural(G)$.*

ii) *The transform $\mathcal{A}^*$ has the transmutation property*

$$D\mathcal{A}^*(g) = \mathcal{A}^*(\gamma(D)g), \quad g \in \mathcal{E}(a)^W, \tag{3.6}$$

*for each $D \in \mathbf{D}(G/K)$.*

iii) *The transform $\mathcal{A}^*$ is connected to the transform $\mathcal{A}$ by the duality relation*

$$\int_A \mathcal{A}(f)(a)g(a)\,da = \int_G f(x)\mathcal{A}^*(g)(x)\,dx, \tag{3.7}$$

*where $f$ is in $\mathcal{D}^\natural(G)$ and $g$ in $\mathcal{E}(A)^W$.*

*Remark* 3.1. When $G$ has rank one, explicit formulas for the Abel transform and its dual are given respectively as Weyl and Riemann-Liouville fractional integrals. (See [**8**]).

**Definition 3.3.** The Abel transform on $\mathcal{E}'_\natural(G)$ denoted also by $\mathcal{A}$ is defined by

$$\langle \mathcal{A}(S), \psi \rangle = \langle S, \mathcal{A}^*(\psi) \rangle, \quad \psi \in \mathcal{E}(A)^W. \tag{3.8}$$

The mapping $\mathcal{A}$ possesses the properties given by the following theorem.

**Theorem 3.3.** i) *The transform $\mathcal{A}$ is a topological isomorphism from $\mathcal{E}'_\natural(G)$ onto $\mathcal{E}'_W(A)$. Its inverse is given by*

$$\langle \mathcal{A}^{-1}(S), \psi \rangle = \langle S, (\mathcal{A}^*)^{-1}(\psi) \rangle, \quad \psi \in \mathcal{E}^\natural(G). \tag{3.9}$$

ii) *Let* $T_f$ *be the distribution in* $\mathcal{E}'_\natural(G)$ *given by the function* $f$, *with* $f$ *in* $\mathcal{D}^\natural(G)$. *Then we have*

$$\mathcal{A}(T_f) = T_{\mathcal{A}(f)}. \tag{3.10}$$

iii) *Let* $T_h$ *be the distribution in* $\mathcal{E}'_W(A)$ *given by the function* $h$ *in* $\mathcal{D}(A)^W$. *Then we have*

$$\mathcal{A}^{-1}(T_h) = T_{\mathcal{A}^{-1}(h)}. \tag{3.11}$$

**4. The spherical transform.** In this section we define the spherical transform of functions and distributions, and we give the main results satisfied by this transform.

**Notations.** We denote by $\mathbf{H}(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$ the space of entire functions on $\boldsymbol{\alpha}^*_{\mathbf{C}}$, which are $W$-invariant, of exponential type and rapidly decreasing. We have

$$\mathbf{H}(\boldsymbol{\alpha}^*_\delta)^W = \cup_{R\geq 0}\mathbf{H}_R(\boldsymbol{\alpha}^*_{\mathbf{C}})^W,$$

where $\mathbf{H}_R(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$ is the space of entire functions $\psi$ on $\boldsymbol{\alpha}^*_{\mathbf{C}}$ satisfying, for all $m \in \mathbf{N}$,

$$\sup_{\lambda\in\boldsymbol{\alpha}^*_{\mathbf{C}}} (1+\|\lambda\|^2)^m|\psi(\lambda)|e^{-R\|\mathrm{Im}\,\lambda\|} < +\infty.$$

$\mathcal{H}(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$ is the space of entire functions on $\boldsymbol{\alpha}^*_{\mathbf{C}}$, which are $W$-invariant, of exponential type and slowly increasing. We have

$$\mathcal{H}(\boldsymbol{\alpha}^\alpha_{\mathbf{C}})^W = \cup_{R\geq 0}\mathcal{H}_R(\boldsymbol{\alpha}^*_{\mathbf{C}})^W,$$

where $\mathcal{H}_R(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$ is the space of entire functions $\Phi$

$$\sup_{\lambda\in\boldsymbol{\alpha}^*_{\mathbf{C}}} (1+\|\lambda\|^2)^{-N}|\Phi(\lambda)|e^{-R\|\mathrm{Im}\,\lambda\|} < +\infty$$

on $\boldsymbol{\alpha}^*_{\mathbf{C}}$ such that there exists an $N \in \mathbf{N}$ and $L^p_\natural(G)$, $p \in [1, +\infty]$, is the space of measurable functions on $G$, bi-invariant under $K$ and such that

$$\|f\|_{G,p} = \left( \int_G |f(x)|^p \, dx \right)^{1/p} < +\infty, \quad p \in [1, +\infty[,$$
$$\|f\|_{G,\infty} = \text{ess} \sup_{x \in G} |f(x)| < +\infty.$$

$L^p_W(\boldsymbol{\alpha}^*)$, $p \in [1, +\infty]$, the space of measurable functions on $\boldsymbol{\alpha}^*$, which are $W$-invariant and such that

$$\|f\|_{\boldsymbol{\alpha}^*,p} = \left( \frac{1}{|W|} \int_{\boldsymbol{\alpha}^*} |f(\lambda)|^p \frac{d\lambda}{|c(\lambda)|^2} \right)^{1/p} < \infty, \quad p \in [1, +\infty[\,,$$
$$\|f\|_{\boldsymbol{\alpha}^*,\infty} = \text{ess} \sup_{\lambda \in \boldsymbol{\alpha}^*} |f(\lambda)| < +\infty.$$

We provide these spaces with classical topology.

### 4.1. The spherical transform of functions (see [4, 5, 6]).

**Definition 4.1.** The spherical transform $\mathcal{F}$, sometimes called the Harish-Chandra's transform, is defined on $\mathcal{D}^\natural(G)$ by

$$\mathcal{F}(f)(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) \, dx, \quad \text{for all } \lambda \in \boldsymbol{\alpha}^*. \tag{4.1}$$

*Remark* 4.1. When $G$ has rank one, the spherical transform can be written for all $\lambda \in \mathbf{R}$ in the form

$$\mathcal{F}(f)(\lambda) = \int_0^\infty f(a_t) \varphi^{(p,q)}_\lambda(t) A_{p,q}(t) \, dt, \quad f \in \mathcal{D}^\natural(G)$$

where

$$A_{p,q}(t) = 2^{2\rho} (sht)^{2p+1} (cht)^{2q+1}.$$

We put

$$f(a_t) = f[t], \quad a_t \in A, \; t \in \mathbf{R}.$$

The function $f[t]$ belongs to $\mathcal{D}_*(\mathbf{R})$ (the space of $C^\infty$-functions on $\mathbf{R}$, even and with compact support).

Using this notation the preceding relation takes for all $\lambda \in \mathbf{R}$ the form

$$\mathcal{F}(f)(\lambda) = \int_0^\infty f[t]\varphi_\lambda^{(p,q)}(t)A_{p,q}(t)\,dt, \quad f \in \mathcal{D}_*(\mathbf{R}). \tag{4.2}$$

Then the spherical transform of $f[t]$ is the Jacobi transform. (See [**8**, page 27]).

The transform $\mathcal{F}$ has the following properties:

i) For $f$ in $L^1_\natural(G)$ the function $\mathcal{F}(f)$ belongs to $C^\natural(G)$ and tends to zero as $\|\lambda\|$ goes to infinity.

ii) For $f$ in $L^1_\natural(G)$ we have

$$\|\mathcal{F}(f)\|_{\boldsymbol{\alpha}^*,\infty} \leq \|f\|_{G,1}. \tag{4.3}$$

**Theorem 4.1.** *The transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{D}^\natural(G)$ onto $\mathbf{H}(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$.*

*The inverse transform is given for all $x \in G$ by*

$$\mathcal{F}^{-1}(h)(x) = \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} h(\lambda)\varphi_\lambda(x)\,\frac{d\lambda}{|c(\lambda)|^2}. \tag{4.4}$$

**Theorem 4.2. i) Plancherel formula for $\mathcal{F}$.** *For all $f$ in $\mathcal{D}^\natural(G)$ we have*

$$\int_G |f(x)|^2 dx = \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} |\mathcal{F}(f)(\lambda)|^2\,\frac{d\lambda}{|c(\lambda)|^2}. \tag{4.5}$$

**ii) Plancherel theorem for $\mathcal{F}$.** *The transform $\mathcal{F}$ can be uniquely extended to an isometric isomorphism from $L^2_\natural(G)$ onto $L^2_W(\boldsymbol{\alpha}^*)$.*

**Proposition 4.1.** *For all $f$ in $\mathcal{D}^\natural(G)$ we have*

$$\mathcal{F}(f) = \mathcal{F}_0 \circ \mathcal{A}(f), \tag{4.6}$$

*where $\mathcal{F}_0$ is the classical Fourier transform given by the relation* (2.1).

**4.2. The spherical transform of distributions.**

**Definition 4.2.** The spherical transform of a distribution $S$ in $\mathcal{E}'_\natural(G)$ is defined for all $\lambda \in \boldsymbol{\alpha}^*$ by

$$\mathcal{F}(S)(\lambda) = \langle S, \varphi_{-\lambda} \rangle. \tag{4.7}$$

The following proposition gives some properties of the transform $\mathcal{F}$ on $\mathcal{E}'_\natural(G)$.

**Proposition 4.2.** *Let $S$ be a distribution in $\mathcal{E}'_\natural(G)$. Then*

i) *The function $\mathcal{F}(S)$ is of class $C^\infty$ on $\boldsymbol{\alpha}^*$ and $W$-invariant.*

ii) *We have that*

$$\mathcal{F}(S) = \mathcal{F}_0 \circ \mathcal{A}(S), \tag{4.8}$$

(*where $\mathcal{F}_0$ is the classical Fourier transform of distributions in $\mathcal{E}'_W(A)$*) *is given for all $\lambda \in \boldsymbol{\alpha}^*$ by*

$$\mathcal{F}_0(U)(\lambda) = \left\langle U_a, e^{-i\lambda(\log a)} \right\rangle. \tag{4.9}$$

From relation (4.8), Theorem 3.3 and the properties of transform $\mathcal{F}_0$, we deduce the following Paley-Wiener theorem for the transform $\mathcal{F}$:

**Theorem 4.3.** *The transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{E}'_\natural(G)$ onto $\mathcal{H}(\boldsymbol{\alpha}^*_{\mathbf{C}})^W$.*

**5. Convolution product of functions and distributions.** In this section we give first the definition and properties of the generalized translation operators on $G$, and next we study the convolution product of functions and distributions on $G$.

**5.1. Generalized translation operators on $G$.**

**Definition 5.1.** The generalized translation operators $\tau_x$, $x \in G$, are defined on $C^\natural(G)$ by

$$\tau_x f(y) = \int_K f(xky)\, dk, \quad \text{for all } y \in G. \tag{5.1}$$

**Proposition 5.1.** i) *The function $\tau_x f(y)$ is bi-invariant under $K$ with respect to the variables $x$ and $y$.*

ii) *The operators $\tau_b$, $b \in A^+$, possesses the following integral representation*

$$\tau_b f(c) = \int_{A^+} f(a)\, d\mu_{b,c}(a), \quad \textit{for all } c \in A^+ \tag{5.2}$$

*where $\mu_{b,c}$ is the measure given by the relation* (2.23) *and $f$ in $C(A)^W$.*

The operators $\tau_x$, $x \in G$, also satisfy the following properties.

i) For all $x \in G$, the operator $\tau_x$ is continuous from $\mathcal{E}^\natural(G)$ into itself.

ii) Let $f$ be in $\mathcal{D}^\natural(G)$. For all $x \in G$, the function $\tau_x f$ belongs to $\mathcal{D}^\natural(G)$.

iii) For all $f$ in $C^\natural(G)$ and $x, y \in G$, we have

$$\tau_e f(x) = f(x); \quad \tau_x f(y) = \tau_y f(x). \tag{5.3}$$

iv) For all $x, y \in G$ and $\lambda \in \boldsymbol{\alpha}^*_{\mathbf{C}}$ we have the product formula

$$\tau_x \varphi_\lambda(y) = \varphi_\lambda(x)\varphi_\lambda(y). \tag{5.4}$$

v) For all $f$ in $\mathcal{D}^\natural(G)$ and $x \in G$, we have for all $\lambda \in \boldsymbol{\alpha}^*$,

$$\mathcal{F}(\tau_x f)(\lambda) = \varphi_\lambda(x)\mathcal{F}(f)(\lambda). \tag{5.5}$$

vi) Let $f$ be in $L^p_\natural(G)$, $p \in [1, +\infty]$. For all $x \in G$, the function $\tau_x f$ belongs to $L^p_\natural(G)$, $p \in [1, +\infty]$, and we have

$$\|\tau_x f\|_{G,p} \leq \|f\|_{G,p}. \tag{5.6}$$

*Remark* 5.1. We denote by $\sigma_a$, $a \in A$, the classical translation operators on $A$ defined on $C(A)^W$ by

$$\sigma_a f(b) = f(ab), \quad \text{for all } b \in A. \tag{5.7}$$

**5.2. Convolution product on $G$.**

**Definition 5.2.** The convolution product of $f$ and $g$ in $C_c^\natural(G)$ is the function $f * g$ defined by

$$f * g(x) = \int_G f(xy^{-1})g(y)\,dy, \quad \text{for all } x \in G. \tag{5.8}$$

This relation can also be written for all $x \in G$ in the form

$$f * g(x) = \int_G \tau_x f(y^{-1})g(y)\,dy. \tag{5.9}$$

*Remark* 5.2. The classical convolution product of functions on $A$ is defined for $f_1$ and $g_1$ in $C_c(A)^W$ by

$$f_1 *_0 g_1(a) = \int_A \sigma_a f_1(b^{-1})g_1(b)\,db, \quad \text{for all } a \in A. \tag{5.10}$$

**Proposition 5.2.** *Let $f$ and $g$ be in $C_c^\natural(G)$. Then the function $f * g$ is bi-invariant under $K$ and we have, for all $a \in A^+$,*

$$f * g(a) = \int_{A^+} \tau_a f(b^{-1})g(b)\,db. \tag{5.11}$$

The following propositions give some other properties of the convolution product $*$.

**Proposition 5.3.** i) *The convolution product $*$ is commutative and associative.*

ii) *For $f$ and $g$ in $\mathcal{D}^\natural(G)$, the function $f * g$ belongs to $\mathcal{D}^\natural(G)$ and we have, for all $\lambda \in \boldsymbol{\alpha}^*$,*

$$\mathcal{F}(f * g)(\lambda) = \mathcal{F}(f)(\lambda)\mathcal{F}(g)(\lambda). \tag{5.12}$$

iii) *For $f$ in $L^p_\natural(G)$ and $g$ in $L^q_\natural(G)$ with $p, q \in [1, +\infty]$, the function $f * g$ belongs to $L^r_\natural(G)$ with $r \in [1, +\infty]$, such that $1/p + (1/q) - 1 = 1/r$ and we have*

$$\|f * g\|_{G,r} \leq \|f\|_{G,p}\|g\|_{G,q}. \tag{5.13}$$

iv) *For $f$ and $g$ in $\mathcal{D}^\natural(G)$, we have*

$$\mathcal{A}(f * g) = \mathcal{A}(f) *_0 \mathcal{A}(g). \tag{5.14}$$

(*See* [**4, 5, 6**]).

*Remark* 5.3. When the rank of $G$ is one, Koornwinder has given the expression of the generalized translation operators on $G$ and has studied the convolution product of functions in this case. (See [**8**, pages 57–61].)

**Definition 5.3.** The convolution product of a distribution $S$ in $\mathcal{D}'_\natural(G)$ and a function $\psi$ in $\mathcal{D}^\natural(G)$ is the function $S * \psi$ defined for all $x \in G$ by

$$S * \psi(x) = \langle S_y, \tau_x \psi(y^{-1}) \rangle. \tag{5.15}$$

**Proposition 5.4.** i) *Let $S = T_f$ be the distribution in $\mathcal{D}'_\natural(G)$ given by the function $f$ in $C^\natural(G)$. Then, for $\psi$ in $\mathcal{D}^\natural(G)$, we have*

$$S * \psi = f * \psi.$$

ii) *We consider $S$ in $\mathcal{D}'_\natural(G)$ and $\psi$ in $\mathcal{D}^\natural(G)$. Then the function $S*\psi$ belongs to $\mathcal{E}^\natural(G)$.*

iii) *Let $S$ be in $\mathcal{E}'_\natural(G)$ and $\psi$ in $\mathcal{D}^\natural(G)$. Then the function $S*\psi$ is in $\mathcal{D}^\natural(G)$.*

**Definition 5.4.** Let $U$ be a distribution in $\mathcal{D}'_\natural(G)$ and $S$ a distribution in $\mathcal{E}'_\natural(G)$. The convolution product of $U$ and $S$ is the distribution $U*S$ in $\mathcal{D}'_\natural(G)$ defined for all $\psi$ in $\mathcal{D}^\natural(G)$ by

$$\langle U*S,\psi\rangle = \langle U_x,\langle S_y,\tau_x\psi(y)\rangle\rangle = \langle S_y,\langle U_x,\tau_x\psi(y)\rangle\rangle. \tag{5.16}$$

**Proposition 5.5.** *Let $S$ be in $\mathcal{D}'_\natural(G)$ and $T_f$ the distribution in $\mathcal{E}'_\natural(G)$ given by the function $f$ in $\mathcal{D}^\natural(G)$. Then we have*

$$S*T_f = T_{S*f}. \tag{5.17}$$

*Remarks* 5.4. i) The classical convolution product of a distribution $S_1$ in $\mathcal{D}'_W(A)$ and a function $\psi_1$ in $\mathcal{D}(A)^W$ is defined for all $a\in A$ by

$$S_1 *_0 \psi_1(a) = \langle S_{1,b},\sigma_a\psi_1(b^{-1})\rangle. \tag{5.18}$$

ii) The classical convolution product of a distribution $U_1$ in $\mathcal{D}'_W(A)$ and a distribution $S_1$ in $\mathcal{E}'_W(A)$ is given for all $\psi_1$ in $\mathcal{D}(A)^W$ by

$$\begin{aligned}\langle U_1 *_0 S_1,\psi_1\rangle &= \langle U_{1,a},\langle S_{1,b},\sigma_a\psi_1(b)\rangle\rangle\\ &= \langle S_{1,b},\langle U_{1,a},\sigma_a(\psi_1)(b)\rangle\rangle.\end{aligned} \tag{5.19}$$

**Theorem 5.1.** *Let $S$ and $U$ be two distributions in $\mathcal{E}'_\natural(G)$. The convolution product $S*U$ of $S$ and $U$ belongs to $\mathcal{E}'_\natural(G)$, and we have*

$$\mathcal{F}(S*U) = \mathcal{F}(S).\mathcal{F}(U). \tag{5.20}$$

**Theorem 5.2.** *Let $S$ and $U$ be two distributions in $\mathcal{E}'_\natural(G)$. Then we have*

$$\mathcal{A}(S*U) = \mathcal{A}(S) *_0 \mathcal{A}(U). \tag{5.21}$$

**6. Hypoelliptic convolution equations in the space of distributions.** Let $S$ be in $\mathcal{E}'_\natural(G)$. In this section we study convolution equations of the form

$$S * U = V, \tag{6.1}$$

where $U$ and $V$ are distributions in $\mathcal{D}'_\natural(G)$.

We say that equation (6.1) is hypoelliptic if all solutions $U$ are given by a function $f$ in $\mathcal{E}^\natural(G)$ whenever $V$ is given by a function $g$ in $\mathcal{E}^\natural(G)$.

When (6.1) is hypoelliptic we also say that the distribution $S$ is hypoelliptic.

The main result of this section is the characterization of hypoelliptic convolution equations in terms of their spherical transform.

We say that the distribution $S$ in $\mathcal{E}'_\natural(G)$ satisfies the $H$-property if

i) there exist $k, M > 0$ such that $|\mathcal{F}(S)(\lambda)| \geq \|\lambda\|^{-k}$ for all $\lambda \in \boldsymbol{\alpha}^*$ with $\|\lambda\| \geq M$.

ii) $\lim_{\|z\| \to +\infty, z \in Z} \|\mathrm{Im}\, z\| / \log \|z\| = +\infty$, where

$$Z = \{z \in \boldsymbol{\alpha}^*_{\mathbf{C}}, \mathcal{F}(S)(z) = 0\}$$

with $\|z\|^2 = \sum_{j=1}^{\ell} ((\mathrm{Re}\, z_j)^2 + (\mathrm{Im}\, z_j)^2)$.

**Proposition 6.1.** *Let $S$ be in $\mathcal{E}'_\natural(G)$. If $S$ is hypoelliptic, then $S$ satisfies* i) *of the $H$-property.*

*Proof.* We assume that i) of the $H$-property does not hold. Then we can find a sequence $(\lambda_n)_{n\in\mathbf{N}} \subset \boldsymbol{\alpha}^*$ such that $\|\lambda_n\| \geq 2^n$ and, for all $n \in \mathbf{N}$,

$$|\mathcal{F}(S)(\lambda_n)| < \|\lambda_n\|^{-n}. \tag{6.2}$$

We consider the sequence $(U_p)_{p\in\mathbf{N}}$ of distributions in $\mathcal{D}'_\natural(G)$ given by

$$U_p = \sum_{n=0}^{p} T_{\varphi_{-\lambda_n}},$$

where $T_{\varphi_{-\lambda_n}}$ is the distribution in $\mathcal{D}'_\natural(G)$ given by the function $\varphi_{-\lambda_n}$.

Let $\psi$ be in $\mathcal{D}^\natural(G)$. For all $p, q \in \mathbf{N}$ with $p > q$, we have

$$\langle U_p, \psi\rangle - \langle U_q, \psi\rangle = \sum_{n=q+1}^{p} \langle T_{\varphi_{-\lambda_n}}, \psi\rangle.$$

Thus,

$$\langle U_p, \psi\rangle - \langle U_q, \psi\rangle = \sum_{n=q+1}^{p} \mathcal{F}(\psi)(\lambda_n). \tag{6.3}$$

But from Theorem 4.1 the function $\mathcal{F}(\psi)$ is rapidly decreasing. Then there exists a positive constant $C$ such that for all $y \in \boldsymbol{\alpha}^*$,

$$|\mathcal{F}(\psi)(y)| \le \frac{C}{1 + \|y\|}.$$

Thus, for all $n \in \mathbf{N}$,

$$|\mathcal{F}(\psi)(\lambda_n)| \le \frac{C}{\|\lambda_n\|} \le \frac{C}{2^n}. \tag{6.4}$$

By applying this relation to (6.3), we obtain

$$|\langle U_p, \psi\rangle - \langle U_q, \psi\rangle| \le C \sum_{n=q+1}^{p} \frac{1}{2^n} \to 0, \quad \text{as } q \to +\infty.$$

Then

$$\langle U_p, \psi\rangle \to L(\psi), \qquad \text{as} \quad p \to +\infty.$$

We deduce that $L$ is a distribution $U$ in $\mathcal{D}'_\natural(G)$ and $U_p$ converges to $U$ in $\mathcal{D}'_\natural(G)$ as $p$ tends to infinity. Thus,

$$U = \sum_{n=0}^{\infty} T_{\varphi_{-\lambda_n}}, \tag{6.5}$$

and for all $\psi$ in $\mathcal{D}^\natural(G)$ we have

$$\langle U, \psi\rangle = \sum_{n=0}^{\infty} \mathcal{F}(\psi)(\lambda_n). \tag{6.6}$$

We shall prove now that the distribution $S * U$ of $\mathcal{D}'_\natural(G)$ is given by a function $f$ in $\mathcal{E}^\natural(G)$.

From (5.16), (6.2) and (5.5), for all $\psi$ in $\mathcal{D}^\natural(G)$, we have

$$\langle S * U, \psi \rangle = \langle S_y, \langle U_t, \tau_y \psi(t) \rangle \rangle.$$

Thus,

$$\langle S * U, \psi \rangle = \langle S_y, \sum_{n=0}^{\infty} \varphi_{\lambda_n}(y) \mathcal{F}(\psi)(\lambda_n) \rangle.$$

By applying Theorem 4.1, we obtain

$$\langle S * U, \psi \rangle = \sum_{n=0}^{\infty} \mathcal{F}(\psi)(\lambda_n) \langle S_y, \varphi_{\lambda_n}(y) \rangle.$$

Then, Definition 4.2 implies

$$\langle S * U, \psi \rangle = \sum_{n=0}^{\infty} \mathcal{F}(\psi)(\lambda_n) \mathcal{F}(S)(-\lambda_n). \tag{6.7}$$

This relation can also be written in the form

$$\langle S * U, \psi \rangle = \sum_{n=0}^{\infty} \mathcal{F}(S)(-\lambda_n) \int_G \varphi_{-\lambda_n}(t) \psi(t)\, dt.$$

By using (2.19) and the fact that the function $\psi$ belongs to $\mathcal{D}^\natural(G)$ and $\mathcal{F}(S)$ satisfies the relation (6.2) we can exchange the series and the integral, and we obtain

$$\langle S * U, \psi \rangle = \int_G \left[ \sum_{n=0}^{\infty} \mathcal{F}(S)(-\lambda_n) \varphi_{-\lambda_n}(t) \right] \psi(t)\, dt.$$

Thus, the distribution $S * U$ is given by the function $f$ defined by

$$f(t) = \sum_{n=0}^{\infty} \mathcal{F}(S)(-\lambda_n) \varphi_{-\lambda_n}(t), \quad \text{for all } t \in G.$$

From relations (6.2) and (2.20), we deduce that the function $f$ belongs to $\mathcal{E}^\natural(G)$.

In the following we will show that the distribution $U$ is not given by a function $g$ in $\mathcal{E}^\natural(G)$. If not, we take a function $\phi$ in $\mathcal{D}^\natural(G)$ such that $\phi(e) = 1$ and $\mathcal{F}(\phi)$ is positive. For all $\mu \in \boldsymbol{\alpha}^*$, we consider

$$\langle \varphi_\mu U, \phi \rangle = \langle U, \varphi_\mu \phi \rangle. \tag{6.8}$$

By using (6.6) we obtain

$$\langle \varphi_\mu U, \phi \rangle = \sum_{n=0}^{\infty} \mathcal{F}(\varphi_\mu \phi)(\lambda_n). \tag{6.9}$$

But for all $\nu \in \boldsymbol{\alpha}^*$, we have

$$\mathcal{F}(\varphi_\mu \phi)(\nu) = \int_G \varphi_\mu(t)\varphi_{-\nu}(t)\phi(t)\,dt. \tag{6.10}$$

On the other hand, from relation (2.25), for all $t \in G$ we have

$$\varphi_\mu(t)\varphi_{-\nu}(t) = \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} \varphi_\xi(t) K(\mu, -\nu, \xi)\frac{d\xi}{|c(\xi)|^2},$$

with $K(\mu, -\nu, .)$ the positive function on $\boldsymbol{\alpha}^*$ given by (2.24).

By using this relation and Fubini-Tonelli's theorem, for all $\nu \in \boldsymbol{\alpha}^*$, the relation (6.10) can also be written in the form

$$\begin{aligned} \mathcal{F}(\varphi_\mu \phi)(\nu) &= \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} K(\mu, -\nu, \xi)\left(\int_G \phi(t)\varphi_\xi(t)\,dt\right)\frac{d\xi}{|c(\xi)|^2} \\ &= \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} K(\mu, -\nu, \xi)\mathcal{F}(\phi)(-\xi)\,\frac{d\xi}{|c(\xi)|^2}. \end{aligned} \tag{6.11}$$

Thus, for all $\mu, \nu \in \boldsymbol{\alpha}^*$ the function $\mathcal{F}(\varphi_\mu \phi)(\nu)$ is positive. By taking $\nu = \lambda_n$, and by replacing $\mathcal{F}(\varphi_\mu \phi)(\lambda_n)$ by its expression (6.11), we obtain from (6.9):

$$\langle \varphi_\mu U, \phi \rangle = \frac{1}{|W|}\sum_{n=0}^{\infty}\int_{\boldsymbol{\alpha}^*} K(\mu, -\lambda_n, \xi)\mathcal{F}(\phi)(-\xi)\frac{d\xi}{|c(\xi)|^2}.$$

As the function $\mu \to \langle \varphi_\mu U, \phi\rangle$ and those of the second member are positive, then by applying relations (2.25) and (2.26) and by using Fubini-Tonelli's theorem, we deduce that

$$\begin{aligned}
&\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\langle \varphi_\mu U, \phi\rangle \frac{d\mu}{|c(\mu)|^2}\\
&= \frac{1}{|W|}\sum_{n=0}^{\infty}\int_{\boldsymbol{\alpha}^*}\left[\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*} K(\mu,-\lambda_n,\xi)\,\frac{d\mu}{|c(\mu)|^2}\right]\mathcal{F}(\phi)(-\xi)\,\frac{d\xi}{|c(\xi)|^2}\\
&= \frac{1}{|W|}\sum_{n=0}^{\infty}\int_{\boldsymbol{\alpha}^*}\mathcal{F}(\phi)(-\xi)\frac{d\xi}{|c(\xi)|^2}.
\end{aligned}$$

But, from Theorem 4.1, we have

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\mathcal{F}(\phi)(-\xi)\frac{d\xi}{|c(\xi)|^2} = \phi(e) = 1.$$

Thus,

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\langle \varphi_\mu U, \phi\rangle\,\frac{d\mu}{|c(\mu)|^2} = +\infty. \tag{6.12}$$

On the other hand, as the distribution $U$ is given by the function $g$ in $\mathcal{E}^\natural(G)$, then from (6.8) we have

$$\langle \varphi_\mu U, \phi\rangle = \langle T_g, \varphi_\mu \phi\rangle = \int_G \varphi_\mu(t) g(t)\phi(t)\,dt = \mathcal{F}(g\,\phi)(-\mu).$$

Thus,

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\langle \varphi_\mu U, \phi\rangle \frac{d\mu}{|c(\mu)|^2} = \frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\mathcal{F}(g\phi)(-\mu)\,\frac{d\mu}{|c(\mu)|^2}.$$

By applying Theorem 4.1, we obtain

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\langle \varphi_\mu U, \phi\rangle \frac{d\mu}{|c|\mu|^2} = g(e)\phi(e).$$

Then

$$\frac{1}{|W|}\int_{\boldsymbol{\alpha}^*}\langle \varphi_\mu U, \phi\rangle \frac{d\mu}{|c(\mu)|^2} = g(e).$$

This contradicts (6.12). Hence, the distribution $U$ is not given by a function $g$ in $\mathcal{E}^\natural(G)$.

**Proposition 6.2.** *Let $S$ be in $\mathcal{E}'_\natural(G)$. If $S$ is hypoelliptic, then $S$ satisfies* ii) *of the $H$-property.*

*Proof.* Suppose that ii) of the $H$-property does not hold. Then there exist a sequence $(z_n)_{n\in\mathbf{N}} \subset \boldsymbol{\alpha}^*_{\mathbf{C}}$ and a positive constant $M$ such that, for all $n \in \mathbf{N}$, $\mathcal{F}(S)(z_n) = 0$ and $\|\operatorname{Im} z_n\| \leq M \log \|z_n\|$.

Let $\phi$ be in $\mathcal{D}^\natural(G)$. According to Theorem 4.1, there exists an $R \in \mathbf{N}$ such that, for every $p \in \mathbf{N}$, we can find $C_p > 0$ for which, for all $z \in \boldsymbol{\alpha}^*_{\mathbf{C}}$ such that $\|z\| > 1$, we have

$$|\mathcal{F}(\phi)(z)| \leq C_p e^{R\|\operatorname{Im} z\| - p \log \|z\|}.$$

If we take $p \in \mathbf{N}$ such that $p > MR + 1$, we get for all $n \in \mathbf{N}$,

$$\|z_n\| \, |\mathcal{F}(\phi)(z_n)| \leq C_p. \tag{6.13}$$

Let $(a_n)_{n\in\mathbf{N}}$ be a complex sequence such that the series $\sum_{n=0}^{\infty} |a_n|$ is convergent.

We consider the sequence $(V_q)_{q\in\mathbf{N}}$ of distributions in $\mathcal{D}'_\natural(G)$ given by

$$V_q = \sum_{n=0}^{q} a_n T_{\|z_n\|\varphi_{z_n}}.$$

For all $q, r \in \mathbf{N}$ with $q > r$, we have

$$\begin{aligned} \langle V_q, \phi\rangle - \langle V_r, \phi\rangle &= \Big\langle \sum_{n=r+1}^{q} a_n T_{\|z_n\|\varphi_{z_n}}, \phi \Big\rangle \\ &= \sum_{n=r+1}^{q} a_n \|z_n\| \mathcal{F}(\phi)(-z_n). \end{aligned}$$

Thus, using (6.13) we obtain

$$|\langle V_q, \phi\rangle - \langle V_r, \phi\rangle| \leq C_p \sum_{n=r+1}^{q} |a_n| \longrightarrow 0, \quad \text{as } r \mapsto +\infty. \tag{6.14}$$

Then

$$\langle V_q, \phi\rangle \longrightarrow L(\phi), \quad \text{as } q \mapsto +\infty.$$

We deduce that $L$ is a distribution $V$ in $\mathcal{D}'_\natural(G)$ and $V_q$ converges to $V$ in $\mathcal{D}'_\natural(G)$ as $q$ tends to infinity. Then

$$V = \sum_{n=0}^{\infty} a_n T_{\|z_n\|\varphi_{z_n}}, \tag{6.15}$$

and from (6.14) we deduce that

$$|\langle V, \phi\rangle| \le C_p \sum_{n=0}^{\infty} |a_n|. \tag{6.16}$$

On the other hand, by making a proof similar to those which has given the relation (6.7), we obtain

$$\langle S * V, \phi\rangle = \sum_{n=0}^{\infty} a_n \|z_n\| \mathcal{F}(S)(z_n)\mathcal{F}(\phi)(-z_n) = 0.$$

Thus,

$$S * V = 0.$$

As $S$ is hypoelliptic, we deduce that the distribution $V$ is given by a function $f$ in $\mathcal{E}^\natural(G)$. Then we have

$$V = T_f. \tag{6.17}$$

From (2.15), for all $n \in \mathbf{N}$, we have

$$\varphi_{z_n}(e) = 1.$$

Thus, for all closed balls $B = \{x \in G / |x| \le r\}$, and for all $n \in \mathbf{N}$, we have

$$\sup_{x\in B} |\varphi_{z_n}(x)| \ge 1. \tag{6.18}$$

On the other hand, using (6.15), (6.16) and (6.17), we obtain

$$\sup_{x\in B} |f(x)| \le C_p \sum_{n=0}^{\infty} |a_n|.$$

Thus, for all $n \in \mathbf{N}$,

$$\|z_n\| \sup_{x \in B} |\varphi_{z_n}(x)| \leq C_p. \tag{6.19}$$

From this relation and (6.18), we deduce that, for all $n \in \mathbf{N}$,

$$\|z_n\| \leq C_p, \tag{6.20}$$

which is a contradiction with our choice of the sequence $(z_n)_{n \in \mathbf{N}}$. This completes the proof of the proposition. □

**Proposition 6.3.** *Let $S$ be in $\mathcal{E}'_\natural(G)$. If $S$ satisfies the $H$-property, then there exists a parametrix for $S$, that is, there exist a $V$ in $\mathcal{E}'_\natural(G)$ and a $\psi$ in $\mathcal{D}^\natural(G)$ such that $\delta_e = S * V + T_\psi$, where $\delta_e$ represents the Dirac distribution at $e$.*

*Proof.* Using (4.8), the $H$-property can also be written in the form

i) there exist $k, M > 0$ such that $|\mathcal{F}_0(\mathcal{A}(S))(\lambda)| \geq \|\lambda\|^{-k}$, for all $\lambda \in \boldsymbol{\alpha}^*$ with $\|\lambda\| \geq M$.

ii) $\lim_{\|z\| \to +\infty, z \in Z} \|\operatorname{Im} z\| / \log \|z\| = +\infty$, where

$$Z = \{z \in \boldsymbol{\alpha}^*_{\mathbf{C}}, \mathcal{F}_0(\mathcal{A}(S))(z) = 0\}.$$

We see that the $H$-property is true for the distribution $\mathcal{A}(S)$ of $\mathcal{E}'_W(A)$ in the case of the classical Fourier transform $\mathcal{F}_0$ on $A$. Then, from [**7**], there exists a parametrix for $\mathcal{A}(S)$, that is, there exist a $V_0$ in $\mathcal{E}'_W(A)$ and $\psi_0$ in $\mathcal{D}(A)^W$ such that

$$\delta_e = \mathcal{A}(S) *_0 V_0 + T_{\psi_0}. \tag{6.21}$$

As the operator $\mathcal{A}$ is a topological isomorphism from $\mathcal{E}'_\natural(G)$ onto $\mathcal{E}'_W(A)$, we deduce from (6.21) and (3.11) that

$$\delta_e = \mathcal{A}(S) *_0 \mathcal{A}(\mathcal{A}^{-1}(V_0)) + \mathcal{A}(\mathcal{A}^{-1}(T_{\psi_0})).$$

Thus,

$$\delta_e = \mathcal{A}(S) *_0 \mathcal{A}(V) + \mathcal{A}(T_\psi) \tag{6.22}$$

with

$$\mathcal{A}^{-1}(V_0) = V \quad \text{and} \quad \mathcal{A}^{-1}(\psi_0) = \psi.$$

The distribution $V$ and the function $\psi$ belong respectively to $\mathcal{E}'_\natural(G)$ and $\mathcal{D}^\natural(G)$.

On the other hand, from Theorem 5.2, we have

$$\mathcal{A}(S * V) = \mathcal{A}(S) *_0 \mathcal{A}(V).$$

Thus, relation (6.22) can also be written in the form

$$\mathcal{A}^{-1}(\delta_e) = S * V + T_\psi.$$

But

$$\mathcal{A}^{-1}(\delta_e) = \delta_e.$$

Thus,

$$\delta_e = S * V + T_\psi. \qquad \square$$

**Theorem 6.1.** *We assume that the distribution $S$ in $\mathcal{E}'_\natural(G)$ is such that $Z = \{z \in \boldsymbol{\alpha}^*_{\mathbf{C}}, \mathcal{F}(S)(z) = 0\}$ is infinite. The following assertions are equivalent.*

i) *$S$ is hypoelliptic.*

ii) *$S$ satisfies the $H$-property.*

iii) *There exists a parametrix for $S$, that is, there exist a $V$ in $\mathcal{E}'_\natural(G)$ and a $\psi$ in $\mathcal{D}^\natural(G)$ such that*

$$\delta_e = S * V + T_\psi.$$

*Proof.* From Propositions 6.1 and 6.2 it suffices to prove that iii) $\Rightarrow$ i). Assume that the distribution $U$ is in $\mathcal{D}'_\natural(G)$ and that $S * U$ is given by a function $f$ in $\mathcal{E}^\natural(G)$. From iii) we have

$$\delta_e = S * V + T_\psi,$$

with $V$ in $\mathcal{E}'_\natural(G)$ and $\psi$ in $\mathcal{D}^\natural(G)$.

Thus,

$$\begin{aligned} U &= U * \delta_e \\ &= U * (S * V + T_\psi). \end{aligned}$$

Using commutativity and associativity of the convolution product of distributions, we obtain

$$\begin{aligned} U &= V * (S * U) + U * T_\psi \\ &= V * T_f + U * T_\psi. \end{aligned}$$

By applying (5.17), we deduce that

$$\begin{aligned} U &= T_{V*f} + T_{U*\psi} \\ &= T_{(V*f+U*\psi)} \end{aligned}$$

But, from Proposition 5.4 ii), the function $V * f + U * \psi$ belongs to $\mathcal{E}^\natural(G)$. Thus, $S$ is hypoelliptic.

*Remark* 6.1. In [**2, 9**] the authors have proved the analogue of Theorem 6.1 in the case of the Jacobi transform. Their result implies Theorem 6.1 when the rank of $G$ is one.

**Example 6.1.** We suppose that the rank $\ell \geq 2$, and we consider the equation

$$\Delta U = V, \tag{6.23}$$

with $U$ and $V$ in $\mathcal{D}'_\natural(G)$ and $\Delta$ is the Laplace-Beltrami operator on $\mathcal{D}'_\natural(G)$ defined by

$$\langle \Delta U, \phi \rangle = \langle U, \Delta \phi \rangle, \quad \phi \in \mathcal{D}^\natural(G).$$

We say that the operator $\Delta$ is hypoelliptic if all solutions $U$ of (6.23) are given by a function $f$ in $\mathcal{E}^\natural(G)$ whenever $V$ is given by a function $g$ in $\mathcal{E}^\natural(G)$. We have

$$\Delta U = (\Delta \delta_e) * U,$$

where $\delta_e$ is the Dirac distribution at $e$.

Then the hypoellipticity of $\Delta$ is equivalent to the hypoellipticity of the distribution $\Delta\,\delta_e$ in $\mathcal{E}'_\natural(G)$, given by

$$\langle \Delta\,\delta_e, \phi\rangle = \langle \delta_e, \Delta\phi\rangle = \Delta\,\phi(e), \quad \phi \in \mathcal{E}^\natural(G).$$

Relation (2.21) implies that, for all $z \in \boldsymbol{\alpha}^*_{\mathbf{C}}$,

$$\mathcal{F}(\Delta\delta_e)(z) = -\Big(\sum_{j=1}^{\ell} z_j^2 + \|\rho\|^2\Big). \tag{6.24}$$

i) From (6.24) we deduce that, for all $\lambda \in \boldsymbol{\alpha}^*$,

$$\mathcal{F}(\Delta\,\delta_e)(\lambda) = -(\|\lambda\|^2 + \|\rho\|^2).$$

Thus, for $\|\lambda\| \geq 1$, we have

$$|\mathcal{F}(\Delta\,\delta_e)(\lambda)| \geq \|\lambda\|^{-1}. \tag{6.25}$$

ii) Relation (6.24) also implies that

$$\begin{aligned} Z &= \{z \in \boldsymbol{\alpha}^*_{\mathbf{C}}, \mathcal{F}(\Delta\,\delta_e)(z) = 0\} \\ &= \{(\operatorname{Re} z, \operatorname{Im} z) \in \boldsymbol{\alpha}^* \times \boldsymbol{\alpha}^*, \|\operatorname{Re} z\|^2 = \|\operatorname{Im} z\|^2 + \|\rho\|^2 \\ &\qquad \text{and } \langle \operatorname{Re} z, \operatorname{Im} z\rangle = 0\}. \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{\|z\|\to+\infty, z\in Z} \frac{\|\operatorname{Im} z\|}{\log\|z\|} &= \lim_{\|\operatorname{Im} z\|\to+\infty} \frac{2\|\operatorname{Im} z\|}{\log(2\|\operatorname{Im} z\|^2 + \|\rho\|^2)} \\ &= +\infty. \end{aligned} \tag{6.26}$$

Relations (6.25) and (6.26) show that distribution $\Delta\,\delta_e$ satisfies the $H$-property. Thus, Theorem 6.1 implies that the distribution $\Delta\delta_e$ is hypoelliptic. The Laplace-Beltrami operator $\Delta$ is then hypoelliptic.

## REFERENCES

**1.** M. Belhadji and J.J. Betancor, *Hypoellipticity of Hankel convolution equations on Schwartz distributions*, Arch. Math. **79** (2002), 188–196.

**2.** J.J.Betancor, J.D. Betancor and J.R.M. Mendez, *Hypoelliptic Jacobi convolution operators on Schwartz distributions*, Positivity **8** (2004), 407–422.

**3.** L. Ehrenpreis, *Solution of some problems of divisions* IV. *Invertible and elliptic operators*, Amer. J. Math. **82** (1960), 522–588.

**4.** R. Gangolli and V.S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*, Ergebnisse der Mathematik under ihrer Genzgebiete **101**, Springer-Verlag, Berlin, 1988.

**5.** S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Academic Press, New York, 1984.

**6.** ———, *Geometric analysis on symmetric spaces*, American Mathematical Society, Providence, RI, 1994.

**7.** L. Hörmander, *Hypoelliptic convolution equations*, Math. Scand. **9** (1961), 178–181.

**8.** T.H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, in *Special functions, group theoretical aspects and applications*, R.A. Askey, T.H. Koornwinder and W. Schempp, eds., Reidel, Dordrecht, 1984.

**9.** K. Trimèche, *Hypoelliptic distributions on Chébli-Trimèche hypergroups*, Global J. Pure Appl. Math. **1** (2005), 251–271.

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