

ORDERS OF VANISHING OF ZEROS OF CHARACTERISTIC p ZETA FUNCTION

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ABSTRACT. Orders of vanishing of zeros of zeta functions have much arithmetic information encoded in them. For the absolute zeta function, Dinesh Thakur gave sufficient conditions for the order of vanishing of its zeros when the finite field has two elements. Such conditions consider only principal ideals. This result was generalized by Thakur and Diaz-Vargas. Now the conditions involve not only the principal ideals but all the classes of ideals, still in the field of two elements. In this work, we generalize these results to arbitrary finite fields, using similar proofs of Thakur and Diaz-Vargas.

1. Introduction. One of the most important topics in the study of zeta functions is the order of vanishing of its zeros. Some results have been found for the characteristic p zeta function and the “trivial” zeros that we analyze. In [7], Thakur gave sufficient conditions for a hyperelliptic function field over the finite field \mathbf{F}_q , $q = 2$, to have order of vanishing 2 at the negative integer $-s$. An interesting phenomenon is that such conditions involve the sum of the digits in the expansion base 2 of s , $l_2(s)$, see Theorem 5. This result was generalized by Thakur and Diaz-Vargas in [2] considering now all the ideal classes in the definition of the zeta function, and not only the principal ideals. The conditions to have order of vanishing at least 2, depend again on the decomposition base 2 of es , where e is the exponent of the ideal class group, see Theorem 8.

In [7], Thakur says succinctly how to deal with the general case $q = p^n$ and arbitrary function fields, when one considers only principal ideals. The conditions in order to have multiplicity q at the zeros, depend now on Weierstrass gaps at ∞ . We add an extra condition and give the proof of Theorem 7, which is a generalization of Thakur’s theorem for $q = 2$. We analyze also what it means for a function field to have an r -gap structure at ∞ .

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Finally, in Theorem 9 we obtain, by joining Theorems 7 and 8, our principal result. This is a partial generalization to any value of q and, considering all the classes of ideals, of conditions that guarantee multiplicity at least q at the trivial zeros, for a very specific class of function fields.

We also present examples of the phenomena described in the theorems, wherever possible.

2. Basic definitions. Let K be a function field of one variable with constant field \mathbf{F}_q of characteristic p , ∞ a place of degree 1. Then its residue field at ∞ is \mathbf{F}_q and its completion K_∞ at ∞ is the Laurent series field $\mathbf{F}_q((u_\infty))$, where u_∞ is any uniformizer at ∞ , i.e., a generator of the maximal ideal of the valuation ring at ∞ .

Let A be the ring of elements of K that have no poles outside ∞ . Let \mathbf{C}_∞ the completion of the algebraic closure of K_∞ . We denote by h the class number of K , so the class number of A is h too.

For $x \in K$, we define the degree of x as $\deg(x) = -v_\infty(x)$. By convention, $\deg(0) = -\infty$. Similarly, for the \mathcal{I} ideal $\in A$, define $\deg \mathcal{I} = \dim_{\mathbf{F}_q} A/\mathcal{I}$.

We define now the relevant zeta function, see [8, page 156].

$$\zeta(s, X) := \zeta_A(s, X) := \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg a=d \\ a \text{ monic}}} \frac{1}{a^s} \in K[[X]],$$

$$\zeta(s) := \zeta_A(s) := \zeta(s, 1) \in K_\infty.$$

In this work, we will focus on orders of vanishing of trivial zeros of the characteristic p absolute zeta function. We begin by giving the definition of order of vanishing.

Definition 1. The order of vanishing of ζ , at the zero s , is defined as the order of vanishing of $\zeta(s, X)$ at $X = 1$.

3. Zeta values at negative integers. For a nonnegative integer $k = \sum k_i q^i$, with $0 \leq k_i < q$, we let $l_q(k) = \sum k_i$, that is, $l_q(k)$ is the sum of base q digits of k .

Theorem 1 [7]. *Let W be an \mathbf{F}_q -vector space of dimension d inside a field (or ring) \mathcal{F} over \mathbf{F}_q . Let $f \in \mathcal{F} - W$. If $d > (l_q(k)/q - 1)$, then*

$$\sum_{w \in W} (f + w)^k = 0.$$

This proposition is useful for proving the following result, about the “trivial zeros” of the absolute zeta function. The theorem is analogue to the classical result, since multiples of $q - 1$ are playing the role, in this context, of even integers.

Theorem 2 [3]. *For a positive integer s , $\zeta(-s) \in A$ and $\zeta(-(q-1)s) = 0$.*

The reciprocal is true in the following situation.

Theorem 3 [3]. *Let $A = \mathbf{F}_q[x]$, and let s be a positive integer. Then $\zeta(-s) = 0$ if and only if s is a multiple of $q - 1$.*

It is not known for general A if the values at odd integers are not zero.

4. Order of vanishing. An immediate consequence of Theorem 2 is that the negative integers are zeros of the zeta function for $A = \mathbf{F}_2[x]$. In fact, more is true; from Definition 1, we get the following

Theorem 4 [6]. *The negative integers are zeros of the zeta function for $A = \mathbf{F}_2[x]$, whose order of vanishing is one.*

4.1. Considering only principal ideals: Case $q = 2$. In this and in the following subsections, we will give some theorems related with the order of vanishing of the zeta function. We will do it in two ways: first considering only principal ideals and then considering all the ideals (principal or not). Observe that in the first case, the absolute zeta function is a complete zeta function.

We have the following, easy to prove, lemma.

Lemma 1. *We have*

$$W_d = \{a \in A \mid \deg(a) < d\} = \mathcal{L}((d-1)\infty).$$

Then a simple application of Theorem 1 and the Riemann-Roch theorem to W_d , together with the simplicity of the zeros, gives [7] the following

Theorem 5 (Thakur). *If $q = 2$ and K is hyperelliptic, then the order of vanishing of $\zeta(-s)$, s a positive integer, is 2 if $l_2(s) \leq g$, where g is the genus of K .*

Example 1. Let $A = \mathbf{F}_2[x, y]$, where $y^2 + (x^2 + x + 1)y + (x^2 + x + 1)(x^5 + x^2 + 1) = 0$.

This is Example 26 of [1]; we have that $g = 3$. So, by Theorem 5, the order of vanishing is 2 if $s = (2^l + 2^m + 2^n)$, $l \geq 0$, $m \geq 0$, $n \geq 0$ (possibly with $l = m = n$). \square

Theorem 5 is not valid for $q > 2$, that is, we cannot assure that if $s = k(q-1) > 0$, and $-s$ is a zero of ζ with $l_q(s) \leq g$, then $-s$ is a zero with order of vanishing 2. Let's see it in the next example. Let $S_A(d)$ be the coefficient of X^d in $\zeta_A(-s, X)$.

Example 2. Consider $A = \mathbf{F}_3[x, y]$ where $y^2 = x(x+1)(x+2)(x^2+1)$.

This is Example 36 from [1]; we have that $g = 2$, $\deg(x) = 2$, and $\deg(y) = 5$. By Theorem 2, $\zeta(-2) = 0$. Now, $l_3(2) = 2 = g$. Then W_3 has dimension 2, and so we have $l_3(2) > \dim W_3 / (q-1) = 1$, and by Theorem 1, $S_A(d) = 0$ for $d \geq 3$. Therefore,

$$\zeta_A(-2, X) = (1 + X)(1 - X).$$

Then, the order of vanishing of -2 is 1. \square

For more background and perspective on zeta functions, see the books by Goss [4] and Thakur [8].

4.2. Considering only principal ideals: Case $q = p^n$. Theorem 5 gives us sufficient conditions for a hyperelliptic function field to have order of vanishing 2 when $q = 2$. Now, we try to generalize this result to the case $q = p^n$. For that, we need to define what it means for K to have an r -gap structure at ∞ .

Definition 2. Let g be the genus of K over \mathbf{F}_q . We say that K has an r -gap structure at ∞ if there is an r such that $l(iq\infty) = i + 1$, for $1 \leq i \leq r$, $l((g + r)\infty) = r + 1$, and $rq \leq g + r$.

Example 3. Let K be a function field of genus $g = 5$ over \mathbf{F}_3 . Then K does not have a 1-gap structure at ∞ .

Suppose that K has a 1-gap structure at ∞ , i.e., $l(3\infty) = 2$ and $l(6\infty) = 2$. Consider the following diagram:

1	2	3	4	5	6	7	8	9
1		2				2		

where the first row is filled with the possible gaps (that lies between 1 and $2g - 1 = 9$) and the second row with the values of $l(i\infty)$.

Recall that i is a gap if and only if $l((i - 1)\infty) = l(i\infty)$, and there are g gaps. We know that 1 is always a gap and that 4, 5, 6 are gaps by assumption.

Then

(1)	2	3	(4)	(5)	(6)	7	8	9
1		2	2	2	2			

Observe that 3 is a gap since if it were not a gap then an element of degree 6 could exist, which does not exist because we know that 6 is a gap. By the same reason, 2 is a gap and 1 is always a gap. Then K has gaps at 1, 2, 3, 4, 5, 6, which is a contradiction because it only has $g = 5$ gaps. So there is not such a function field. \square

The following example shows that if the function field K has genus 5 over \mathbf{F}_3 and K has a 2-gap structure, then it has an element of degree 3.

Example 4. Let K be a function field of genus $g = 5$ over \mathbf{F}_3 that has a 2-gap structure at ∞ . Then its gaps are: 1, 2, 4, 5, 7.

In order to prove this, a 2-gap structure means that: $l(3\infty) = 2, l(6\infty) = l(7\infty) = 3$. Then 7 is a gap. Consider the following diagram:

$$\begin{array}{cccccccccc} (1) & 2 & 3 & 4 & 5 & 6 & (7) & 8 & 9 \\ & 1 & & 2 & & & 3 & 3 & \end{array}$$

Now assume that 2 is not a gap, then 4, 6 and 8 are not either. Therefore,

$$\begin{array}{cccccccccc} (1) & 2 & 3 & 4 & 5 & 6 & (7) & 8 & 9 \\ & 1 & & 2 & 3 & & 3 & 3 & 4 \end{array}$$

We observe that this is a contradiction, because this force to $l(5\infty) = 3$, saying that 6 is a gap, which is absurd. Therefore, 2 is a gap.

$$\begin{array}{cccccccccc} (1) & (2) & 3 & 4 & 5 & 6 & (7) & 8 & 9 \\ & 1 & 1 & 2 & & & 3 & 3 & \end{array}$$

From the previous diagram, we see that 3 is not a gap. Suppose now that 4 is not a gap, then 8 is not either.

$$\begin{array}{cccccccccc} (1) & (2) & 3 & 4 & 5 & 6 & (7) & 8 & 9 \\ & 1 & 1 & 2 & 3 & & 3 & 3 & 4 \end{array}$$

This forces 5 and 6 to be gaps. This is a contradiction, since there is an element of degree 3. Then 4 is a gap.

$$\begin{array}{cccccccccc} (1) & (2) & 3 & (4) & 5 & 6 & (7) & 8 & 9 \\ & 1 & 1 & 2 & 2 & & 3 & 3 & \end{array}$$

Lastly, assume that 5 is not a gap,

$$\begin{array}{cccccccccc} (1) & (2) & 3 & (4) & 5 & 6 & (7) & 8 & 9 \\ & 1 & 1 & 2 & 2 & 3 & 3 & 3 & \end{array}$$

and we have that 6 is a gap, which is absurd since there is an element of degree 3, that again must exist. In conclusion, we have the complete diagram:

$$\begin{array}{cccccccccc} (1) & (2) & 3 & (4) & (5) & 6 & (7) & 8 & 9 \\ & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 5 \end{array}$$

and the gaps are: 1, 2, 4, 5, 7. \square

The following theorem “generalizes” the previous example.

Theorem 6. *Let K be of genus g over \mathbf{F}_q . Suppose that K has an r -gap structure at ∞ with $r \geq q - 1$. Then K has an element of degree q . Moreover, K does not have elements of degree $\neq iq$ between 1 and rq .*

Proof. Since K has an r -gap structure, we have the following diagram:

$$\begin{array}{cccccccccccc}
 (1) & 2 & \dots & q-1 & q & q+1 & \dots & 2q & \dots & rq & rq+1 & \dots & g+r \\
 & 1 & & & 2 & & & 3 & & r+1 & & & r+1
 \end{array}$$

Observe that by definition, K has as gaps $rq + 1, \dots, g + r$ since $rq \leq g + r$ and $l(rq\infty) = r + 1$ and $l((g + r)\infty) = r + 1$. Therefore, we have $g - r(q - 1)$ gaps. Notice also that between 1 and rq there are exactly r numbers that are not gaps. Suppose that there exists an element of degree $1 < k \leq q - 1$, that is, suppose that k is not a gap. Let c be the greatest number such that $ck \leq rq$. If $c \leq r$, then $rq = c(q - 1) + res \leq r(q - 1) + res$ with $res < q - 1$. Then $rq \leq r(q - 1) + res$ implies that $r \leq res$, and this is a contradiction with the fact that $r \geq q - 1$. Therefore, $c > r$ and this implies that we have more than r numbers that are not gaps; this is a contradiction. Therefore, $1, 2, \dots, q - 1$ are gaps. We have the following diagram:

$$\begin{array}{cccccccccccc}
 (1) & (2) & \dots & (q-1) & q & q+1 & \dots & 2q & \dots & rq & (rq+1) & \dots & (g+r) \\
 & 1 & 1 & \dots & 1 & 2 & & 3 & & r+1 & & & (r+1)
 \end{array}$$

Therefore, q is not a gap and from here $2q, 3q, \dots, rq$ are also not gaps. \square

In the previous theorem the hypothesis that $r \geq q - 1$ assures us the existence of an element of degree q .

The next theorem is essentially due to Thakur, compare with [7].

Theorem 7. *If $q = p^n$ and K is of genus g with an r -gap structure at ∞ with $r \geq q - 1$, then the order of vanishing of $\zeta(-s)$ at positive integers s , s a multiple of $q - 1$, is q if $(l_q(s)/q - 1) \leq r$.*

Proof. Let x be an element of degree q (it exists by the previous theorem). Recall that $S_A(d)$ is the coefficient of X^d in $\zeta_A(-s, X)$,

and that $W_d = \mathcal{L}((d-1)\infty)$ (Lemma 1). For $d > (l_q(s)/q - 1) + g$, by Riemann-Roch's theorem (since $h = \dim(H - (d-1)\infty)$, H a canonical divisor), we have

$$\begin{aligned} \dim W_d &= \dim \mathcal{L}((d-1)\infty) = l((d-1)\infty) \\ &= (d-1) + 1 - g + h = d - g + h > \frac{l_q(s)}{q-1}. \end{aligned}$$

Then $S_A(d) = 0$ for $d > (l_q(s)/q - 1) + g$. So, for $d > g + r$, as $r \geq (l_q(s)/q - 1)$ we have that $S_A(d) = 0$. Since A does not have elements of degree $\neq iq$ we only consider, in the summation, elements of degree dq . From here,

$$\zeta_A(-s, X) = \sum_{d=0}^r S_A(dq) X^{dq}.$$

Now, as x has degree q in A but one in $\mathbf{F}_q[x]$, we have that $S_A(dq) = S_{\mathbf{F}_q[x]}(d)$ for $0 \leq d \leq r$. As $(l_q(s)/q - 1) \leq r$, then $S_{\mathbf{F}_q[x]}(d) = 0$ for $d > (l_q(s)/q - 1)$. From here, we have

$$\zeta_A(-s, X) = \zeta_{\mathbf{F}_q[x]}(-s, X^q).$$

We know that the order of vanishing at zeros for $\zeta_{\mathbf{F}_q[x]}(-s, X)$ is one and therefore is q for $\zeta_{\mathbf{F}_q[x]}(-s, X^q)$, and, from here, we get the result. \square

An example of the situation described in the theorem is the following, which appears in [7] and is due to José Felipe Voloch:

Example 5. Let $q = 3$, and consider $x^2(x-1)^3 - y^2 + yx^3 + y^3 + y^5 = 0$ with ∞ being $(1, 0)$.

We found the next example using the computer program Kant/Kash.

Example 6. Let $q = 3$, and consider $y^5 + x^3y^3 + x^2 + x + 1 = 0$.

In this case, $q = 3$, $g = 6$ and we take $r = 2$. The gaps are $1, 2, 4, 5, 7, 8$. Furthermore, $l(3\infty) = 2$ and $l(6\infty) = l(8\infty) = 3$.

Therefore, we have a 2-gap structure. If we take $s = 3^m + 3^n$, then the order of vanishing of the function in $-s$ is 3. \square

4.3. Considering all the ideals: Case $q = 2$. Now, we need to define a zeta function that involves not only the principal ideals but all the ideals of A . We do it as it is suggested in [7]. Let e be the exponent of the ideal class group of A . Let s be a multiple of e , and define \mathcal{I}^s as $a^{s/e}$, where a is the monic generator of \mathcal{I}^e . Then, we can define the zeta function as follows: for s , an integer multiple of e ,

$$\zeta(s, X) := \zeta_A(s, X) := \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \text{ ideal of } A}} \mathcal{I}^{-s},$$

$$\zeta(s) := \zeta_A(s) := \zeta(s, 1).$$

Thakur and Diaz-Vargas, in [2], generalize Theorem 5, including now in the calculation of the zeta function all the ideals, not only the principal ones.

Theorem 8. *Let e be the exponent of the ideal class group and $q = 2$. Let $A = \mathbf{F}_2[x, y]$ be given by $y^2 - a(x)y = b(x)$ or $y^2 - y = b(x)$, where degree of $x = 2$ and degree of $y = N$ is an odd number. Assume that \mathcal{I}_k , $k = 1, \dots, h - 1$, are the integral ideals representing all the nontrivial classes of ideals, with $\deg(\mathcal{I}_k) = d_k$ of order e_k , as an element of the ideal class group. Moreover, assume that $\mathcal{I}_k^{e_k} = f_k$ is an irreducible polynomial and that it divides to $b(x)$. If $N > 2\mu + e_k d_k$ for all k , then $\zeta(-es)$, where s is a positive integer, has order of vanishing at least two, if $l_q(es) \leq \mu$.*

The next example satisfies the conditions of the theorem (for details, see [2]).

Example 7. Let $A = \mathbf{F}_2[x, y]/(y^2 + x(x + 1)y + x(x + 1)(x^5 + x^3 + x^2 + x + 1))$.

4.4. Considering all the ideals: Case $q = p^n$. The main result of this work is in the following theorem which is a generalization of Theorem 7, but now considering all the ideals. The proof is a partial “generalization” of the proof of Theorem 8.

Theorem 9. *Let e be the exponent of the ideal class group and $q = p^n$. Let $K = \mathbf{F}_q(x, y)$ be given by $y^q - a(x)^{q-1}y = b(x)$ with an r -gap structure at ∞ . Suppose that $\deg y = N > q \deg a(x)$ is relatively prime to p , and let $u = (b(x)/a(x)^q)$. Let $m(x)$ be an irreducible polynomial in $\mathbf{F}_q[x]$ such that, if $u = m(x)^n z$, $z \in \mathbf{F}_q(x)$, $m(x)$ does not divide z , and $n < 0$, then n is relatively prime to p . Assume that \mathcal{I}_k , $k = 1, 2, \dots, h - 1$, are integral ideals representing all the nontrivial ideal classes, with $\deg(\mathcal{I}_k) = d_k$, and order e_k , as an element of the ideal class group. Moreover, assume that $\mathcal{I}_k^{e_k} = f_k$ is irreducible and divides $b(x)$. If $N > q\mu + e_k d_k$ for all k , then $\zeta(-es)$, where s is a positive integer, s a multiple of $q - 1$, has multiplicity at least q , if $(l_q(es)/q - 1) \leq \mu$.*

Proof. It follows from [5, page 117] that x has order q and ∞ is a place of degree 1. Furthermore, we have

$$\begin{aligned} \zeta(-es, X) &= \sum_{d=0}^{\infty} X^d \sum_{\deg \mathcal{I}=d} \mathcal{I}^{es} = \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg a=d \\ a \text{ monic}}} a^{es} \\ &\quad + \sum_{k=1}^{h-1} \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \sim \mathcal{I}_k^{-1}}} \mathcal{I}^{es}. \end{aligned}$$

Since K has an r -gap structure, the principal part

$$\sum_{d=0}^{\infty} X^d \sum_{\substack{\deg a=d \\ a \text{ monic}}} a^{es}$$

contributes to the order of vanishing in q by Theorem 7. In fact, it was proved that

$$\sum_{d=0}^{\infty} X^d \sum_{\substack{\deg a=d \\ a \text{ monic}}} a^{es} = \zeta_{\mathbf{F}_q[x]}(-es, X^q).$$

Also, $\mathcal{I}_k^{e_k} = f_k$ has $\deg = e_k d_k < N$, and so $f_k = f_k(x)$, that is, $f_k \in \mathbf{F}_q[x]$. Write $e/e_k = s_k$. From here, the k -term is

$$\sum_{d=0}^{\infty} X^d \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \sim \mathcal{I}_k^{-1}}} \mathcal{I}^{e s} = \frac{1}{f_k^{s s_k}} \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \sim \mathcal{I}_k^{-1}}} (\mathcal{I}_k \mathcal{I})^{e s}.$$

We want to show that the vanishing order, for each k , is q (hence, the total order of vanishing is at least q). Here, $f_k^{s s_k}$ is independent of d , and, therefore, we can ignore it for vanishing considerations.

Now, $\mathcal{I}_k \mathcal{I}$ are integral of degree $d + d_k$ of the form $g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x)$ if and only if

$$\frac{g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x)}{\mathcal{I}_k} = \mathcal{I}$$

is integral, if and only if,

$$\begin{aligned} & \frac{(g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x))^{e_k}}{\mathcal{I}_k^{e_k}} \\ &= \frac{(g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x))^{e_k}}{f_k} \in A, \end{aligned}$$

i.e.,

$$(g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x))^{e_k} \equiv 0 \pmod{f_k}.$$

Then, $y^q - a(x)y^{q-1} = b(x)$ implies that

$$y^{q+j} \equiv a(x)y^{q-1}y^{1+j} \pmod{f_k},$$

and therefore,

$$\begin{aligned} & (g_0(x) + y g_1(x) + y^2 g_2(x) + \dots + y^{q-1} g_{q-1}(x))^{e_k} \equiv \\ & g_0^{e_k}(x) + y g_1'(x) + y^2 g_2'(x) + \dots + y^{q-1} g_{q-1}'(x) \pmod{f_k}. \end{aligned}$$

So, if this is congruent to zero (mod f_k), $f_k \mid g_0^{e_k}$ which implies that $f_k \mid g_0$, since f_k is an irreducible polynomial in x . So, we have

$$\begin{aligned} & \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \sim \mathcal{I}_k^{-1}}} (\mathcal{I}_k \mathcal{I})^{es} \\ &= \sum_{\deg f_k \bar{g}_0 + yg_1 + \dots + y^{q-1} g_{q-1} = d + d_k} (f_k \bar{g}_0 + yg_1 + \dots + y^{q-1} g_{q-1})^{es}, \end{aligned}$$

where \bar{g}_0 is a polynomial in x , and the bar denotes the action of “dividing by f_k .” We now examine sufficient conditions for the vanishing of the previous sum. If $q\mu + e_k d_k < d + d_k$, then $d + d_k - e_k d_k > q\mu > \mu > (l_q(es)/q - 1)$ implies that the sum vanishes. So, without loss of generality, $d + d_k \leq q\mu + e_k d_k < N$, and these are the only terms which can give nonzero contributions. As $d + d_k < N$, then $g_1 = \dots = g_{q-1} = 0$. Then, degree $\bar{g}_0 = d + d_k - e_k d_k \leq q\mu$. Therefore, the k th term of the zeta sum (up to $f_k^{-ess_k}$ factor which can be ignored as already mentioned) is the following:

$$\begin{aligned} & \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg \mathcal{I}=d \\ \mathcal{I} \sim \mathcal{I}_k^{-1}}} (\mathcal{I}_k \mathcal{I})^{es} \\ &= \sum_{d=0}^{\infty} X^d \sum_{\deg f_k \bar{g}_0 = d + d_k} (f_k \bar{g}_0)^{es} \\ &= X^{e_k d_k - d_k} f_k^{es} + \dots + X^{e_k d_k - d_k + q\mu} \\ & \quad \times \sum_{a_{\mu-1}, \dots, a_0 \in \mathbf{F}_q} f_k^{es} (x^\mu + a_{\mu-1} x^{\mu-1} + \dots + a_0)^{es} \\ &= f_k^{es} X^{e_k d_k - d_k} \left(1 + \dots + X^{q\mu} \right. \\ & \quad \left. \times \sum_{a_{\mu-1}, \dots, a_0 \in \mathbf{F}_q} (x^\mu + a_{\mu-1} x^{\mu-1} + \dots + a_0)^{es} \right) \\ &= f_k^{es} X^{e_k d_k - d_k} \left(\sum_{d=0}^{\mu} X^{qd} \sum_{\deg_x \bar{g}_0 = d} \bar{g}_0^{es} \right) \\ &= f_k^{es} X^{e_k d_k - d_k} \zeta_{\mathbf{F}_q[x]}(-es, X^q). \end{aligned}$$

In order to prove the last equality, observe that by Theorem 7, since $\mu > (l_q(es)/q - 1)$ implies that

$$\sum_{d=0}^{\mu} X^{qd} \sum_{\deg_x \bar{g}_0=d} \bar{g}_0^{es} = \sum_{d=0}^{\infty} X^{qd} \sum_{\deg_x \bar{g}_0=d} \bar{g}_0^{es} = \zeta_{\mathbb{F}_q[x]}(-es, X^q).$$

Now, since the order of vanishing of $\zeta_{\mathbb{F}_q[x]}(-es, X)$ is one, we have that, for each k , the order of vanishing of the k th term is q . \square

Remark 1. From the proof of Theorem 9, it follows that when $(l_q(es)/q - 1) \leq \mu$, we have

$$\zeta(-es, X) = \zeta_{\mathbb{F}_q[x]}(-es, X^q) \left(1 + \sum_{k=1}^{h-1} f_k^{(es/e_k)(e_k-1)} X^{e_k d_k - d_k} \right).$$

Then, the order of vanishing is exactly q when

$$1 + \sum_{k=1}^{h-1} f_k^{(es/e_k)(e_k-1)} \neq 0.$$

In particular, when $h = 2$ the order of vanishing is exactly q .

In our search for examples of Theorem 9 we looked first at $q = 2$. We have

Proposition 1. *Suppose that $q = 2$ and K is hyperelliptic. If K has an r -gap structure at ∞ , then $g - 1 \leq r \leq g$, where g is the genus of K .*

Proof. Assume that $r = g - k$, with $k \geq 2$. Then, since K has an r -gap structure, $l(2r\infty) = l((2g - 2k)\infty) = g - k + 1$, and $l((g + r)\infty) = l((2g - k)\infty) = g - k + 1$. So, between $2g - 2k + 1 \leq n \leq 2g - k$ all the numbers n are gaps. Observe that, as $k \geq 2$, then there is a gap at an even number, a contradiction. \square

Remark 2. Still, we do not have an example of the phenomenon described in Theorem 9. We contacted J.F. Voloch, whose comment

was that it is probable that a field satisfying the hypothesis of the theorem exists, but that the method used by him to find Example 5 is no longer useful in our case; because, apart from asking for the curve geometric conditions, we are asking also for conditions that are not geometric, for example, the type of curve, the class number, etc. So, he recommended that we do a computational search using the computer program Kant/Kash. But we have not yet been successful.

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