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ON OSCILLATION PROPERTIES FOR LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. In this paper, by using the nonlinear functional called negativity-preserving, some new oscillation criteria for linear matrix Hamiltonian systems are established. Our main theorems are of the form that the upper limit of the largest eigenvalue of the coefficient matrices are bounded away from a positive constant, rather than tending to infinity. Our results are generalizations of a recent paper due to Sun [**24**].

1. Introduction. We consider oscillatory properties for the linear Hamiltonian system

$$(1.1)\qquad \begin{cases} x' = A(t)x + B(t)u, \\ u' = C(t)x - A^*(t)u \quad t \geq t_0, \end{cases}$$

where $A(t)$, $B(t)$ and $C(t)$ are real $n \times n$ matrix-valued functions, B and C are Hermitian and B is positive definite. By M^* we mean the conjugate transpose of the matrix M; for any $n \times n$ Hermitian matrix M, its eigenvalues are real numbers. We always denote them by $\lambda_1[M] \geq \lambda_2[M] \geq \cdots \geq \lambda_n[M]$. The trace of M is denoted by $\operatorname{tr}(M)$ and $\operatorname{tr}(M) = \sum_{k=1}^n \lambda_k(M)$.

We also consider the corresponding matrix system

$$(1.2)\qquad \begin{cases} X' = A(t)X + B(t)U, \\ U' = C(t)X - A^*(t)U \quad t \geq t_0. \end{cases}$$

For any two solutions $(X_1(t), U_1(t))$ and $(X_2(t), U_2(t))$ of system (1.2), the Wronskian matrix $X_1^*(t)U_2(t) - U_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution $(X(t), U(t))$ of system (1.2), $X^*(t)U(t) - U^*(t)X(t)$ is a constant matrix.

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A solution $(X(t), U(t))$ of system (1.2) is said to be nontrivial if $\det X(t) \neq 0$ is fulfilled for at least one $t \geq t_0$. A nontrivial solution $(X(t), U(t))$ of system (1.2) is said to be conjoined (prepared) if $X^*(t)U(t) - U^*(t)X(t) \equiv 0$, $t \geq t_0$. A conjoined solution $(X(t), U(t))$ of (1.2) is said to be a conjoined basis of (1.1), or (1.2), if the rank of the $2n \times n$ matrix $\binom{X(t)}{U(t)}$ is n.

Two distinct points a, b in $[t_0, \infty)$ are said to be (mutually) conjugate with respect to (1.1) if there exists a solution $(x(t), u(t))$ of (1.1) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ on the subinterval with endpoints a and b. The system (1.1) is said to be conjugate on a subinterval J of $[t_0, \infty)$ if no two distinct points are conjugate. If (1.1) is conjugate on J and $(X(t), U(t))$ is the conjoined basis of (1.2) satisfying $X(a) = 0, U(a) = I$, the identity $n \times n$ matrix, $a \in J$, then $\det X(t) \neq 0$ for $t \in J\{a\}$. A conjoined basis $(X(t), U(t))$ of system (1.2) is said to be oscillatory in case the determinant of $X(t)$ vanishes on $[T, \infty)$ for each $T \geq t_0$. We note that the definition of oscillation agrees with the nondisconjugacy of system (1.1), or (1.2), on any neighborhood of $+\infty$.

When $A(t) \equiv 0$, system (1.2) reduces to the second order self-adjoint matrix differential system

$$(P(t)Y')' + Q(t)Y = 0, \quad t \geq t_0 \tag{1.3}$$

with $P(t) = B^{-1}(t)$ is positive definite, and $Q(t) = -C(t)$. Oscillation and nonoscillation of system (1.3) and its special cases

$$Y'' + Q(t)Y = 0, \quad t \geq t_0 \tag{1.4}$$

have been extensively studied by many authors, see [**1**–**8**, **15**, **20**, **21**, **25**] and the references contained therein. Many of these criteria are modeled on the criteria for the scalar equation

$$(p(t)y')' + q(t)y = 0, \quad t \geq t_0 \tag{1.5}$$

$$y'' + q(t)y = 0, \quad t \geq t_0. \tag{1.6}$$

Here we list some known criteria for equation (1.6):

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)\, ds = \infty, \quad \text{(Fite-Wintner-Leighton [\textbf{15}])}; \tag{1.7}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\, d\tau\, ds = \infty, \quad \text{(Wintner [}\mathbf{22}\text{]);} \tag{1.8}$$

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\, d\tau\, ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\, d\tau\, ds \le \infty; \tag{1.9}$$

see Hartman [**13**]; and

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds = \infty \tag{1.10}$$

for some integer $n > 1$, see Kamenev [**11**]. In 1989, Kamenev's theorem was extended by Philos, see [**23**], with the function class $H(t,s)$. However, all these criteria cannot be applied to the Euler differential equation

$$y'' + \frac{\gamma}{t^2} y = 0, \quad \gamma > 0. \tag{1.11}$$

In 1995, Li [**17**] improved Kamenev's theorem (including Philos's theorem) by using a generalized Riccati transformation, which is applicable to the oscillation for equation (1.11) with $\gamma > 1/4$. In 1999, Kong [**13**] introduced a new interval criteria for linear equation, which is also applicable for equation (1.11).

For systems (1.3) and (1.4), Etgen and Pawlowski [**8**] showed that system (1.3) is oscillatory provided

$$[g(P(t))y']' + g(Q(t))y = 0 \tag{1.12}$$

is oscillatory, where g is a positive linear functional. So all the results can be generalized to system (1.3). Using the nonlinear functional, it was conjectured by Hinton and Lewis [**9**] that (1.4) is oscillatory if

$$\lim_{t\to\infty} \lambda_1 \left(\int_{t_0}^{t} Q(s)\, ds \right) = \infty. \tag{1.13}$$

This conjecture was partially proved by several authors and finally settled by Byers, Harris and Kwong [**2**]. Coles [**3, 5**] extended this result by applying the weighted average method. Butler, Erbe and Mingarelli [**1**] showed that (1.4) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \lambda_1\left[\int_{t_0}^{s}(Q_1(\tau)\,d\tau\right]ds = \infty \tag{1.14}$$

provided

$$\liminf_{t\to\infty} \frac{1}{t}\int_{a}^{t}\int_{a}^{s} \operatorname{tr}(Q(\tau))\,d\tau\,ds > -\infty.$$

In 1995, Erbe, Kong and Ruan [**7**] gave the following Kamenev type oscillation criteria.

Theorem 1.1. *Suppose that there exists a constant $\alpha > 1$ such that*

$$\limsup_{t\to\infty} \frac{1}{t^\alpha}\lambda_1\left[\int_{t_0}^{t}(t-s)^\alpha Q(s)\,ds\right] = \infty. \tag{1.15}$$

Then system (1.4) *is oscillatory.*

However, the results mentioned above cannot be applied to the second order Euler differential system

$$Y'' + \operatorname{diag}\left(\frac{\alpha}{t^2},\frac{\beta}{t^2}\right)Y = 0, \quad t \ge 1, \quad \alpha \ge \beta > 0. \tag{1.16}$$

In 1998, Meng, Wang and Zheng [**21**] generalized Theorem 1.1 which can be applied to the Euler differential system. In 1999, Kong [**13**] obtained the following theorem which is also applicable for oscillation of the Euler differential system.

Theorem 1.2. *System* (1.4) *is oscillatory provided that for each $r \ge t_0$ and for some $\lambda \ge 1$, either*

i) *the following two inequalities hold*:

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}}\int_{r}^{t}(s-r)^\lambda \operatorname{tr}\left(Q(s)\right)ds \ge \frac{n\lambda^2}{4(\lambda-1)} \tag{1.17}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \int_r^t (t-s)^\lambda \mathrm{tr}\,(Q(s))\,ds \geq \frac{n\lambda^2}{4(\lambda-1)}; \tag{1.18}$$

or

ii) *the following inequality holds*:

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda-1}} \lambda_1 \bigg(\int_r^t (s-r)^\lambda [Q(s)+Q(2t-s)]\,ds \bigg) \geq \frac{\lambda^2}{2(\lambda-1)}. \tag{1.19}$$

In 2004, with the auxiliary function $(t-s)^{2\alpha}(s-r)^2$, $\alpha > 1/2$, Sun [**24**] improved Kong's theorem and obtained oscillation criteria which was the upper limit of the coefficients larger than some constant; Dube and Mingarelli [**6**] generalized Sun's theorem with the auxiliary function $(t-s)^p(s-r)^q$, $p, q > 1$ to obtain oscillation criteria.

The oscillation for the Hamiltonian system (1.2) also has been investigated by many authors, see [**14, 19, 20, 24, 28, 29, 30, 31** et al.]. Most of these oscillation criteria involve the fundamental matrix $\Phi(t)$ for the linear system $v' = A(t)v$. This eliminates the applications of these criteria because such a system cannot be solved if $A(t)$ is of variation. Moreover, by using the transformation

$$\begin{bmatrix} \overline{X} \\ \overline{U} \end{bmatrix} = \begin{pmatrix} \Phi^{-1}(t) & 0 \\ 0 & \Phi^*(t) \end{pmatrix} \begin{bmatrix} X \\ U \end{bmatrix}, \tag{1.20}$$

we can transform system (1.2) into the following Hamiltonian system

$$\begin{cases} \overline{X}' = \Phi^*(t)B(t)\Phi(t)\overline{U}, \\ \overline{U}' = \Phi^{-1}(t)C(t)\Phi^{*-1}(t)\overline{X} \quad t \geq t_0. \end{cases} \tag{1.21}$$

Now system (1.21) becomes (1.3) with $P(t) = \Phi^{-1}(t)B^{-1}(t)\Phi^{*-1}(t)$ and $Q(t) = -\Phi^{-1}(t)C(t)\Phi^{*-1}(t)$. So those criteria are similar to that of system (1.3). In paper [**20**], the authors obtain oscillation criteria with the fundamental matrix $\Phi(t)$ as follows:

Theorem 1.3 [**20**, Corollary 2.5]**.** *Suppose that there exist* $a(t) \in C^1([t_0,\infty);\mathbf{R}^+)$ *and* $f(t) = -a'(t)/2a(t)$ *such that* fB^{-1} *is differentiable. If, for each* $r \geq t_0$,

(1.22)

$$\limsup_{t\to\infty} \lambda_1 \bigg\{ \int_r^t [(t-s)^2(s-r)^2 C_2(s) - (t+r-2s)^2 B_2(s)]\,ds \bigg\}\,ds > 0.$$

Then system (1.2) *is oscillatory, where* $B_2(t) = a(t)\Phi^*(t)B^{-1}(t)\Phi(t)$ *and*

$$C_2(t) = -a(t)\Phi^*(t)\{C+f(A^*B^{-1}+B^{-1}A)-f^2B^{-1}+(fB^{-1})'\}(t)\Phi(t).$$

We note that most of the results mentioned above depend on the linear functionals or the largest eigenvalue of the matrices under consideration. In 2002, Meng and Mingarelli [**18**] obtained oscillation criteria of Kamenev type for system (1.2) with the fundamental matrix $\Phi(t)$ by introducing a monotone subhomogeneous functional of degree c, $c > 0$, on a suitable matrix space. Mingarelli [**22**] introduced a new nonlinear functional called negativity-preserving and obtained oscillation criteria for system (1.3), which also can be applied to system (1.2).

On the other hand, without the fundamental matrix $\Phi(t)$, Kumari and Umanaheswaram [**6**] obtained oscillation criteria of Kamenev type, which generalized the results due to Erbe, Kong and Ruan [**7**], also Meng, Wang and Zheng [**21**]; using linear functional and integral average methods, Yang, Mathsen and Zhu [**29**] obtained oscillation criteria of Wintner type; and Zheng [**31**] obtained oscillation criteria of interval type.

In paper [**24**], by multiplying a ternary function $\phi(t,s,r) = (t-s)^2(s-r)^\alpha$, the author obtained oscillation for system (1.2). Here we list the main result as follows.

Theorem 1.4 [**7**, Theorem 2]**.** *Suppose that there exists an* $a(t) \in C([t_0,\infty);\mathbf{R}^+)$ *such that* $a(t)B^{-1}(t) \le I$ *and* $f(t)B^{-1}(t)$ *are differentiable, where* $f(t) = -a'(t)/a(t)$*. Then system* (1.2) *is oscillatory provided for some* $\alpha > 1/2$ *and, for each* $r \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}}\lambda_1\bigg\{\int_r^t (t-s)^2(s-r)^{2\alpha} \times \left(D_1(s) + 2\frac{\alpha t-(\alpha+1)s+r}{(t-s)(s-r)}K(s)\right)ds\bigg\} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{1.23}$$

where $D_1(t) = D(t) - (aA^*B^{-1}A)(t)$, $K(t) = (a(t)/2)(A^*B^{-1} + B^{-1}A)(t)$ *and* $D(t) = \{a[-C-2fK+f^2B^{-1}-(fB^{-1})']\}(t)$.

Theorem 1.5 [**29**, Theorem 3]. *Suppose that there exists an* $a(t) \in C([t_0, \infty); \mathbf{R}^+)$ *such that* $a(t)B^{-1}(t) \leq I$ *and* $f(t)B^{-1}(t)$ *is differentiable, where* $f(t) = -a'(t)/a(t)$. *Then system* (1.2) *is oscillatory provided for some* $\alpha > 1/2$ *and, for each* $r \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \bigg\{ \int_r^t (t-s)^{2\alpha}(s-r)^2 \times \bigg(D_1(s) + 2\frac{t-(\alpha+1)s+\alpha r}{(t-s)(s-r)} K(s) \bigg)\, ds \bigg\} > \frac{\alpha}{(2\alpha-1)(2\alpha+1)},$$

where $D_1(t)$ *and* $K(t)$ *are defined as above.*

In this paper, using a more generalized Riccati transformation and matrix analysis technique, we obtain some new oscillation criteria for system (1.2), which extend and improve the oscillation criteria mentioned above.

2. Main results. In what follows, we denote by $\mathbf{M}$ the linear space of $n \times n$ real matrices, and by $\mathbf{S}$ the subspace of all symmetric matrices in $\mathbf{M}$. For any $A, B \in \mathbf{S}$, $A \geq B$ means that $A - B \geq 0$ is positive semi-definite, and $A > B$ means that $A - B > 0$ is positive definite.

A nonlinear (and possibly discontinuous) functional $q : \mathbf{S} \to \mathbf{R}$ with $q(A) \leq 0$ whenever $A \leq 0$ is called negativity-preserving and the class of all such negativity-preserving functionals on $\mathbf{S}$ is denoted by $\mathcal{N}(\mathbf{S})$. The negativity-preserving functionals $\mathcal{N}(\mathbf{S})$ contain most known functionals used in oscillation, for example, $q(A) = \lambda_1(A)$; $q(A) = \mathrm{tr}\,(A-P)$ where P is positive semi-definite and fixed, and are of negativity-preserving functionals. In addition,

$$q(A) = \frac{\lambda_1(A)}{1-\lambda_1(A)}; q(A) = a_{ii}, \quad 1 \leq i \leq n,$$

are also negativity-preserving functionals. We also note that any positive linear functional is negativity-preserving. Thus, functionals in the class $\mathcal{N}(\mathbf{S})$ make up all those being used in the current study of matrix oscillation theory.

Firstly, we give the main oscillation criteria for system (1.2) using the largest eigenvalue functional.

Theorem 2.1. *Suppose that there exists an* $a(t) \in C([t_0, \infty); \mathbf{R}^+)$ *such that* $a(t)B^{-1}(t) \leq I$ *and* $f(t)B^{-1}(t)$ *is differentiable, where* $f(t) = -a'(t)/a(t)$*. If for each* $r \geq t_0$ *and, for some* $\mu, \nu > 1$,

$$\limsup_{t\to\infty} \frac{1}{t^{\mu+\nu-1}} \lambda_1 \left\{ \int_r^t (t-s)^\mu (s-r)^\nu \left(D_1(s) + \frac{\nu t - (\mu+\nu)s + \mu r}{(t-s)(s-r)} K(s) \right) ds \right\} > \mu\nu(\mu+\nu-2) \frac{\Gamma(\mu-1)\Gamma(\nu-1)}{4\Gamma(\mu+\nu)},$$

where $D_1(t)$ *and* $K(t)$ *are defined as in Theorem* 1.4. *Then system* (1.2) *is oscillatory.*

Proof. Suppose to the contrary that there exists a conjoined basis $(X(t), U(t))$ of (1.2) which is not oscillatory. Without loss of generality, we may suppose that $\det X(t) \neq 0$ for $t \geq t_0$. Define

$$W(t) = a(t) \left\{ U(t)X^{-1}(t) + f(t)B^{-1}(t) \right\}, \quad t \geq t_0. \tag{2.2}$$

Then $W(t)$ is Hermitian, and satisfies the Riccati equation

$$W' = -\frac{1}{a} WBW - A^*W - WA - D. \tag{2.3}$$

Let $V(t) = W(t) + (aB^{-1}A)(t)$. Then we have

$$\frac{1}{2}(V + V^*) = W + K, \tag{2.4}$$

$$\frac{1}{a} V^*BV = \frac{1}{a} W^*BW + A^*W + WA + aA^*B^{-1}A. \tag{2.5}$$

So we have by (2.3), (2.4) and (2.5) that for $t \geq t_0$,

$$W' = -\frac{1}{a} V^*BV - D_1. \tag{2.6}$$

For simplicity, set $g(t,s,r) = (t-s)^{\mu}(s-r)^{\nu}$. We get

$$\begin{aligned} g_s'(t,s,r) =& g(t,s,r)\frac{\nu t - (\mu+\nu)s + \mu r}{(t-s)(s-r)} \\ &= (t-s)^{\mu/2}(s-r)^{\nu/2} \\ &\times \left[\nu(t-s)^{\mu/2}(s-r)^{(\nu/2)-1} - \mu(t-s)^{(\mu/2)-1}(s-r)^{\nu/2}\right] \\ &:= \sqrt{g(t,s,r)}h(t,s,r), \end{aligned}$$

where $h(t,s,r) = \nu(t-s)^{\mu/2}(s-r)^{(\nu/2)-1} - \mu(t-s)^{(\mu/2)-1}(s-r)^{\nu/2}$.

Since $B(t) > 0$, we may define $R(t) = B^{1/2}(t)$. Set

$$\Phi(t,s,r) = R(s)\left[\sqrt{g(t,s,r)}V(s) - \frac{1}{2}a(s)h(t,s,r)B^{-1}(s)\right]R(s).$$

We have

$$\frac{1}{a}R^{-1}\Phi^*\Phi R^{-1} = \frac{g}{a}V^*BV - \frac{1}{2}\sqrt{g}h(V^*+V) + \frac{1}{4}ah^2B^{-1}.$$

For each $r \geq t_0$ and $\mu, \nu > 1$, by multiplying (2.6) with $g(t,s,r)$ and integrating it from r to t, $t > r$, we get

$$\begin{aligned} \int_r^t g(t,s,r)D_1(s)\,ds &= -\int_r^t g(t,s,r)W'(s)\,ds \\ &\quad - \int_r^t \frac{g(t,s,r)}{a(s)}(V^*BV)(s)\,dt \\ &= -g(t,s,r)W(s)|_r^t \\ &\quad + \int_r^t \sqrt{g(t,s,r)}h(t,s,r)W(s)\,ds \\ &\quad - \int_r^t \frac{g(t,s,r)}{a(s)}(V^*BV)(s)\,ds \\ &= \int_r^t \frac{1}{2}\sqrt{g}h(V^*+V-2K)\,ds \\ &\quad - \int_r^t \frac{g}{a}V^*BV\,ds. \end{aligned} \tag{2.7}$$

Consequently,

$$\begin{aligned}
(2.8)\qquad & \int_r^t (t-s)^\mu (s-r)^\nu \left(D_1(s) + \frac{\nu t-(\mu+\nu)s+\mu r}{(t-s)(s-r)}K(s)\right)\,ds \\
&= \int_r^t \left[g(t,s,r)D_1(s) + \sqrt{g(t,s,r)}h(t,s,r)K(s)\right]\,ds \\
&= \int_r^t \frac{1}{2}\sqrt{g}h(V^*+V)ds - \int_r^t \frac{g}{a}V^*BV\,ds \\
&= -\int_r^t \frac{1}{a}R^{-1}\Phi^*\Phi R^{-1}\,ds + \int_r^t \frac{1}{4}ah^2B^{-1}\,ds \\
&\le \frac{1}{4}\int_r^t a(s)h^2(t,s,r)B^{-1}(s)\,ds \le \frac{1}{4}\int_r^t h^2(t,s,r)I\,ds \\
&= \frac{1}{4}\int_r^t \left[\nu(t-s)^{\mu/2}(s-r)^{(\nu/2)-1} - \mu(t-s)^{(\mu/2)-1}(s-r)^{\nu/2}\right]^2 I\,ds \\
&= \frac{1}{4}\bigg[\int_r^t \nu^2(t-s)^\mu(s-r)^{\nu-2}\,ds \\
&\qquad - 2\mu\nu\int_r^t (t-s)^{\mu-1}(s-r)^{\nu-1}\,ds \\
&\qquad + \mu^2\int_r^t (t-s)^{\mu-2}(s-r)^\nu\,ds\bigg]I.
\end{aligned}$$

Now we compute these three integrals in (2.8) using Euler's beta function

$$(2.9)\qquad \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}\,(\alpha,\beta)>0.$$

We have

$$(2.10)\qquad \begin{aligned}
\int_r^t (t-s)^\mu(s-r)^{\nu-2}\,ds &= \int_0^{t-r} w^{\nu-2}(t-r-w)^\mu\,dw \\
&= (t-r)^{\mu+\nu-1}\int_0^1 s^{\nu-2}(1-s)^\mu\,ds \\
&= (t-r)^{\mu+\nu-1}\frac{\Gamma(\nu-1)\Gamma(\mu+1)}{\Gamma(\mu+\nu)} \\
&= \mu(t-r)^{\mu+\nu-1}\frac{\Gamma(\nu-1)\Gamma(\mu)}{\Gamma(\mu+\nu)}.
\end{aligned}$$

Similarly, we have the second and the third integral in (2.8) as follows

$$\int_r^t (t-s)^{\mu-1}(s-r)^{\nu-1}\,ds = (t-r)^{\mu+\nu-1}\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\mu+\nu)}, \tag{2.11}$$

$$\begin{aligned}\int_r^t (t-s)^{\mu-2}(s-r)^{\nu}\,ds &= (t-r)^{\mu+\nu-1}\frac{\Gamma(\nu+1)\Gamma(\mu-1)}{\Gamma(\mu+\nu)}\\ &= \nu(t-r)^{\mu+\nu-1}\frac{\Gamma(\nu)\Gamma(\mu-1)}{\Gamma(\mu+\nu)}.\end{aligned} \tag{2.12}$$

Replacing the three integrals in (2.8) by (2.10), (2.11) and (2.12), we have

$$\begin{aligned}&\int_r^t (t-s)^{\mu}(s-r)^{\nu}\left(D_1(s)+\frac{\nu t-(\mu+\nu)s+\mu r}{(t-s)(s-r)}K(s)\right)ds\\ &\quad\le \frac{1}{4}(t-r)^{\mu+\nu-1}\left[\frac{\mu\nu^2\Gamma(\nu-1)\Gamma(\mu)}{\Gamma(\mu+\nu)}\right.\\ &\qquad\qquad\left.-\frac{2\mu\nu\Gamma(\nu)\Gamma(\mu)}{\Gamma(\mu+\nu)}+\frac{\mu^2\nu\Gamma(\nu)\Gamma(\mu-1)}{\Gamma(\mu+\nu)}\right]I\\ &\quad= (t-r)^{\mu+\nu-1}\frac{\mu\nu}{4\Gamma(\mu+\nu)}\\ &\quad\times\left[\nu\Gamma(\nu-1)\Gamma(\mu)-2\mu\nu\Gamma(\nu)\Gamma(\mu)+\mu\Gamma(\nu)\Gamma(\mu-1)\right]I\\ &\quad= (t-r)^{\mu+\nu-1}\mu\nu(\mu+\nu-2)\frac{\Gamma(\mu-1)\Gamma(\nu-1)}{4\Gamma(\mu+\nu)}I.\end{aligned} \tag{2.13}$$

Dividing both sides with $t^{\mu+\nu-1}$ and taking the largest eigenvalue, we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t^{\mu+\nu-1}}\lambda_1\Bigg\{\int_r^t (t-s)^{\mu}(s-r)^{\nu}\\ \times\left(D_1(s)+\frac{\nu t-(\mu+\nu)s+\mu r}{(t-s)(s-r)}K(s)\right)ds\Bigg\}\\ \le \mu\nu(\mu+\nu-2)\frac{\Gamma(\mu-1)\Gamma(\nu-1)}{4\Gamma(\mu+\nu)}.\end{aligned}$$

This contradicts condition (2.14). The proof of Theorem 2.1 is completed. □

Remark 2.1. When $A(t) \equiv 0$, $f(t) \equiv 0$, Theorem 2.1 coincides with [**6**, Theorem 6] for system (1.4).

Remark 2.2. When $\mu = 2$ and $\nu = 2\alpha$, $\alpha > 1/2$, Theorem 2.1 reduces to Theorem 1.4. In fact, by (2.14), we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{2\alpha+1}} \lambda_1 \\
&\times \left\{ \int_r^t (t-s)^2 (s-r)^{2\alpha} \left(D_1(s) + 2\frac{\alpha t - (\alpha+1)s + r}{(t-s)(s-r)} K(s) \right) ds \right\} \\
&> 2\alpha \cdot 2(2 + 2\alpha - 2) \frac{\Gamma(2-1)\Gamma(2\alpha-1)}{4\Gamma(2\alpha+2)} \\
&\qquad = \frac{8\alpha^2 \Gamma(2\alpha-1)\Gamma(1)}{4(2\alpha+1)\cdot 2\alpha \cdot (2\alpha-1)\Gamma(2\alpha-1)} \\
&= \frac{\alpha}{(2\alpha-1)(2\alpha+1)}.
\end{aligned}$$

Similarly, if we choose $\mu = 2\alpha$, $\alpha > 1/2$ and $\nu = 2$, then Theorem 2.1 reduces to Theorem 1.5.

Remark 2.3. From the proof of Theorem 2.1, we note that our theorem cannot be applied to the critical case $\mu = \nu = 1$.

By Theorem 2.1, if $1 < \mu < 2$ and $\nu = 3 - \mu$, using the equality

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad \text{for} \quad 0 < p < 1,$$

we obtain the following useful corollary.

Corollary 2.1. *Suppose that there exists an $a(t) \in C([t_0, \infty); \mathbf{R}^+)$ such that $a(t)B^{-1}(t) \le I$ and $f(t)B^{-1}(t)$ are differentiable, where $f(t) = -a'(t)/a(t)$. If for each $r \ge t_0$ and, for some $1 < \mu < 2$,*

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t^2} \lambda_1 &\left\{ \int_r^t (t-s)^\mu (s-r)^{3-\mu} \right. \\
&\left. \left(D_1(s) + \frac{(3-\mu)t - 3s + \mu r}{(t-s)(s-r)} K(s) \right) ds \right\} \\
&\qquad > \frac{\mu(3-\mu)\pi}{8 \sin(\mu-1)\pi},
\end{aligned} \tag{2.14}$$

where $D_1(t)$ and $K(t)$ are defined as in Theorem 1.4. *Then system* (1.2) *is oscillatory.*

In a similar manner, by using the negativity-preserving functional, we have the following theorem.

Theorem 2.2. *Suppose that there exists an $a(t) \in C([t_0, \infty); \mathbf{R}^+)$ such that $a(t)B^{-1}(t) \leq I$ and $f(t)B^{-1}(t)$ are differentiable, where $f(t) = -a'(t)/a(t)$. If, for each $r \geq t_0$ and for some $\mu, \nu > 1$, there exists a q in $\mathcal{N}(\mathbf{S})$ such that*

$$(2.15) \quad \limsup_{t\to\infty} q\left(\frac{1}{t^{\mu+\nu-1}}\int_r^t (t-s)^\mu (s-r)^\nu \left[D_1(s) + \frac{\nu t - (\mu+\nu)s + \mu r}{(t-s)(s-r)} K(s)\right] ds - \mu\nu(\mu+\nu-2)\frac{\Gamma(\mu-1)\Gamma(\nu-1)}{4\Gamma(\mu+\nu)}\left(1-\frac{r}{t}\right)^{\mu+\nu-1} I\right) > 0,$$

where $D_1(t)$ and $K(t)$ are defined as in Theorem 1.1. *Then system* (1.2) *is oscillatory.*

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