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DISTORTION MINIMAL MORPHING: THE THEORY FOR STRETCHING

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ABSTRACT. A morph between two isotopic compact connected oriented smooth n-manifolds without boundary embedded into $\mathbf{R}^{n+1}$ is an isotopy between them together with all the intermediate states equipped with the volume forms generated by the usual volume form on $\mathbf{R}^{n+1}$. We consider the problem of distortion minimal morphing of two smooth n-manifolds. Distortion involves bending and stretching. In this paper, distortion (with respect to stretching) is defined as the total infinitesimal relative change in volume. We prove the existence of minimal distortion diffeomorphisms and morphs between isotopic manifolds.

1. Introduction. A morph is a transformation between two shapes through a set of intermediate shapes. A minimal morph is such a transformation that minimizes distortion.

There are important applications of minimal morphing in manufacturing [**9, 13**], computer graphics [**11, 12**], movie making [**6**], and mesh construction [**5, 8**]. To address these applications would require a theory of minimal morphing that includes bending and stretching together with algorithms to compute minimal morphs. In this paper, we formulate and solve the mathematical problem of minimal morphing with respect to stretching.

Minimal morphing is often considered as a numerical problem where a cost functional is minimized over a finite number of intermediate shapes, see [**9, 13**]. We introduce a theory of distortion minimal morphing over a continuous family of states in the context of morphs between n-dimensional oriented compact connected smooth manifolds without boundary embedded in $\mathbf{R}^{n+1}$ whose volume forms are generated by the usual volume form on $\mathbf{R}^{n+1}$. The natural cost functional (for stretching) measures the total relative change of volume with respect to a

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family of diffeomorphisms that defines the morph. This functional is invariant under compositions with volume preserving diffeomorphisms; hence, the corresponding minimal morphs are not unique. On the other hand, we prove that the extremals of our functional are (global) minimizers. Our main result is the existence of a distortion minimal morph (with respect to stretching) between every pair of isotopic submanifolds.

2. Minimal distortion diffeomorphisms. In this section we prove the existence of distortion minimal diffeomorphisms between diffeomorphic oriented compact connected n-manifolds M and N (which are not necessarily embedded in $\mathbf{R}^{n+1}$) with respective volume forms ω_M and ω_N.

Recall that the Jacobian determinant of a diffeomorphism $h : M \to N$ is defined by the equation

$$h^*\omega_N = J(\omega_M, \omega_N)(h)\,\omega_M,$$

where $h^*\omega_N$ denotes the pullback of the volume form ω_N on N by the diffeomorphism h, see [**1**]. The Jacobian determinant $J(h) := J(\omega_M, \omega_N)(h)$ depends on the diffeomorphism and the volume forms.

The distortion (due to stretching) $\xi(m)$ at $m \in M$, with respect to a diffeomorphism $h : M \to N$, is defined by

$$\xi(m) = \lim_{\varepsilon \to 0^+} \frac{\left| \int_{h(A_\varepsilon)} \omega_N \right| - \left| \int_{A_\varepsilon} \omega_M \right|}{\left| \int_{A_\varepsilon} \omega_M \right|} = \big|J(h)(m)\big| - 1, \tag{1}$$

where $A_\varepsilon \subset M$, for $\varepsilon > 0$, is a nested family of (open) neighborhoods of the point $m \in M$ such that $A_\alpha \supseteq A_\beta$ whenever $\alpha > \beta > 0$ and $\cap_{\varepsilon > 0} A_\varepsilon = m$.

In other words, *distortion* is the infinitesimal relative change of volume with respect to h. It is easy to see that the definition of distortion does not depend on the family of nested sets A_ε.

We denote the set of all smooth diffeomorphisms between manifolds M and N by $\mathrm{Diff}\,(M, N)$. The *total distortion* functional $\Phi : \mathrm{Diff}\,(M, N) \to \mathbf{R}$, with respect to oriented manifolds (M, ω_M) and (N, ω_N), is defined by

$$\Phi(h) = \int_M \Big(\big|J(h)\big| - 1\Big)^2\,\omega_M. \tag{2}$$

We will establish necessary and sufficient conditions for a diffeomorphism $h : M \to N$ to be a minimizer of the functional Φ. Also, we will show that a minimizer always exists in $\mathrm{Diff}\,(M, N)$.

As a useful notation, we let $\mathfrak{X}(M)$ denote the set of smooth vector fields on the manifold M. Also, we recall a basic fact from global nonlinear analysis: $\mathrm{Diff}\,(M, N)$ *is a Banach manifold and its tangent space at* $h \in \mathrm{Diff}\,(M, N)$ *can be identified with* $\mathfrak{X}(N)$, see [**7**]. Indeed, an element of $T_h\mathrm{Diff}\,(M, N)$ is an equivalence class of curves $[h_\varepsilon]$, represented by a family of diffeomorphisms h_ε with $h_0 = h$, where two curves passing through h are equivalent if they have the same derivative at h. For each $q \in N$, this family defines a curve $\epsilon \mapsto h_\epsilon(h^{-1}(q))$ in N that passes through q at $\varepsilon = 0$; hence, it defines a vector $Y(q) \in T_qN$ by

$$Y(q) := \frac{d}{d\varepsilon} h_\epsilon(h^{-1}(q))\Big|_{\varepsilon=0}.$$

The vector field $Y \in \mathfrak{X}(N)$ is thus associated with the equivalence class $[h_\varepsilon]$. In fact, the vector field Y does not depend on the choice of the representative of the equivalence class. On the other hand, for $Y \in \mathfrak{X}(N)$ with flow ϕ_t, we associate the curve $h_t = \phi_t \circ h$ in $\mathrm{Diff}\,(M, N)$. The (tangent) equivalence class of this curve is an element in $T_h\mathrm{Diff}\,(M, N)$.

Proposition 2.1 (Euler-Lagrange equation). *Suppose that* (M, ω_M) *and* (N, ω_N) *are smooth diffeomorphic connected compact oriented* n*-manifolds without boundary. A smooth diffeomorphism* $h : M \to N$ *is a critical point of the total distortion functional* Φ *if and only if* $J(h)$ *is constant.*

Proof. Let $h_\varepsilon : (-1, 1) \to \mathrm{Diff}\,(M, N)$ be a curve of diffeomorphisms from M to N such that $h_0 = h$. By definition, $h \in \mathrm{Diff}\,(M, N)$ is a critical point of the functional $\Phi(h)$, if $(d/dt)\Phi(h_t)|_{t=0} = 0$. Using the formula

$$\Phi(h) = \int_M J(h)^2 \omega_M - 2\ \mathrm{Vol}\,(N) + \mathrm{Vol}\,(M), \tag{3}$$

we note that h is a critical point of Φ if and only if

$$2 \int_M J(h) \frac{d}{dt} J(h_t)|_{t=0}\, \omega_M = 0.$$

Moreover, using the calculus of differential forms (see [**1**] and note in particular that L_Y is used to denote the Lie derivative in the direction of the vector field Y), we have that for $h^t = \psi_t \circ h$, where ψ_t is the flow of $Y \in \mathfrak{X}(N)$,

$$\begin{aligned}
\frac{d}{dt}(J(\psi_t \circ h)\omega_M)\Big|_{t=0} &= \frac{d}{dt}\big((\psi_t \circ h)^* \omega_N\big)\Big|_{t=0} \\
&= h^* \frac{d}{dt}(\psi_t^* \omega_N)\Big|_{t=0} \\
&= h^* \psi_t^* L_Y \omega_N\big|_{t=0} \\
&= h^* L_Y \omega_N \\
&= h^* (\operatorname{div} Y \omega_N) \\
&= (\operatorname{div} Y) \circ h J(h) \omega_M.
\end{aligned}$$

We will assume that h is orientation preserving. The proof for the orientation reversing case is similar. By Stokes's theorem and the properties of the $\wedge$-antiderivations d and i_Y, we have that

$$\begin{aligned}
\frac{d}{dt}\Phi(h_t)\Big|_{t=0} &= \int_M J(h)^2 \operatorname{div} Y \circ h \, \omega_M \\
&= \int_N J(h) \circ h^{-1} \operatorname{div} Y \omega_N \\
&= \int_N J(h) \circ h^{-1} L_Y \omega_N \\
&= \int_N J(h) \circ h^{-1} d\, i_Y \omega_N \\
&= \int_N d(J(h) \circ h^{-1} \wedge i_Y \omega_N) \\
&\quad - \int_N d(J(h) \circ h^{-1}) \wedge i_Y \omega_N \\
&= \int_N i_Y (d(J(h) \circ h^{-1}) \wedge \omega_N) \\
&\quad - \int_N i_Y \big(d(J(h) \circ h^{-1})\big)\, \omega_N \\
&= -\int_N d(J(h) \circ h^{-1})(Y)\, \omega_N.
\end{aligned}$$

Hence, $h \in \mathrm{Diff}\,(M,N)$ is a critical point of the functional $\Phi(h)$ if and only if

$$\int_N d(J(h)\circ h^{-1})(Y)\,\omega_N = 0$$

for all $Y \in \mathfrak{X}(N)$. It follows that if $J(h)$ is constant, then h is a critical point of Φ.

To complete the proof it suffices to show that if

$$\int_N df(Y)\,\omega_N = 0 \tag{4}$$

for all $Y \in \mathfrak{X}(N)$, then $df = 0$, where $f := J(h)\circ h^{-1}$.

Suppose, on the contrary, that there exists a continuous vector field $Y \in \mathfrak{X}(N)$ such that (without loss of generality) $df(Y)(q) > 0$ for some point $q \in N$. Because the map $df(Y) : N \to \mathbf{R}$ is continuous, there exists an open neighborhood $U \subset N$ of the point $q \in N$ such that $df(Y)(p) > 0$ for every $p \in U$. After multiplying the vector field Y by an appropriate bump function, see [**1**], we obtain a vector field $Z \in \mathfrak{X}(N)$ supported in U such that $\int_N df(Z)\omega_N = \int_U df(Z)\omega_N > 0$, in contradiction to equality (4). Hence, $df = 0$. □

Definition 2.2. A function $h \in \mathrm{Diff}\,(M,N)$ is called a *distortion minimal map* if it is a critical point of the total distortion functional Φ.

We will show that every distortion minimal map is a minimizer of the functional Φ, see Theorem 2.6.

As an immediate corollary of Proposition 2.1, we have the following theorem.

Theorem 2.3. *A function $h \in \mathrm{Diff}\,(M,N)$ is a distortion minimal map if and only if $|J(h)|$ is the constant function with value* $\mathrm{Vol}\,(N)/\mathrm{Vol}\,(M)$.

We will use the elementary properties of distortion minimal maps stated in the following lemma. The proof is left to the reader.

Lemma 2.4. *Compositions and inverses of distortion minimal maps are distortion minimal maps.*

Also we will use (the strong form) of Moser's theorem on volume forms, which we state here for the convenience of the reader, see [**10**].

Theorem 2.5. *Let τ_t be a family of volume forms defined for $t \in [0,1]$ on a compact connected orientable smooth n-manifold M without boundary. If*

$$\int_c \tau_t = \int_c \tau_0 \tag{5}$$

for every n-cycle c on M, then there exists a one-parameter family of diffeomorphisms $\phi_t : M \to M$ such that

$$\phi_t^* \tau_t = \tau_0 \tag{6}$$

and ϕ_0 is the identity mapping. Moreover, the dependence of $\phi_t(m)$ on $m \in M$ and $t \in [0,1]$ is as smooth as in the family τ_t.

Theorem 2.6. *If (M, ω_M) and (N, ω_N) are diffeomorphic n-dimensional compact connected oriented manifolds without boundary, then*

(i) *there is a distortion minimal map from M to N,*

(ii) *every distortion minimal map from M to N minimizes the functional Φ, and*

(iii) *the minimum value of Φ is*

$$\Phi_{\min} = \frac{\big(\mathrm{Vol}\,(M) - \mathrm{Vol}\,(N)\big)^2}{\mathrm{Vol}\,(M)}. \tag{7}$$

Proof. To prove (i), choose a diffeomorphism $h \in \mathrm{Diff}\,(M,N)$ and note that the differential form $h^*\omega_N$ is a volume on M. Define a new volume on M as follows:

$$\overline{\omega}_M = \frac{\mathrm{Vol}\,(M)}{\int_M h^*\omega_N} h^*\omega_N.$$

Since

$$\int_M \overline{\omega}_M = \int_M \omega_M$$

and M is compact, by an application of Moser's theorem, there exists a smooth diffeomorphism $f : M \to M$ such that $\omega_M = f^*\overline{\omega}_M$. Hence,

$$\frac{\int_M h^*\omega_N}{\mathrm{Vol}\,(M)}\omega_M = (h \circ f)^*\omega_N;$$

and $|J(h \circ f)| = \mathrm{Vol}\,(N)/\mathrm{Vol}\,(M)$ is constant. Thus, $k = h \circ f$ is a distortion minimal map.

To prove parts (ii) and (iii), note that if k is an arbitrary distortion minimal map from M to N, then

$$\Phi(k) = \big(|J(k)| - 1\big)^2\ \mathrm{Vol}\,(M) = \frac{\big(\mathrm{Vol}\,(M) - \mathrm{Vol}\,(N)\big)^2}{\mathrm{Vol}\,(M)}. \tag{8}$$

We claim that this value of Φ is its minimum.

Let $g \in \mathrm{Diff}\,(M, N)$. By the Cauchy-Schwartz inequality,

$$\begin{aligned}\int_M J(g)^2\omega_M &\geq \frac{1}{\mathrm{Vol}\,(M)}\bigg(\int_M |J(g)|\,\omega_M\bigg)^2\\ &= \frac{\mathrm{Vol}\,(N)^2}{\mathrm{Vol}\,(M)}.\end{aligned}$$

The latter inequality together with formulas (3) and (8) implies that $\Phi(g) \geq \Phi(k)$ as required. □

Example 2.7. Let S_r and S_R be two-dimensional round spheres of radii r and R, respectively, centered at the origin in $\mathbf{R}^3$. Define $h : S_r \to S_R$ by $h(p) = R/r\, p$ for $p = (x, y, z) \in S_r$. We will show that h is a distortion minimal map.

Let ω_r, respectively ω_R, be the standard volume forms on S_r, respectively S_R, generated by the usual volume form on $\mathbf{R}^{n+1}$.

Using the parametrizations of S_r and S_R by spherical coordinates, it is easy to show that the Jacobian determinant of h is given by

$$J(\omega_r, \omega_R)(h)(m) = R^2/r^2 = \mathrm{Vol}\,(S_R)/\mathrm{Vol}\,(S_r)$$

for all $m \in S_r$; hence, by Theorem 2.6, h is a distortion minimal map.

Remark 2.8 (Harmonic maps). For $h \in \mathrm{Diff}(M,N)$, the distortion functional (2) has value

$$\Phi(h) = \int_M |J(h)|^2 \omega_M - 2\mathrm{Vol}\,(N) + \mathrm{Vol}\,(M).$$

Thus, it suffices to consider the minimization problem for the reduced functional Ψ given by

$$\Psi(h) = \int_M |J(h)|^2 \omega_M.$$

We note that if M and N are one-dimensional, then Ψ is the same as

$$\Psi(h) = \int_M |Dh|^2 \omega_M.$$

An extremal of this functional is called a harmonic map, see [**2**–**4**]. Thus, for the one-dimensional case, distortion minimal maps and harmonic maps coincide. On the other hand, there seems to be no obvious relationship in the general case.

3. Morphs of embedded manifolds.

We will discuss a minimization problem for morphs of compact connected oriented n-manifolds without boundary embedded in $\mathbf{R}^{n+1}$.

3.1. Pairwise minimal morphs.

Definition 3.1. Let M and N be isotopic compact connected smooth n-manifolds without boundary embedded in $\mathbf{R}^{n+1}$ such that M is oriented. A C^∞ isotopy $H : [0,1] \times M \to \mathbf{R}^{n+1}$ together with all the intermediate manifolds $M^t := H(t,M)$, equipped with the orientations induced by the maps $h^t = H(t,\cdot) : M \to M^t$ and the volume forms ω_t generated by the standard volume form on $\mathbf{R}^{n+1}$, is called a (smooth) *morph* from M to N. We denote the set of all morphs between manifolds M and N by $\mathcal{M}(M,N)$.

For simplicity, we will consider only morphs H such that $p \mapsto H(0,p)$ is the identity map. Each manifold $M^t = H(t,M)$ (with $M^0 = M$ and $M^1 = N$) is equipped with the volume form $\omega_t = i_{\eta_t}\Omega$, where

$$\Omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}$$

is the standard volume form on $\mathbf{R}^{n+1}$ and $\eta_t : M^t \to \mathbf{R}^{n+1}$ is the outer unit normal vector field on M^t with respect to the orientation induced by h^t and the usual metric on $\mathbf{R}^{n+1}$.

Definition 3.2. A morph H is *distortion pairwise minimal* (or, for brevity, *pairwise minimal*) if $h^{s,t} = h^t \circ (h^s)^{-1} : M^s \to M^t$ is a distortion minimal map for every s and t in $[0,1]$. We denote the set of all distortion pairwise minimal morphs between manifolds M and N by $\mathcal{PM}(M,N)$.

By Proposition 2.1 and Theorem 2.6, a morph H is pairwise minimal if and only if each Jacobian determinant $J(\omega_s,\omega_t)(h^{s,t})$ is constant.

Proposition 3.3. *Let $M = M^0$ and $N = M^1$ be n-dimensional manifolds as in Definition 3.1. A morph H between M and N is distortion pairwise minimal if and only if*

$$\frac{J(\omega_0,\omega_t)(h^t)(m)}{\mathrm{Vol}\,(M^t)} = \frac{1}{\mathrm{Vol}\,(M)} \tag{9}$$

for all $t \in [0,1]$ and $m \in M$.

Proof. Using Lemma 2.4 and Theorem 2.6, it suffices to prove that each map $h^t : M \to M^t$ is minimal if and only if the map (9) is constant. An application of Theorem 2.3 finishes the proof. □

Proposition 3.4. *Let M and N be n-dimensional manifolds as in Proposition 3.3. If there is a morph G from M to N, then there is a distortion pairwise minimal morph between M and N.*

Proof. Fix a morph G from M to N with the corresponding family of diffeomorphisms $g^t := G(t,\cdot)$, let $M^t := G(t,M)$, and consider the family of volume forms

$$\overline{\omega}_t = \frac{\mathrm{Vol}\,(M)}{\mathrm{Vol}\,(M^t)}(g^t)^*\omega_t$$

defined for $t \in [0,1]$. It is easy to see that

$$\int_M \overline{\omega}_t = \int_M \overline{\omega}_0;$$

hence, by Moser's theorem, there is a family of diffeomorphisms α^t on M such that $\omega_M = (\alpha^t)^*\overline{\omega}_t$. It follows that

$$(g^t \circ \alpha^t)^*\omega_t = \frac{\text{Vol}\,(M^t)}{\text{Vol}\,(M)}\omega_M;$$

therefore,

$$J(\omega_M, \omega_t)(g^t \circ \alpha^t)(m) = \frac{\text{Vol}\,(M^t)}{\text{Vol}\,(M)}$$

for all $m \in M$. The morph H defined by $H(t,p) := g^t \circ \alpha^t(p)$ for all $t \in [0,1]$ and $p \in M$ is the desired distortion pairwise minimal morph. □

3.2. Minimal morphs. We will define distortion minimal morphs of embedded connected oriented smooth n-manifolds without boundary.

For a morph H from M to N, let $E_{s,t}$ denote the total distortion of $h^{s,t} : M^s \to M^t$. We have that

$$\begin{aligned} E_{s,t} &= \int_{M^s} \Big(J(h^{s,t}) - 1\Big)^2 \omega_s \\ &= \int_M \left(\frac{J(h^t)}{J(h^s)} - 1\right)^2 J(h^s)\,\omega_M. \end{aligned}$$

By Taylor's theorem, $E_{s,t}$ has the representation

$$\begin{aligned} E_{s,t} = E_{t,t} + \frac{d}{ds}(E_{s,t})\big|_{s=t}(s-t) + \frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t}(s-t)^2 \\ + O\big((s-t)^3\big). \end{aligned}$$

Note that $E_{t,t}$ and $(d/ds)(E_{s,t})\big|_{s=t}$ both vanish, and

$$\frac{1}{2}\frac{d^2}{ds^2}(E_{s,t})\big|_{s=t} = \int_M \frac{\Big((d/dt)J(h^t)\Big)^2}{J(h^t)}\omega_M.$$

Definition 3.5. The *infinitesimal distortion* of a smooth morph H from M to N at $t \in [0,1]$ is

$$\varepsilon^H(t) = \lim_{s\to t}\frac{E_{s,t}}{(s-t)^2} = \int_M \frac{\Big((d/dt)J(h^t)\Big)^2}{J(h^t)}\omega_M.$$

The *total distortion functional* $\Psi : \mathcal{M}(M, N) \to \mathbf{R}$ is defined by

$$\Psi(H) = \int_0^1 \varepsilon^H(t)\, dt = \int_0^1 \left(\int_M \frac{\Big((d/dt)J(h^t)\Big)^2}{J(h^t)} \omega_M \right) dt. \tag{10}$$

Definition 3.6. A smooth morph is called a *distortion minimal morph* if it minimizes the functional Ψ.

Lemma 3.7. *For every morph $H \in \mathcal{M}(M, N)$ there exists a pairwise minimal morph $G \in \mathcal{PM}(M, N)$ such that $\Psi(G) \leq \Psi(H)$. In particular, if $H \in \mathcal{M}(M, N)$ is a distortion minimal morph, then there exists a pairwise minimal morph $G \in \mathcal{PM}(M, N)$ such that $\Psi(H) = \Psi(G)$.*

Proof. Let $H \in \mathcal{M}(M, N)$ be a morph with the intermediate states $M^t = H(t, M)$. By Proposition 3.4, there exists a pairwise minimal morph $G \in \mathcal{PM}(M, N)$ with the same intermediate states. The deformation energy of transition maps satisfies the inequality $E_{s,t}(H) \geq E_{s,t}(G)$ for all $s, t \in [0, 1]$ because G is pairwise minimal. Therefore, $\varepsilon^H(t) \geq \varepsilon^G(t)$ for all $t \in [0, 1]$, and, consequently,

$$\Psi(H) \geq \Psi(G) \tag{11}$$

as required.

If H is distortion minimal, the reverse inequality $\Psi(H) \leq \Psi(G)$ holds and $\Psi(H) = \Psi(G)$ as required. □

Corollary 3.8. (i) *The following inequality holds*:

$$\inf_{G \in \mathcal{PM}(M,N)} \Psi(G) \leq \inf_{H \in \mathcal{M}(M,N)} \Psi(H). \tag{12}$$

(ii) *If there exists a minimizer F of the total distortion functional Ψ over the class $\mathcal{PM}(M, N)$, then F minimizes the functional Ψ over the class $\mathcal{M}(M, N)$ as well:*

$$\Psi(F) = \min_{G \in \mathcal{PM}(M,N)} \Psi(G) = \min_{H \in \mathcal{M}(M,N)} \Psi(H). \tag{13}$$

Lemma 3.9. *The total distortion of a pairwise minimal morph H from M to N is*

$$\Psi(H) = \int_0^1 \frac{\big((d/dt)\mathrm{Vol}\,(M^t)\big)^2}{\mathrm{Vol}\,(M^t)}\,dt. \tag{14}$$

Proof. The proof is an immediate consequence of formula (10) and Proposition 3.3. □

Lemma 3.10. *The functional $\overline{\Psi}$ defined by*

$$\overline{\Psi}(\phi) = \int_0^1 \frac{\dot{\phi}^2}{\phi}\,dt \tag{15}$$

on the admissible set

$$Q = \big\{\phi \in C^2\big([0,1];(0,\infty)\big) : \phi(0) = \mathrm{Vol}\,(M), \phi(1) = \mathrm{Vol}\,(N)\big\}$$

attains its infimum

$$\inf_{\rho\in Q} \overline{\Psi}(\rho) = 4\Big(\sqrt{\mathrm{Vol}\,(N)} - \sqrt{\mathrm{Vol}\,(M)}\Big)^2 \tag{16}$$

at the element $\phi \in Q$ given by

$$\phi(t) = \Big[\Big(\sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}\Big)t - \sqrt{\mathrm{Vol}\,(M)}\Big]^2.$$

Proof. The proof is a simple application of the Euler-Lagrange equation and the Cauchy-Schwarz inequality.

The Euler-Lagrange equation for the functional $\overline{\Psi}$ is

$$\frac{2\,\ddot{\phi}\,\phi - \dot{\phi}^2}{\phi^2} = 0.$$

Its solutions have the form

$$\xi(t) = (Ct + D)^2,$$

where the constants C and D must be chosen so that $\xi(0) = \mathrm{Vol}\,(M)$ and $\xi(1) = \mathrm{Vol}\,(N)$. Because $\overline{\Psi}(\xi) = 4C^2$, we determine the values $C = \sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}$ and $D = -\sqrt{\mathrm{Vol}\,(M)}$ by eliminating the other possible choices of these constants that yield larger values of $\overline{\Psi}$. Hence, the function ϕ in the statement of the theorem is the solution of the Euler-Lagrange equation in Q that yields the smallest value of $\overline{\Psi}$.

By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\overline{\Psi}(\eta) &= \int_0^1 \frac{\dot{\eta}^2}{\eta}\,dt \geq \left(\int_0^1 \frac{\dot{\eta}}{\sqrt{\eta}}\,dt\right)^2 \\ &= 4\left(\sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}\right)^2 = \overline{\Psi}(\phi)\end{aligned}$$

for every $\eta \in Q$. Thus, the critical point ϕ in the statement of the lemma minimizes the functional $\overline{\Psi}$ on Q. □

Using Corollary 3.8 and Lemma 3.10, we will minimize the total distortion energy functional Ψ over the set $\mathcal{M}(M, N)$ of all morphs.

Theorem 3.11. *Let M and N be two n-dimensional manifolds satisfying the assumptions of Definition* 3.1. *If M and N are connected by a smooth morph, then there exists a distortion minimal morph. The minimal value of Ψ is*

$$\min_{H\in\mathcal{M}(M,N)} \Psi(H) = 4\left(\sqrt{\mathrm{Vol}\,(N)} - \sqrt{\mathrm{Vol}\,(M)}\right)^2. \tag{17}$$

Proof. Let G be a morph between M and N. Without loss of generality, we assume that G is pairwise minimal, see Proposition 3.4. Set

$$H(t, m) = \lambda(t)G(t, m),$$

where $\lambda : [0, 1] \to \mathbf{R}$ is to be determined.

Note that if $M^t = H(t, M)$ and $W^t = G(t, M)$, then

$$\mathrm{Vol}\,(M^t) = \int_M (h^t)^*\omega_M = \left[\lambda(t)\right]^n \int_M (g^t)^*\omega_M = \left[\lambda(t)\right]^n \mathrm{Vol}\,(W^t).$$

Let $\phi(t)$ be the minimizer of the auxiliary functional Ψ from Lemma 3.10, and define

$$\lambda(t) = \left[\frac{\phi(t)}{\mathrm{Vol}\,(W^t)}\right]^{1/n}.$$

The volume of the intermediate state M^t is given by $\mathrm{Vol}\,(M^t) = \phi(t)$; therefore, by Corollary 3.8 and Lemma 3.10, the morph H minimizes the total distortion functional Ψ over the class $\mathcal{M}(M, N)$ and

$$\Psi(H) = 4\big(\sqrt{\mathrm{Vol}\,(N)} - \sqrt{\mathrm{Vol}\,(M)}\big)^2. \qquad \square$$

The next result provides a basic class of distortion minimal morphs.

Proposition 3.12. *Suppose that M is an n-dimensional manifold embedded in $\mathbf{R}^{n+1}$ that satisfies the assumptions of Definition 3.1. If α is a positive real number and*

$$N := \{\alpha m : m \in M\},$$

then the morph given by the family of maps $h^t(m) = \lambda(t)m$, where

$$\lambda(t) = \mathrm{Vol}\,(M)^{-1/n}\Big[\Big(\sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}\Big)t - \sqrt{\mathrm{Vol}\,(M)}\Big]^{2/n},$$

is distortion minimal.

Proof. Define $h^t(m) = \lambda(t)m$. It is easy to check that h^t defines a morph from M to N. Also, we have that $J(h^t) := J(\omega_M, \omega_t)(h^t) = [\lambda(t)]^n$. Since $J(h^t)$ is constant on M, the family h^t defines a pairwise minimal morph H.

We will determine $\lambda(t)$ so that the morph H becomes a minimizer of Ψ over the class $\mathcal{M}(M, N)$. Indeed, by Lemma 3.10, it suffices to choose λ so that

$$\mathrm{Vol}\,(M^t) = [\lambda(t)]^n \mathrm{Vol}\,(M) = \Big[\Big(\sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}\Big)t - \sqrt{\mathrm{Vol}\,(M)}\Big]^2,$$

which yields

$$\lambda(t) = \mathrm{Vol}\,(M)^{-1/n}\Big[\Big(\sqrt{\mathrm{Vol}\,(M)} - \sqrt{\mathrm{Vol}\,(N)}\Big)t - \sqrt{\mathrm{Vol}\,(M)}\Big]^{2/n}.$$

The corresponding morph $H(t,m) = \lambda(t)m$ satisfies the equality

$$\Psi(H) = \min_{G\in\mathcal{M}(M,N)} \Psi(G). \qquad \square$$

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