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MODULARITY OF SOME NONRIGID DOUBLE OCTIC CALABI-YAU THREEFOLDS

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ABSTRACT. In this paper we discuss four methods of proving modularity of Calabi-Yau threefolds with $h^{12} = 1$: existence of elliptic ruled surfaces inside (Hulek-Verrill), correspondence with a product of an elliptic curve and a K3 surface (Livné-Yui), correspondence with a (modular) rigid Calabi-Yau threefold, and existence of an involution splitting the fourdimensional representation into two-dimensional subrepresentations.

We apply these methods to prove modularity of 17 out of 18 double octic Calabi-Yau threefolds for which "numerical evidence of modularity" was found in the second author's recently published book [**11**].

We observe that modularity holds for those elements in a pencil having certain additional geometric properties. In the proofs we use representations of the considered Calabi-Yau threefolds as a Kummer fibration associated to a fiber product of rational elliptic fibrations.

1. Introduction. The modularity conjecture for Calabi-Yau manifolds predicts that every Calabi-Yau manifold should be modular in the sense that its L-series coincides with the L-series of some automorphic form(s). The case of rigid Calabi-Yau threefolds was (almost) solved by Dieulefait and Manoharmayum in [**6, 7**]. On the other hand, in the nonrigid case it is not even clear which automorphic forms should appear.

Examples of nonrigid modular Calabi-Yau threefolds were constructed by Livné and Yui [**10**], Hulek and Verrill [**8, 9**] and Schütt [**16**]. In these examples modularity means a decomposition of the associated Galois representation into two- and four-dimensional subrepresentations with L-series equal to $L(g_4, s)$, $L(g_2, s-1)$ or $L(g_2 \otimes g_3, s)$, where g_k is a weight k cusp form. The summand with L-series equal to $L(g_2 \otimes g_3, s)$

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is explained by a double cover of a product of a K3 surface and an elliptic curve, see [**10**].

The L-series $L(g_2, s-1)$ is the L-series of the product of the projective line $\mathbf{P}^1$ and an elliptic curve E with $L(E, s) = L(g_2, s)$. A two-dimensional subrepresentation with such an L-series may be identified by a map $\mathbf{P}^1 \times E \to X$ which induces a nonzero map on the third cohomology, see [**8**]. Using an interpretation in terms of deformation theory, we conjecture that a splitting of the Galois action into two-dimensional pieces can happen only for isolated elements of any family of Calabi-Yau threefolds.

In this paper we will study modularity of some nonrigid double octic Calabi-Yau threefolds. We will prove modularity of all examples listed in Table 1 except X_{154}. Apart from the methods of Livné-Yui and Hulek-Verrill, we will use two others: one is based on giving a correspondence with a rigid Calabi-Yau threefold, the other is based on an involution. We also observe that the splitting of the Galois action into two-dimensional pieces holds for those Calabi-Yau threefolds in the studied families having certain additional geometric properties. The Calabi-Yau threefold X_{154} is also the only one which we were not able to represent as a Kummer fibration associated to a fiber product of elliptic fibrations, cf. [**14**].

1. Modular double octics with $h^{12} = 1$. Let D be an arrangement of 8 planes in $\mathbf{P}^3$. If no six of the planes intersect in a point and no four in a line then the double covering of $\mathbf{P}^3$ branched along D admits a resolution of singularities X which is a smooth Calabi-Yau threefold, see [**2**]. The resolution of singularities is performed by blowing up singularities of the branch locus in the following order: fivefold points, fourfold points that do not lie on a triple line, triple lines, double lines. The Euler number of the resulting Calabi-Yau threefold can easily be expressed in terms of numbers of different types of singularities. The Hodge number $h^{1,2}(X)$ (the dimension of the deformation space) can be computed as the dimension of the space of equisingular deformations of D in $\mathbf{P}^3$; it can also be computed as the dimension of the equisingular ideal of D, see [**5**].

An extensive computer search in [**11**] produced 18 double octic Calabi-Yau threefolds with $h^{12} = 1$ (in 11 one-parameter families) for which

$$\operatorname{tr}(\mathrm{Frob}_p^*|H^3_{\text{ét}}(X)) = a_p + p \cdot b_p,$$

for all primes $5 \leq p \leq 97$, where a_p, respectively b_p, are the coefficients of a weight four, respectively two, cusp form. This is strong numerical evidence for modularity in the sense of splitting into two two-dimensional subrepresentations. We list all these examples in Table 1. We include the number of the arrangement (as in [**11**, page 59]), the equation, the expected modular form of levels 4 and 2 (using Stein's notation from [**17**]) and the Picard number h^{11}. Since the Calabi-Yau threefolds in the table coming from arrangements with the same number are birational, see Lemma 3.1, we will use the notation X_n for any Calabi-Yau threefold in the table constructed from arrangement number n.

The Picard groups of all listed Calabi-Yau threefolds are generated by divisors defined over $\mathbf{Q}$, so Frobenius acts on $H^2_{\text{ét}}$ by multiplication with p. In fact, in all the examples except X_{244}, the skew-symmetric part of the Picard group is zero, whereas for X_{244} it is generated by a divisor coming from the contact plane $x + y - z + t = 0$.

TABLE 1.

no.	equation: $u^2 = xyzt \cdot \ldots$	wt. 4	wt. 2	h^{11}
4	$(x+y)(y+z)(x-y-z-t)$ $(x+y-z-t)$	32k4A1	32A1	61
4	$(x+y)(y+z)(x+2y+2z-t)$ $(x+y+2z-t)$	32k4A1	32A1	61
4	$2(x+y)(y+z)(2x+y+z-2t)$ $(2x+2y+z-2t)$	32k4A1	32A1	61
8	$(x+y)(y+z)(-z+t)(3x-y-z+t)$	24k4A1	24A1	61
13	$(x+y)(y+z)(x-z-t)(x-z-2t)$	32k4A1	32A1	61
13	$(x+y)(y+z)(x-z-t)(x-z+t)$	32k4A1	32A1	61
13	$(x+y)(y+z)(x-z-t)(2x-2z-t)$	32k4A1	32A1	61
21	$(x+y)(y+z)(2x+y-t)(2x-z-2t)$	32k4B1	32A1	53
53	$(x+y)(z+t)(x-y-z-t)$ $(x+y-z+t)$	32k4B1	32A1	53
154	$(x+y+z)(x+y+z-t)$ $(-2x+y-3z+3t)(2x+3z-2t)$	8k4A1	72A1	41

TABLE 1. (Continued)

no.	equation: $u^2 = xyzt \cdot \ldots$	wt. 4	wt. 2	h^{11}
244	$(x+y+z+t)(x+y-z-t)(y-z+t)$ $(x-z+t)$	12k4A1	48A1	39
249	$(x+y+z)(x+z+t)(2x+3y-z+2t)$ $(y-z+2t)$	24k4A1	24A1	37
249	$(x+y+z)(x+z+t)(2x-y+3z+2t)$ $(-3y+3z+2t)$	24k4A1	24A1	37
267	$(x+y-2z)(x-y-z+t)(2y-z+t)$ $(x+y+z+t)$	96k4B1	96B1	37
267	$(x+y+z)(x+2y-z+t)$ $(-y+2z-2t)(2x+2y-z+2t)$	96k4B1	96B1	37
267	$(2x+2y-z)(2x+y-2z+2t)$ $(y+z-t)(x+y-2z+t)$	96k4B1	96B1	37
274	$(x+y+z)(-x-z+t)(x+2y-z+t)$ $(x+y-z+2t)$	96k4E1	96B1	37
275	$(x+y+z)(2x-2z-t)(8y+4z+t)$ $(2x+4y+t)$	96k4B1	96B1	37

2. Double quartic elliptic fibrations. In this section we will shortly review some information about rational elliptic fibrations that can be realized as a resolution of a double covering of $\mathbf{P}^2$ branched along a sum of four lines. The structure of the elliptic fibration is determined by the choice of a point in $\mathbf{P}^2$. Some of these surfaces where described in [**4**]; we will omit here all the details explained in that paper.

The double covering is rational exactly when the lines do not intersect in one point. We can have the following combinations of singular fibers (the Picard number $\rho(S_w)$ of a generic fiber can be computed from the Zariski lemma):

	singular fibers	$\rho(S_w)$
S_1	D_4^*, D_4^*	1
S_2	I_2, I_2, D_6^*	1
S_3	I_2, I_2, I_4, I_4	1
S_4	I_2, I_2, I_2, D_4^*	2
S_5	I_2, I_2, I_2, I_2, I_4	2
S_6	$I_2, I_2, I_2, I_2, I_2, I_2$	3

A double covering of S_1 branched along the two singular fibers is birational to a product of $\mathbf{P}^1$ and an elliptic curve E, and all smooth fibers are isomorphic to E. This elliptic fibration depends on the j-invariant of E.

The surfaces S_2 and S_3 are extremal, i.e., they have $\rho(S_w) = 1$. Consequently, they are uniquely defined as fiber spaces. Moreover, the parameters corresponding to the singular fibers of S_3 form a harmonic quadruple, i.e., their cross-ratio equals -1; they can be chosen as

-1	0	1	∞
I_2	I_4	I_2	I_4

Denote by S_3' the pullback of S_3 via the involution $t \mapsto t - 1/t + 1$ of $\mathbf{P}^1$, so S_3' has the following singular fibers:

-1	0	1	∞
I_4	I_2	I_4	I_2

Thus S_3 and S_3' have singular fibers at the same points but of different types. There exists an isogeny $\Psi : S_3 \mapsto S_3'$ which is a degree 2 unbranched covering on a smooth fiber.

Fibration S_4 is not extremal, so we can chose arbitrary coordinates of singular fibers. The configuration of lines is not uniquely determined by the coordinates of singular fibers. In fact, there are exactly two types: one with a triple point and one with a "vertical line."

The Picard number of the generic fiber of fibration S_5 equals two, so we cannot choose arbitrary coordinates of singular fibers. In fact, there is an involution of $\mathbf{P}^1$ which preserves the fiber I_4 and exchanges two pairs of I_2s. The configuration of lines is uniquely determined.

Fibration S_6 is the most complicated one. In this case the configuration of lines is not uniquely determined. There can be several choices coming from automorphisms of $\mathbf{P}^1$ preserving the singular fibers.

3. Kummer fibrations. All examples in Table 1 except X_{154} can be realized as a Kummer fibration associated to a fiber product of elliptic fibrations, cf. [**15**]. Contrary to Schoen we do not require that the involution on the fiber product lifts to a resolution, so the resulting Calabi-Yau threefold is not necessarily a blow-up of the Kummer fibration.

To see the fibration, we reorder the planes such that the first four and the last four intersect in a point. Then, after change of coordinates in $\mathbf{P}^3$, we may assume that these points of intersection are $(0,0,0,1)$ and $(1,0,0,0)$, or equivalently that the double octic is given in weighted projective space $\mathbf{P}(1,1,1,1,4)$ by the equation

$$w^2 = f_1(x,y,z) \cdot \ldots \cdot f_4(x,y,z) f_5(y,z,t) \cdot \ldots \cdot f_8(y,z,t).$$

Consequently the double octic is birational to the quotient of the fiber product of elliptic fibrations

$$u^2 = f_1(x,y,z) \cdot \ldots \cdot f_4(x,y,z)$$

and

$$v^2 = f_5(y,z,t) \cdot \ldots \cdot f_8(y,z,t)$$

by the involution

$$(x,y,z,t,u,v) \longmapsto (x,y,z,t,-u,-v).$$

In the following table we list descriptions of Calabi-Yau threefolds from Table 1 as Kummer fibrations. For each Kummer fibration, we give coordinates and types of singular fibers. In some cases we were able to find two different representations as a Kummer fibration.

Lemma 3.1. *The Calabi-Yau threefolds in Table* 1 *defined by arrangements of the same type are birational. The Calabi-Yau threefolds X_{21} and X_{53} are birational; and the Calabi-Yau threefolds X_{267} and X_{275} are birational. There exists a correspondence between the Calabi Yau-threefolds X_8 and X_{249}.*

	0	1	2	3	∞
X_4	I_2	I_2	I_2	I_2	I_4
	I_0	I_2	I_2	I_0	D_6^*

	-1	0	1	∞
X_4	I_4	I_2	I_4	I_2
	D_6^*	I_2	I_0	I_2

	0	1	4	∞
X_8	D_4^*	I_2	I_2	I_2
	I_2	I_2	I_0	D_6^*

	0	1	∞
X_{13}	D_4^*	D_4^*	I_0
	I_2	I_2	D_6^*

	-1	0	1	∞
X_{13}	I_2	I_4	I_2	I_4
	I_0	D_4^*	I_0	D_4^*

	-1	0	1	∞
X_{21}	I_2	I_4	I_2	I_4
	D_6^*	I_0	I_2	I_2

	-1	0	1	∞
X_{21}	I_2	I_2	D_4^*	I_2
	D_4^*	I_2	I_2	I_2

	-1	0	1	∞
X_{53}	I_2	D_6^*	I_2	I_0
	I_2	I_0	I_2	D_6^*

	-1	0	1	∞
X_{53}	I_2	I_2	I_2	D_4^*
	I_2	D_4^*	I_2	I_2

	-1	0	1	2	∞
X_{244}	I_0	I_2	I_4	I_2	I_4
	I_4	I_2	I_4	I_0	I_2

	-1	0	$1/3$	1	3	∞
X_{244}	I_2	I_2	I_2	I_4	I_0	I_2
	I_2	I_2	I_0	I_4	I_2	I_2

	-1	0	$1/3$	1	3	∞
X_{249}	I_0	I_2	I_2	I_4	I_2	I_2
	I_2	I_4	I_0	I_2	I_0	I_4

	-1	0	$1/2$	1	2	∞
X_{267}	I_2	I_2	I_2	I_2	I_2	I_2
	I_2	I_2	I_2	I_2	I_2	I_2

	-1	0	$1/2$	1	2	∞
X_{274}	I_2	I_4	I_2	I_2	I_0	I_2
	I_4	I_2	I_2	I_0	I_2	I_2

	-1	0	$1/2$	1	2	∞
X_{275}	I_2	I_2	I_2	I_2	I_2	I_2
	I_2	I_2	I_2	I_2	I_2	I_2

Proof. From the explicit description of the fiber products in local coordinates it easily follows that the Calabi-Yau threefolds defined by arrangements of the same type with different parameters are in fact projectively equivalent.

Arrangement number 21 is projectively equivalent to

$$x(x-z)(x+z)(x+y)y(t+z)(t-z)(t+y)=0.$$

Substituting the birational involution of $\mathbf{P}^3$ given by

$$(x,y,z,t)\longmapsto(yz,xz,xy,tx),$$

we obtain

$$(xzy^2)^2x(x-z)(x+z)(x+y)z(t+y)(t-y)(t+z)=0,$$

and since arrangement number 53 is projectively equivalent to

$$x(x-z)(x+z)(x+y)z(t+y)(t-y)(t+z)=0,$$

we conclude that the resulting Calabi-Yau threefolds are birational.

To prove that X_{267} and X_{275} are birational, observe that the corresponding arrangements are projectively equivalent to

$$\begin{aligned}
\text{Arr. no. 267:}\quad & x(x-z)(2x-2z+y)(2x-z-y)\times\\
&\times t(t+z-y)(2y-z-2t)(2z-y+2t)=0\\
\text{Arr. no. 275:}\quad & x(x-z)(2x-2z+y)(2x-z-y)\times\\
&\times t(2t-y)(2t-z)(3t-y-z)=0.
\end{aligned}$$

Simple computations show that the cross ratios of the quadruples

$$\begin{array}{llll}
0, & y-1, & y-(1/2), & (y-z)/2\\
0, & y/2, & z/2, & y/3+z/3
\end{array}$$

are equal so there is a birational transformation in y,z,t that maps one of them to the other.

To see the correspondence between the Calabi-Yau threefolds X_8 and X_{249}, first pull back arrangement number 8 by the map $t\mapsto(t+1/t-1)^2$, obtaining

-1	0	$1/3$	1	3	∞
I_0	I_2	I_2	I_4	I_2	I_2
I_4	I_2	I_0	I_4	I_0	I_2

Now it is enough to compose this map with the isogeny of the elliptic fibration with fibers I_4, I_4, I_2, I_2 that exchanges I_2 fibers with I_4 fibers, see [**4**]. □

Remark 3.2. Arrangement numbers 267 and 275 are not projectively equivalent, they come from different twisted self-fiber products of the same elliptic fibration. The self-fiber product (without twist) of this elliptic fibration gives a nonbirational Calabi-Yau threefold with $h^{12} = 2$ (see Example 1).

4. Ruled surfaces over elliptic curves. In this section we will use elliptic ruled surfaces to prove modularity of four Calabi-Yau threefolds from Table 1.

Proposition 4.1. *The Calabi-Yau threefolds X_4, X_8, X_{244} and X_{249} are modular, with modular forms as listed in Table* 1.

Consider a Calabi-Yau threefold X such that an L-series of the form $L(g_2, s-1)$ (where g_2 is a weight two modular form corresponding to an elliptic curve E) appears in the Galois representation. Then by the Tate conjecture we can expect that there is a correspondence between X and the product $E \times \mathbf{P}^1$ which induces the isomorphism of representations.

Hulek and Verrill proved in [**8**] that when a smooth ruled surface over an elliptic curve $S \to E$ is contained in a Calabi-Yau threefold X then the map on third cohomology $H^3(X) \to H^3(S)$ is surjective. The map can be represented by a direct sum of $H^1(\mathcal{T}_X) \to H^1(\mathcal{N}_{S|X})$ and its complex conjugate. The map $H^1(\mathcal{T}_X) \to H^1(\mathcal{N}_{S|X})$ associates to a deformation of X the obstruction to lift it to a deformation of E (inside X). Therefore, if this map is nonzero, then E deforms inside X only over a codimension one submanifold of the Kuranishi space of X, cf. [**19**, Proposition 4.1].

Now, if we have ruled surfaces $E_1, \dots, E_r$, with $r = h^{21}(X)$, such that the map

$$H^3(X) \longrightarrow \bigoplus_i H^3(E_i) \tag{1}$$

is surjective, then the obstructions are independent and the surfaces do not deform simultaneously over any subvariety of the Kuranishi space of X of positive dimension. This explains why in a family there were always only finitely many examples where one was able to prove modularity in that way.

If we have several ruled surfaces over elliptic curves, it is usually difficult to determine whether the map (1) is surjective. In case we know the Kuranishi space of X we can try to invert the above argument. For each elliptic fibration we consider the hypersurface V_i of the Kuranishi space over which E_i deforms, knowing that the kernel of (1) is the tangent to the intersection of the V_is plus its complex conjugate (see example at the end of this section).

To use this method in our examples we need to find elliptic fibrations inside the double octics. If a plane S in $\mathbf{P}^3$ contains two double lines and the other four arrangement planes intersect at a point in S, then the pullback of S to the double covering is an elliptic fibration. On the Kummer fibration these planes are recognized as corresponding to the product of fibers I_0 and I_4.

We were able to find such a plane only for two arrangements:

Arrangement number 4: the plane S has equation $x - z = 0$, respectively $y + 2z - t = 0$, respectively $2x + y - 2t = 0$ (for the three arrangements in the table).

Arrangement number 244: the plane S has equation $x + y + z - t = 0$.

To prove modularity of X_8 and X_{249} we will study an auxiliary Calabi-Yau threefold X_{269} with $h^{12} = 2$. Modularity of this Calabi-Yau threefold follows from existence of some elliptic ruled surfaces and their behavior under deformations.

Example 1. Consider the double octic Calabi-Yau threefold X_{269} defined by the following arrangement of eight planes (arrangement number 269 in [**11**]):

$$xyzt(x+y+z)(x+2y-z+t)(y+z-t)(x+y-2z+t)=0.$$

It has $h^{2,1}(X_{269})=2$. Substituting $y=y-z, z=z+t$ we can represent this Calabi-Yau threefold as the following Kummer fibration:

-1	0	$1/3$	1	3	∞
I_2	I_2	I_2	I_2	I_2	I_2
I_2	I_2	I_2	I_2	I_2	I_2

On the other hand, substituting $x=x+2z-4y$, $z=x-2y$, we can also obtain the following Kummer fibration:

-1	0	$1/3$	1	3	∞
I_0	I_2	I_2	I_4	I_2	I_2
I_4	I_2	I_0	I_4	I_0	I_2

Hence, using the isogeny between S_3 and S_3' from Section 2, we can find correspondences between this Calabi-Yau threefold and the Calabi-Yau threefolds X_8 and X_{249}.

Observe that the planes $z=x+2y$ and $y=2z-t$ contain two double lines and a fourfold point, so they give two ruled surfaces E_1, E_2 over an elliptic curve with conductor 24.

The Kuranishi space of the Calabi-Yau threefold X_{269} may be parametrized by the equation

$$\begin{aligned}&xyzt(x+y+z)(Bx+Cy-Az+At)\times\\&\qquad\times(y+z-t)(Bx+By+(-A+B-C)z+At)=0.\end{aligned}$$

By [**9**] both elliptic fibrations give nonzero maps

$$H^3(X)\longrightarrow H^3(E_i)$$

so they deform over curves in $\mathbf{P}^2$. One easily checks that they deform over the lines given by

$$\begin{aligned}A+B-C&=0,\\C&=2B,\end{aligned}$$

which intersect only at the point $(1,1,2)$ corresponding to the equation we started with. Consequently, the obstructions are independent and the map

$$H^3(X) \longrightarrow H^3(E_1) \oplus H^3(E_2)$$

is surjective, giving a splitting of the representation on H^3 into two-dimensional pieces. Counting points over $\mathbf{F}_p$ for $p \leq 97$, one checks that X is modular and that the coefficients of the L-series are given by $b_p + 2pc_p$, where b_p, respectively c_p, are the coefficients of the unique cusp form of level 24 and weight 4, respectively 2.

There is a degree two correspondence between the above Calabi-Yau threefold and X_{249}, hence also X_8. These correspondences prove the modularity of X_8 and X_{249}.

5. Correspondences with rigid double octics. In this section we will use correspondences between rigid and nonrigid Calabi-Yau threefolds to prove modularity of the latter.

Proposition 5.1. *The Calabi-Yau threefolds X_4, X_{21}, X_{53} and X_{244} are modular, with modular forms as listed in Table* 1.

In [**3**] we checked the modularity and computed modular forms of some rigid double octic Calabi-Yau threefolds. Now we will use correspondences between some rigid and nonrigid Calabi-Yau threefolds to show the modularity of the latter.

We first recall the considered rigid examples. As before, we will use the equations and numbers of arrangements from [**11**, page 52] (the numbers from [**3**] are given in brackets).

Arrangement number 3 (old number 6) is given by the equation

$$xyzt(x+y)(y+z)(z+t)(t+x) = 0.$$

The corresponding fiber product of elliptic fibrations has singular fibers

$$\begin{matrix} I_4 & I_4 & I_2 & I_2 \\ D_6^* & I_2 & I_2 & I_0 \end{matrix}$$

Arrangement number 19 (old number 23) is given by the equation

$$xyzt(x+y)(y+z)(x-z-t)(x+y+z-t)=0.$$

The corresponding fiber product of elliptic fibrations has singular fibers

$$\begin{array}{cccc} I_2 & I_2 & I_4 & I_4 \\ I_0 & D_6^* & I_2 & I_2 \end{array}$$

Arrangement number 239 (old number 86^a) is given by the equation

$$xyzt(x+y+z)(x+y+t)(x+z+t)(y+z+t)=0.$$

The corresponding fiber product of elliptic fibrations has singular fibers

$$\begin{array}{ccccc} I_2 & I_2 & I_4 & I_4 & I_0 \\ I_0 & I_4 & I_2 & I_4 & I_2 \end{array}$$

Lemma 5.2. *There are correspondences between the Calabi-Yau threefolds given by the following arrangements*:

(1) *Number* 4 *and number* 19,

(2) *Number* 21 *and number* 3,

(3) *Number* 53 *and number* 3,

(4) *Number* 244 *and number* 239.

Proof. All the correspondences are in fact defined on the level of the fiber products of elliptic fibrations. They are given by applying the isogeny of the elliptic fibration with fibers I_2, I_2, I_4, I_4 that exchanges the fibers I_2 and I_4. $\square$

Assume that we have a generically finite correspondence between two Calabi-Yau threefolds X and Y. Then this correspondence induces an isomorphism between $H^{3,0}(X)$ and $H^{3,0}(Y)$ coming from a pullback of the canonical form. If Y is rigid, then taking this isomorphism plus its complex conjugate we obtain a splitting of the Galois representation on $H^3(X)$ into a two-dimensional representation isomorphic to $H^3(Y)$ and its complement. Using the correspondences from the above lemma and counting points in $\mathbf{F}_p$ for $p \leq 97$ we obtain Proposition 5.

6. Kummer construction. In this section we will use the Kummer construction studied by Livné and Yui [**10**].

Proposition 6.1. *The Calabi-Yau threefold X_{13} is modular, with modular forms as listed in Table* 1.

We will consider a two-dimensional family of double octic Calabi-Yau threefolds which are the quotient by an involution of a product of a $K3$ surface studied in [**1**] and an elliptic curve. Take the elliptic curve

$$E_\mu = \{(x,t,u) \in \mathbf{P}(1,1,2) : u^2 = (x-t)(x^2-\mu t^2)t\}$$

and the K3 surface

$$S_\lambda = \{(y,z,t,v) \in \mathbf{P}(1,1,1,3) : v^2 = yzt(y+t)(z+t)(y+\lambda z)\}.$$

On the product $Y_{\lambda,\mu} := E_\mu \times S_\lambda$ we have a natural involution

$$((x,t,u),(y,z,t,v)) \longmapsto ((x,t,-u),(y,z,t,-v)).$$

The quotient $X_{\lambda,\mu}$ of $Y_{\lambda,\mu}$ by this involution has a Calabi-Yau nonsingular model. To show this, observe that $Y_{\lambda,\mu}$ is birational to the double covering of $\mathbf{P}^3$ branched along the octic $D_{\lambda,\mu}$ given by the equation

$$(x-t)(x^2-\mu t^2)yz(y+t)(z+t)(y+\lambda z) = 0.$$

The birational map can be given in appropriate affine coordinates $(t=1)$ by

$$(x,1,u),(y,z,1,v) \longmapsto (x,y,z,uv).$$

The octic itself is defined over $\mathbf{Q}$. Over $\mathbf{Q}[\sqrt{\mu}]$ it splits into a sum of eight planes (for general μ, two of them are not defined over $\mathbf{Q}$).

Using [**2**] we conclude that $X_{\lambda,\mu}$ has a nonsingular model $\widetilde{X}_{\lambda,\mu}$ which is a Calabi-Yau threefold.

For general values of λ and μ, the arrangement $D_{\lambda,\mu}$ is arrangement number 52 in [**11**], so $\widetilde{X}_{\lambda,\mu}$ has the invariants $h^{11}(\widetilde{X}_{\lambda,\mu}) = 56$ and $h^{12}(\widetilde{X}_{\lambda,\mu}) = 2$.

For $\lambda \neq 0, -1$, the rank of the symmetric part of the Picard group of the $K3$ surface S_λ is 19; denote by $H^2_{\text{skew}}(S)$ the three dimensional skew-symmetric part. Thus, there is a Shioda-Inose structure on S_λ, namely, there exists an involution on S_λ such that the quotient of S_λ by that involution is a Kummer surface.

In [**1**] it is proved that the surface S_λ, with $\lambda \in \mathbf{Q}\setminus\{0, -1\}$, is modular exactly when $\lambda \in \{1, 8, 1/8, -4, -1/4, -64, -1/64\}$, and the modular form for S_λ is computed. We have the following diagram of rational maps

$$\begin{tikzcd} Y_{\lambda,\mu} \arrow[rr, dashed] \arrow[dr] & & \widetilde{X}_{\lambda,\mu} \arrow[dl] \\ & X_{\lambda,\mu} & \end{tikzcd}$$

The rational map $Y_{\lambda,\mu} \to \widetilde{X}_{\lambda,\mu}$ can be resolved by blowing up at points and lines so it induces a well-defined map in cohomologies $H^3(X_{\lambda,\mu}) \to H^3(Y_{\lambda,\mu})$. The image of the map is invariant under the involution on $Y_{\lambda,\mu}$, so in fact we obtain a map $H^3(X_{\lambda,\mu}) \to H^1(E_\mu) \otimes H^2_{\text{skew}}(S_\lambda)$. From the description of deformations of double coverings of smooth algebraic varieties [**5**], it follows that this map is surjective; moreover, both vector spaces have dimension 6, so it is an isomorphism. We obtain

Proposition 6.2. $H^3(\widetilde{X}_{\lambda\mu}) \cong H^1(E_\mu) \otimes H^2_{\text{skew}}(S_\lambda)$.

Corollary 6.3. *The Calabi-Yau threefold $\widetilde{X}_{\lambda,\mu}$ is modular for $\lambda \in \{1, 8, 1/8, -4, -1/4, -64, -1/64\}$ and $\mu \in \mathbf{Q} \setminus \{0, 1\}$.*

For the seven values of λ the L-series of S_λ corresponds to a cusp form for $S_3(\Gamma_1(8))$, $S_3(\Gamma_1(16))$, $S_3(\Gamma_1(12))$, $S_3(\Gamma_1(7))$ (for λ and $1/\lambda$ the

L-series differ only by a twist). They are the only η-product weight 3 modular forms. The modular form of the surface S_λ corresponds to the symmetric power of the modular form associated to the elliptic curve $E_{1/(\lambda+1)}$, see [**1**]. For the seven special values of λ, the elliptic curve $E_{1/(\lambda+1)}$ has complex multiplication. Denoting by a_p, respectively b_p, the Fourier coefficients of the level 2, respectively level 3, modular forms we get

$$b_p = \begin{cases} a_p^2 - 2p & \left(\frac{-(\lambda+1)}{p}\right) = 1 \\ 0 & \left(\frac{-(\lambda+1)}{p}\right) = -1. \end{cases}$$

The Fourier coefficient of the L-series of S_λ equals $\left(\frac{-(\lambda+1)}{p}\right)(b_p + p)$.

The third symmetric power of a weight 2 form yields also a weight 4 modular form with Fourier coefficients

$$c_p = a_p^3 - 3pa_p,$$

so we obtain

$$a_p b_p = c_p + pa_p.$$

Consequently, we get much better modularity properties for the threefolds $X_\lambda := \widetilde{X}_{\lambda,(1/\lambda+1)}$.

Proposition 6.4. *The L-series of the Calabi-Yau threefold X_λ has Fourier coefficients equal to*

$$c_p + 2pa_p.$$

In the table we collect the data for the four Calabi-Yau threefolds, the L-series of which do not only differ by a twist:

	$\lambda = 1$	$\lambda = 8$	$\lambda = -4$	$\lambda = -64$
wt 2 form	256k2D	32k2A	144k2B	49k2A
wt 3 form	8k3A[1,1]	16k3A[1,0]	12k3A[0,1]	7k3A[3]
wt 4 form	256k4H	32k4A	144k4A	49k4D
$b_p = a_p^2 - 2p$	$p \equiv 1,3(8)$	$p \equiv 3(4)$	$p \equiv 1(3)$	$p \equiv 1,2,4(7)$
$b_p = 0$	$p \equiv 5,7(8)$	$p \equiv 1(4)$	$p \equiv 2(3)$	$p \equiv 3,5,6(7)$
η-products (wt 3)	$\eta^2(z)\eta(2z)$ $\eta(4z)\eta^2(8z)$	$\eta^6(4z)$	$\eta^3(2z)\eta^3(6z)$	$\eta^3(z)\eta^3(7z)$
η-products (wt 2)	—	$\eta^2(8z)\eta^2(4z)$	$\dfrac{(\eta^{12}(12z))}{(\eta^4(24z)\eta^4(6z))}$	—

6.1. Singular $K3$. From the above considerations we excluded the case of $\lambda = -1$. There are two reasons for this. First, in this case all divisors on the $K3$ surface are symmetric and consequently $h^{12}(\widetilde{X}_{-1,\mu}) = 1$ (this is arrangement number 13). Second, $1/(\lambda+1)$ makes no sense. We can however take in that case also the curve $E_{1/9}$, as the modular forms appearing in S_{-1} and S_8 are the same. Hence, for the Calabi-Yau threefold $\widetilde{X}_{-1,1/9}$ the modular form has coefficients $c_p + pa_p$, where c_p, respectively a_p, are coefficients of a weight 4, respectively 2 level, 32 newform.

In the above considerations we can replace the elliptic curve $E_{1/(\lambda+1)}$ by another elliptic curve with the same modular form, or replace both E_μ and S_λ by some twist.

Now fix $\lambda \in \{1, 8, 1/8, -4, -1/4, -64, -1/64\}$. Using [**1**] we can compute the characteristic polynomial of Frobenius on H^3 for the Calabi-Yau threefold $\widetilde{X}_{\lambda,\mu}$ for any rational $\mu \neq 0, -1$. Denoting by $\alpha_p, \bar{\alpha}_p$, respectively $\beta_p, \bar{\beta}_p$, the eigenvalues of Frobenius on $H^1(E_{\lambda,p})$, respectively $H^1(E_{\mu,p})$, we find that the characteristic polynomial of Frobenius acting on $H^3(\widetilde{X}_{\lambda,\mu})$ is (up to sign)

$$(T - p\beta_p)(T - p\bar{\beta}_p) \cdot (T - \alpha_p^2\beta_p)(T - \alpha_p^2\bar{\beta}_p)(T - \bar{\alpha}_p^2\beta_p)(T - \bar{\alpha}_p^2\bar{\beta}_p).$$

This polynomial splits over $\mathbf{Z}$ into the characteristic polynomial of the Frobenius action on $H^2((\mathbf{P}^1 \times E_\mu)_p)$ and the degree 4 polynomial $(T - \alpha_p^2\beta_p)(T - \alpha_p^2\bar{\beta}_p)(T - \bar{\alpha}_p^2\beta_p)(T - \bar{\alpha}_p^2\bar{\beta}_p)$. In the construction, this splitting comes from the Cartesian product of E_μ and a transcendental cycle on the $K3$ surface S_μ; it should have a better geometric interpretation via the Shioda-Inose structure.

If the elliptic curves E_λ and E_μ are nonisogenous, the degree 4 polynomial does not divide by the characteristic polynomial of $\mathbf{P}^1 \times E$, for any elliptic curve E. To see this, denote the eigenvalues of Frobenius on $H^1(E)$ by $\gamma_p, \bar{\gamma}_p$ and assume that $p\gamma_p = \bar{\beta}_p\alpha_p^2$. Multiplying by β_p and dividing by $p = |\beta_p|^2$, we get $\beta_p\gamma_p = \alpha_p^2$. Since E_λ has complex multiplication, looking at the sets of primes p for which the coefficients α_p, β_p and γ_p equal $\pm ip^{1/2}$ we easily see that the other two elliptic curves have complex multiplication by the same quadratic field and so up to a twist the three weight two forms coincide. In particular, E_λ and E_μ are isogenous.

7. Involutions. In this section we will use an involution on a Calabi-Yau threefold to split the cohomology group H^3. Note that van Geemen and Nygaard [**18**] were the first to use an automorphism of a Calabi-Yau manifold to split the Galois representation and prove modularity.

Proposition 7.1. *Calabi-Yau threefolds X_{53}, X_{244}, X_{267}, X_{274} and X_{275} are modular, with modular forms as listed in Table* 1.

On some of the Calabi-Yau threefolds considered in this paper we can find an involution. On the middle cohomology the involution may have only eigenvalues ± 1. If both 1 and -1 are eigenvalues, then the map gives us a splitting of H^3. Since the splitting is compatible with the Frobenius morphism it is in fact a splitting of the Galois representation into two-dimensional subrepresentations.

We can use the Lefschetz formula to compute the trace of Frobenius composed with the involution. This trace is equal to the trace of Frobenius on the $+1$-eigenspace minus the trace of Frobenius on the -1-eigenspace. Together with the trace of Frobenius on H^3 this gives the traces on the two subspaces.

Assume that we have a $\mathbf{Q}$-linear involution on $\mathbf{P}^3$ which preserves the arrangement of eight planes. This map induces an involution $\Phi : X \to X$ on the Calabi-Yau threefold X defined by this arrangement. We will compute the trace

$$d_p = \operatorname{tr}\left((\mathrm{Frob}_p \circ \Phi)^* | H^3(\overline{X}_p, \mathbf{Q}_l)\right)$$

of Frobenius composed with Φ. Since this map acts by multiplication with $\pm p$ on H^2 and with $\pm p^2$ on H^4 the Lefschetz fixed-point formula relates d_p to the number N_p of fixed points of $\mathrm{Frob}_p \circ \Phi$.

Lemma 7.2. *If Φ is a linear involution on $\mathbf{P}^N(\overline{\mathbf{F}}_p)$ defined over $\mathbf{F}_p$, then the fixed points of $\mathrm{Frob}_p \circ \Phi$ are $\mathbf{F}_{p^2}$-rational.*

Proof. The Frobenius morphism Frob_p commutes with any linear involution defined over $\mathbf{F}_p$, so any fixed point of $\mathrm{Frob}_p \circ \Phi$ is also a fixed point of Frob_{p^2}. □

Using the lemma we reduce the counting of fixed points over the infinite field $\overline{\mathbf{F}}_p$ to counting of points over the finite field $\mathbf{F}_{p^2}$, which can easily be done using a computer.

From the representation as a Kummer fibration, we can easily recognize some linear involutions preserving the arrangement:

Arrangement number 53: $(x, y, z, t) \mapsto (y, x, -t, -z)$.

Arrangement number 244: $(x, y, z, t) \mapsto (y, x, -t, -z)$.

Arrangement number 267: $(x, y, z, t) \mapsto (t, -z, -y, x)$.

Arrangement number 274: $(x, y, z, t) \mapsto (z, -t, x, -y)$.

Simple computations show that the above involutions are not equal to identity on the deformation space $H^1(\mathcal{T}_X) \cong H^{12}(X)$, hence they split the Galois representations. In fact, it is easy to observe that $H^{12}(X) \oplus H^{21}(X)$ must be (-1)-eigenspaces. Counting fixed points on the singular double octic yields, for all primes $5 \le p \le 97$:

$$\begin{aligned}
X_{53} :& 1 + p^3 - a_p + pb_p + p^2 + p \\
X_{244} :& \begin{cases} 1 + p^3 - a_p + pb_p + 2p^2 - p & p \equiv 1 \mod 4 \\ 1 + p^3 - a_p + pb_p + 3p & p \equiv 3 \mod 4 \end{cases} \\
X_{267} :& 1 + p^3 - a_p + pb_p + p^2 - p \\
X_{274} :& \begin{cases} 1 + p^3 - a_p + pb_p + p^2 - p & p \equiv 1 \mod 4 \\ 1 + p^3 - a_p + pb_p + p^2 + 3p & p \equiv 3 \mod 4 \end{cases}
\end{aligned}$$

Analyzing the action of Frobenius on the generators of the Picard group and the space of curves H^4 gives the traces of Frobenius of the two-dimensional Galois subrepresentations. Applying the Faltings-Serre-Livné method finishes the proof.

Schütt suggested to us that counting points in $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$ we can compute the characteristic polynomial, which factors into two degree two polynomials. Since we know that the representation splits we get the traces of both actions. It is however not straightforward that the numbers a_p, respectively pb_p, will correspond to the $+1$-eigenspace, respectively the -1-eigenspace.

Remark 7.3. The described involutions act on singular double octics. Since the resolution of singularities of a double octic is not unique (it depends on the order in which we blow up lines in a triple point) it may happen that an involution maps to a birational Calabi-Yau threefold. Since two smooth models differ by a sequence of flops, we can compose the involution with these flops or we can consider a threefold that dominates both smooth models. The action on H^3 is well defined.

If we know that we can choose such a resolution of singularities of the double covering to which the involution lifts, then the quotient will be (after resolution) a rigid Calabi-Yau threefold.

Example 2. Consider the arrangement of planes (arrangement number 287 in [**11**]) given by

$$xyzt(x+y+z-3t)(x+y-3z+t) \\ \times (x-3y+z+t)(-3x+y+z+t)=0.$$

The corresponding Calabi-Yau threefold X_{287} has Hodge numbers $h^{11}(X_{287})=37$, $h^{12}(X_{287})=3$. Counting points in $\mathbf{F}_p$ shows that, for $5 \leq p \leq 97$, the trace of Frobenius on the middle cohomology equals a_p+3b_p, where a_p, respectively b_p, are the coefficients of the weight 4 level 6, respectively weight 2 level 24, cusp form. The arrangement has many linear symmetries. We can use the induced involutions on X to decompose the Galois representation.

We can also use the elliptic fibrations on X described in [**12**, page 62] and apply the deformation argument from Example 1 to prove modularity of X_{287}.

In fact the full permutation group S_4 acts on this Calabi-Yau threefold. If we consider the action of permutations of order 3, then the eigenvalues will be defined in $\mathbf{F}_p$ only for some p, so the decomposition of Frobenius action will depend on p.

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