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SIMPLE MODELS FOR AVIAN INFLUENZA

XINGYANG YE AND XUEPENG LI

ABSTRACT. Simple models for avian influenza are constructed and analyzed. These models are based on the standard SEIQ model, but include constant immigration of latent class and an additional property of the avian influenza, namely, that asymptomatic individuals in the latent period have an infectious force.

The general quarantine-adjusted incidence and a special incidence $\lambda_1 + (\lambda_2/N)$ are studied, respectively. The models not only show the importance of strengthening quarantine work to recruitment and treatment for infections, but also indicate that hunting and isolation are helpful to the epidemic control.

Finally, an eco-epidemiological system of two noninteraction species is proposed and investigated to study human avian influence. The model gives an important indication that the most effective way to control the disease spreads among human beings is to inhibit the influenza virus from spreading among animals.

1. Introduction. Avian influenza is a serious disease of poultry occurring more and more frequently all over the world. Many methods such as hunting and isolation have been taken to control the spread of avian influenza. However, it is observed that poultry without any symptom can excrete much highly pathogenic virus, which makes it more difficult to inhibit the H5N1 type virus from spreading.

Several studies [**2, 4, 5, 8**] have examined models to attempt to control such a pandemic influenza at the source where it should develop.These models are based on networks and stochastic simulations. Very recently,it is shown in [**1**] that many of the predictions of the above models can also be obtained from simple deterministic compartmental models. It is suggested in [**1**] that simple models may be a

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better way for a threatening pandemic with location and parameters as yet unknown.

Following the idea of [**1**], we develop a general epidemic model for avian influenza based on the standard SEI model, including quarantine and an additional property of the influenza described above, namely, that the asymptomatic individuals in the latent period have infectious force.

In our model,the total population $\overline{N}$ is divided into four compartments with $\overline{N}(t) = S(t) + E(t) + I(t) + Q(t)$, where $S(t)$ is the number of individuals in the susceptible class, $E(t)$ is the number of latent or exposed individuals who are asymptomatic but infective, $I(t)$ is the number of infective individuals who are infective and symptomatic, and $Q(t)$ is the number of individuals who are quarantined. It is assumed that these quarantined individuals do not mix with others, so that they do not infect susceptibles. Thus, throughout our paper,we always consider the quarantine-adjusted incidence [**6**], that is, the incidence is dependent on $S(t)$, $E(t)$ and $I(t)$ but independent of $Q(t)$.

Specifically,we make the following assumptions.

(i) There is a constant flow of A new individuals into the population in unit time, of which a fraction p $(0 \leq p \leq 1)$ is latent.

(ii) There is a constant per capita natural death rate constant $d > 0$ in each class.

(iii) The transmission coefficients of the latent and infective are $\beta_1(N)$ and $\beta_2(N)$, respectively. The two coefficients satisfy the following conditions:

$$\beta_i(N) > 0, \qquad \beta_i'(N) \leq 0, \qquad [\beta_i(N)N]' \geq 0$$

where $i = 1, 2$.

(iv) A fraction ε of latent individuals (E) becomes infective and proceeds to the infective class(I).

(v) A fraction η of the infective (I) is quarantined, and a fraction μ of the infective is hunted, while a fraction γ of the infective recovers and goes directly to the susceptible class (S).

(vi) The disease-related death rate constants of the infective and quarantine are α_1 and α_2, respectively.

These assumptions lead to the following model

$$(1)\quad \begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = (1-p)A - dS - \beta_1(N)SE - \beta_2(N)SI + \gamma I, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = pA + \beta_1(N)SE + \beta_2(N)SI - (d+\varepsilon)E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon E - (d+\alpha_1+\mu+\eta+\gamma)I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \eta I - (d+\alpha_2)Q. \end{cases}$$

Here, $N(t) = S(t) + E(t) + I(t)$ is the total population size except the quarantine. It is convenient to use N as one of the model variables rather than S, especially if a more general incidence function depending on total population size is assumed. Thus, substituting $S = N - E - I$ into (1), we have

$$(2)\quad \begin{cases} \dfrac{\mathrm{d}E}{\mathrm{d}t} = pA + [\beta_1(N)E + \beta_2(N)I](N - I - E) - (d+\varepsilon)E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon E - (d+\alpha_1+\mu+\eta+\gamma)I, \\ \dfrac{\mathrm{d}N}{\mathrm{d}t} = A - dN - (\alpha_1+\mu+\eta)I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \eta I - (d+\alpha_2)Q. \end{cases}$$

The special case $p = 0$, $\gamma = 0$, $\eta = 0$ and $\mu = 0$, which gives $Q = 0$, is the SEI model without input to the latent class which has been studied in [**7**].

In the next section, we shall consider (2) in two cases: $p = 0$ and $0 < p < 1$, which imply that there is no input of the infections class and that there is the constant input, respectively. To obtain more detailed property of the epidemic equilibrium, in Section 3 we study the special incident $\beta(N) = \lambda_1 + (\lambda_2/N)$ which includes both the standard incidence and the simple mass action incidence. In Section 4, an epidemic model of two noninteraction species which describes the human avian influence is proposed and investigated. A final discussion concludes the paper.

To keep matters simple, we denote $\omega = d+\alpha_1+\mu+\eta+\gamma$, $\delta = \alpha_1+\mu+\eta$.

2. The avian flu model with general quarantine-adjusted incidence.

2.1. Case: $p = 0$. $p = 0$ implies that there is no input to the class E, that is, the input to the population is all susceptible. For this case, denote

$$R_0 = \frac{A}{d}\frac{\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)}{\omega(d+\varepsilon)},$$

then we have that the following holds.

Theorem 1. *If $p = 0$, for model (2), there is always the disease-free equilibrium $P_0(0, 0, (A/d), 0)$, and there is also a unique endemic equilibrium $P_1^* = (E^*, I^*, N^*, Q^*)$ if and only if $R_0 > 1$ where*

$$I^* = \frac{A - dN^*}{\delta}, \qquad E^* = \frac{\omega}{\varepsilon}I^*, \qquad Q^* = \frac{\eta}{d+\alpha_2}I^*.$$

N^* is the unique root of equation

$$[\omega\beta_1(N) + \varepsilon\beta_2(N)]\{[\varepsilon\delta + d(\omega+\delta)]N - (\omega+\delta)A\} = \varepsilon\delta\omega(d+\varepsilon) \tag{3}$$

in the interval $(0, (A/d))$.

Proof. If $p = 0$, it is easy to see that $P_0(0, 0, (A/d), 0)$ is always the disease-free equilibrium of (2). The endemic equilibrium is determined by equations

$$\begin{cases} \left[\beta_1(N) + \beta_2(N)\dfrac{I}{E}\right](N - I - E) - (d+\varepsilon) = 0 \\ \varepsilon E - \omega I = 0 \\ A - dN - \delta I = 0 \\ \eta I - (d+\alpha_2)Q = 0. \end{cases} \tag{4}$$

From the last three equations of (4) we can obtain the following

$$I = \frac{A - dN}{\delta}, \qquad E = \frac{\omega}{\varepsilon}I, \qquad Q = \frac{\eta}{d+\alpha_2}I. \tag{5}$$

Substituting (5) into the first equation of (4) gives (3). Denote

$$f(N) = [\omega\beta_1(N) + \varepsilon\beta_2(N)]\{[\varepsilon\delta + d(\omega+\delta)]N - (\omega+\delta)A\}.$$

Due to $f(A/d) = \varepsilon\delta(A/d)[\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)]$, $R_0 > 1$ is equivalent to $f(A/d) > \varepsilon\delta(d+\varepsilon)$. Since $\beta_i'(N) \le 0$, $[N\beta_i(N)]' \ge 0$, $i = 1, 2$, we have

$$\begin{aligned} f'(N) &= [\varepsilon\delta + d(\omega+\delta)]\{\omega[N\beta_1(N)]' + \varepsilon[N\beta_2(N)]'\} \\ &\quad - (\omega+\delta)A[\omega\beta_1'(N) + \varepsilon\beta_2'(N)] \\ &\ge 0; \end{aligned}$$

then $f(N)$ is a nondecreasing function of N. $\beta_i(N) > 0$, $i = 1, 2$, implies that $f(N) < 0$ for sufficiently small N. Therefore, (3) has a unique root N^* in the interval $(0, (A/d))$ if and only if $R_0 > 1$. Substituting $N = N^*$ into (5) gives I^*, E^* and Q^*.

The proof is complete. □

Theorem 2. *If $p = 0$, the disease-free equilibrium P_0 of (2) is globally asymptotically stable if $R_0 \le 1$ and unstable if $R_0 > 1$.*

Proof. If $p = 0$, analysis of the Jacobian matrix of system (2) at equilibrium P_0 shows that it is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. In order to prove global stability when $R_0 \le 1$, consider a Liapunov function

$$V(E, I) = \frac{\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)}{(d+\varepsilon)\omega} E + \frac{\beta_2(A/d)}{\omega} I,$$

with the Liapunov derivative

$$\begin{aligned} V'|_{(2)} &= \frac{\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)}{(d+\varepsilon)\omega}\big[\beta_1(N)SE + \beta_2(N)SI - (d+\varepsilon)E\big] \\ &\quad + \frac{\beta_2(A/d)}{\omega}(\varepsilon E - \omega I) \\ &= \frac{\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)}{(d+\varepsilon)\omega}\big[\beta_1(N)E + \beta_2(N)I\big]S \\ &\quad - \left[\beta_1\left(\frac{A}{d}\right)E + \beta_2\left(\frac{A}{d}\right)I\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{A}R_0\big[\beta_1(N)E + \beta_2(N)I\big]S - \left[\beta_1\left(\frac{A}{d}\right)E + \beta_2\left(\frac{A}{d}\right)I\right] \\
&\le \frac{d}{A}R_0\big[\beta_1(N)NE + \beta_2(N)NI\big] - \left[\beta_1\left(\frac{A}{d}\right)E + \beta_2\left(\frac{A}{d}\right)I\right] \\
&\le \frac{d}{A}R_0\left[\beta_1\left(\frac{d}{A}\right)\frac{d}{A}E + \beta_2\left(\frac{d}{A}\right)\frac{d}{A}I\right] - \left[\beta_1\left(\frac{A}{d}\right)E + \beta_2\left(\frac{A}{d}\right)I\right] \\
&= \left[\beta_1\left(\frac{A}{d}\right)E + \beta_2\left(\frac{A}{d}\right)I\right](R_0 - 1) \le 0,
\end{aligned}
$$

since $R_0 \le 1$. By the Liapunov-Lasalle theorem, solutions of system (2) approach the largest positively invariant subset of the set where $V' = 0$, which is the set where $E = I = 0$. In this set, $S' = A - dS$ and $Q' = -(d + \alpha_2)Q$, so that

$$
S = \frac{A}{d} + \left(S(0) - \frac{A}{d}\right)e^{-dt} \longrightarrow \frac{A}{d}, \qquad Q(t) = Q(0)e^{-(d+\alpha_2)t} \longrightarrow 0
$$

as $t \to +\infty$. Thus, all solutions in the set where $E = I = 0$ go to the disease-free equilibrium P_0. By the theory of limit systems, all solutions must also approach P_0.

The local stability of the endemic equilibrium P^* will be obtained similarly for the case $p > 0$.

2.2. Case: $0 < p < 1$. $0 < p < 1$ implies that there is the input of latent individuals. For this case,we have the following results.

Theorem 3. *Suppose $p > 0$. For (2) there exists no disease-free equilibrium, but there is always the endemic equilibrium $P^*(E^*, I^*, N^*, Q^*)$, where*

$$
I^* = \frac{A - dN^*}{\delta}, \qquad E^* = \frac{\omega}{\varepsilon}I^*, \qquad Q^* = \frac{\eta}{d + \alpha_2}I^*,
$$

and N^* is the unique positive root of equation

$$
\begin{aligned}
(6)\quad &[\omega\beta_1(N) + \varepsilon\beta_2(N)]\big\{[\varepsilon\delta + d(\omega + \delta)]N - (\omega + \delta)A\big\} \\
&\qquad = \varepsilon\delta\omega\left[d + \varepsilon - \frac{p\varepsilon\delta A}{\omega(A - dN)}\right]
\end{aligned}
$$

in the interval $(0, (A/d))$.

Proof. Equilibriums of (2) are given by

$$(7)\qquad \begin{cases} pA + [\beta_1(N)E + \beta_2(N)I](N - I - E) - (d+\varepsilon)E = 0, \\ \varepsilon E - \omega I = 0, \\ A - dN - \delta I = 0, \\ \eta I - (d+\alpha_2)Q = 0. \end{cases}$$

If we let $E = 0$ and $I = 0$, then the first equation of (7) becomes $pA = 0$ which is impossible since $p > 0$. Thus, there is no disease-free equilibrium of (2) if $p > 0$. The endemic equilibrium of (2) is determined by equations

$$(8)\qquad \begin{cases} [\beta_1(N) + \beta_2(N)(I/E)](N - I - E) - (d+\varepsilon) + (pA/E) = 0 \\ \varepsilon E - \omega I = 0 \\ A - dN - \delta I = 0 \\ \eta I - (d+\alpha_2)Q = 0. \end{cases}$$

From the last three equations of (8) we have

$$(9)\qquad I = \frac{A - dN}{\delta}, \qquad E = \frac{\omega}{\varepsilon} I, \qquad Q = \frac{\eta}{d+\alpha_2} I.$$

Substituting (9) into the first equation of (8) gives (6). Define

$$f(N) = [\omega\beta_1(N) + \varepsilon\beta_2(N)]\{[\varepsilon\delta + d(\omega+\delta)]N - (\omega+\delta)A\}$$
$$g(N) = \varepsilon\delta\omega\left[d + \varepsilon - \frac{p\varepsilon\delta A}{\omega(A - dN)}\right].$$

Then $f(N)$ is a nondecreasing function of N which has been obtained in the proof of Theorem 2. $f(A/d) = \varepsilon\delta(A/d)[\omega\beta_1(A/d) + \varepsilon\beta_2(A/d)] > 0$ and $f(N) < 0$ for sufficiently small N. Again, $g(N)$ is a strictly decreasing function of N, $g(0) > 0$ and $\lim_{N\to(A/d)^-} g(N) = -\infty$. Therefore, (6) has a unique root N^* in the interval $(0, (A/d))$. Substituting $N = N^*$ into (9) gives I^*, E^* and Q^*.

Theorem 4. *If $0 \le p \le 1$, the endemic equilibrium P^* is locally asymptotically stable if it exists.*

Proof. The Jacobian matrix at the endemic equilibrium is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ \varepsilon & -\omega & 0 & 0 \\ 0 & -\delta & -d & 0 \\ 0 & \eta & 0 & -(d+\alpha_2) \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= \beta_1(N^*)S^* - [\beta_1(N^*)E^* + \beta_2(N^*)I^*] - (d+\varepsilon) \\ a_{12} &= \beta_2(N^*)S^* - [\beta_1(N^*)E^* + \beta_2(N^*)I^*] \\ a_{13} &= [\beta_1'(N^*)E^*]S^* + [\beta_1(N^*)E^* + \beta_2(N^*)I^*]. \end{aligned}$$

The characteristic equation at the endemic equilibrium is a fourth degree polynomial given by

$$(\lambda + d + \delta_2)(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3) = 0,$$

where the coefficients are

$$\begin{aligned} b_1 &= \omega + d - a_{11} \\ b_2 &= d\omega - (d+\omega)a_{11} - \varepsilon a_{12} \\ b_3 &= \varepsilon\delta a_{13} - \varepsilon d a_{12} - \omega d a_{11}. \end{aligned}$$

Since

$$pA + [\beta_1(N^*)E^* + \beta_2(N^*)I^*]S^* - (d+\varepsilon)E^* = 0,$$

we get

$$\beta_1(N^*)S^* = (d+\varepsilon) - \frac{pA}{E^*} - \frac{\varepsilon}{\omega}\beta_2(N^*)S^*. \tag{10}$$

Therefore, by means of (10), after some calculation, the coefficients b_i, $i = 1, 2, 3$, can be rewritten as follows

$$\begin{aligned} b_1 &= (\omega + d) + \frac{pA}{E^*} + [\beta_1(N^*)E^* + \beta_2(N^*)I^*] + \frac{\varepsilon}{\omega}\beta_2(N^*)S^* \\ b_2 &= \omega d + (d + \omega + \varepsilon)[\beta_1(N^*)E^* + \beta_2(N^*)I^*] \\ &\quad + (\omega + d)\frac{pA}{E^*} + \frac{d\varepsilon}{\omega}\beta_2(N^*)S^* \\ b_3 &= (\varepsilon\delta + \varepsilon d + \omega d)[\beta_1(N^*)E^* + \beta_2(N^*)I^*] \\ &\quad + \omega d\frac{pA}{E^*} + \varepsilon\delta S^*[\beta_1'(N^*)E^* + \beta_2'(N^*)I^*]. \end{aligned}$$

From the above, we know that all of the coefficients of b_i are positive. Further, notice that the first term of b_1 multiplied by the second term of b_2 then minus the first term of b_3 is positive, and the second term of b_1 multiplied by the first term of b_2 then minus the second term of b_3 is zero, the third term of b_3 is negative and the other terms of b_1, b_2 are positive. All of this analysis leads to $b_1 b_2 - b_3 > 0$. Thus, the Routh-Hurwitz criteria is satisfied which means local asymptotic stability of the endemic equilibrium has been achieved.

3. The avian flu model with a special quarantine-adjusted incidence $\lambda_1 + (\lambda_2/N)$ [3]. In Section 2, we only obtained the necessary and sufficient condition for the local asymptotic stability of the endemic equilibrium. To obtain more detailed properties of the epidemic equilibrium, in this section we study a special incidence $\beta(N) = \lambda_1 + (\lambda_2/N)$ which includes both the standard incidence and the simple mass action incidence.

Let $\beta_1(N) = \lambda_1 + (\lambda_2/N)$ and $\beta_2(N) = \lambda_3 + (\lambda_4/N)$. Then system (1) becomes

(11)
$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = (1-p)A - dS - \lambda_1 SE - \lambda_2 \dfrac{SE}{N} - \lambda_3 SI - \lambda_4 \dfrac{SI}{N} + \gamma I, \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = pA + \lambda_1 SE + \lambda_2 \dfrac{SE}{N} + \lambda_3 SI + \lambda_4 \dfrac{SI}{N} - (d+\varepsilon)E, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \varepsilon E - \omega I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \eta I - (d+\alpha_2)Q. \end{cases}$$

Since system (11) is a special case of (1), we have the following result for (11).

Theorem 5. *The endemic equilibrium P^* of system* (11) *is locally asymptotically stable if it exists.*

Next we are concerned with the global property of the endemic equilibrium P^*. The variable Q does not appear in model (11) except in the equation for Q. Thus, Q is determined when the other variables are known, and the equation for Q may be discarded from the model.

We now consider the following subsystem of system (11)

(12)
$$\begin{cases} S'(t) = (1-p)A - dS - \lambda_1 SE - \lambda_2 \dfrac{SE}{N} - \lambda_3 SI - \lambda_4 \dfrac{SI}{N} + \gamma I, \\ E'(t) = pA + \lambda_1 SE + \lambda_2 \dfrac{SE}{N} + \lambda_3 SI + \lambda_4 \dfrac{SI}{N} - (d+\varepsilon)E, \\ I'(t) = \varepsilon E - \omega I. \end{cases}$$

To keep matters simple, let $x = (dS/A)$, $y = (dE/A)$, $z = (dI/A)$, $\tilde{t} = dt$ and $\widetilde{N} = x + y + z = (dN/A)$. Then (12) is equivalent to

(13)
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tilde{t}} = (1-p) - x - \tilde{\lambda}_1 xy - \tilde{\lambda}_2 \dfrac{xy}{\widetilde{N}} - \tilde{\lambda}_3 xz - \tilde{\lambda}_4 \dfrac{xz}{\widetilde{N}} + \tilde{\gamma} z, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tilde{t}} = p + \tilde{\lambda}_1 xy - \tilde{\lambda}_2 \dfrac{xy}{\widetilde{N}} - \tilde{\lambda}_3 xz - \tilde{\lambda}_4 \dfrac{xz}{\widetilde{N}} - (\tilde{\varepsilon}+1)y, \\ \dfrac{\mathrm{d}z}{\mathrm{d}\tilde{t}} = \tilde{\varepsilon} y - \tilde{\omega} z, \end{cases}$$

where

$$\tilde{\lambda_1} = \frac{\lambda_1}{d}, \qquad \tilde{\lambda_2} = \frac{\lambda_2 A}{d^2}, \qquad \tilde{\lambda_3} = \frac{\lambda_3}{d}, \qquad \tilde{\lambda_4} = \frac{\lambda_4 A}{d^2},$$
$$\tilde{\alpha} = \frac{\alpha}{d}, \qquad \tilde{\gamma} = \frac{\gamma}{d}, \qquad \tilde{\varepsilon} = \frac{\varepsilon}{d}, \qquad \widetilde{\omega} = \frac{\omega}{d}.$$

The equation of variable N is now

$$\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}\tilde{t}} = 1 - \widetilde{N} - \tilde{\alpha} y. \tag{14}$$

The invariant set is now $\widetilde{D} = \{(x,y,z) \in R^3_+ : x + y + z \le 1\}$. Letting

$$\begin{aligned} E &= \{(\widetilde{N}, y, z) \in \widetilde{D} : \widetilde{N} = 1 - \tilde{\alpha} y\} \\ &= \{(x,y,z) \in \widetilde{D} : x + (1+\tilde{\alpha})y + z = 1\}, \end{aligned}$$

we have

Theorem 6. *There is no periodic solution of system* (13) *in domain* E.

Proof. Obviously, the boundary curve of domain E cannot form the periodic solution of system (13). We consider the following in the interior of E.

Assuming that system (13) has a periodic solution $\phi(t) = \{x(t), y(t), z(t)\}$, the image Γ of $\phi(t)$ is the boundary of a plane domain Π which is in the interior of domain E.

Let f_1, f_2 and f_3 denote the first three formulas of the righthand side in system (13), respectively. Let $f = (f_1, f_2, f_3)^{\mathrm{T}}$ (T denotes transpose), $g(x, y, z) = 1/(xyz)r \times f$, (where $r = (x, y, z)^{\mathrm{T}}$). Then

$$f \cdot g = 0.$$

Denote $g = (g_1, g_2, g_3)$ and

$$\mathrm{Curl}\, g = \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right).$$

By calculating straightforwardly, we get in the interior of domain E,

$$\begin{aligned}(\mathrm{Curl}\, g) \cdot (1, 1 + \tilde{\alpha}, 1)^{\mathrm{T}} = &- \frac{p}{y^2 z} - \frac{\tilde{\varepsilon}}{xz^2} - \frac{(1-p)(1+\tilde{\alpha})}{xz} - \frac{p}{xy^2} \\ &- \frac{1 - p + \tilde{\gamma}}{x^2 y} - \frac{\tilde{\lambda}_1 (1 + \tilde{\alpha})(1 + y)}{z} \\ &- \frac{\tilde{\lambda}_3 (1 + \tilde{\alpha} y)}{y^2} - \frac{\tilde{\lambda}_4 (1 + 2\tilde{\alpha}^2 y^2)}{(1 - \tilde{\alpha} y)^2} < 0\end{aligned}$$

for any $0 \le p \le 1$.

If we choose the direction of plane domain Π upward, the direction of the image Γ conforms to the righthand rule with the direction of plane domain Π. Vector $(1, 1 + \tilde{\alpha}, 1)$ is the normal vector of plane domain Π. Then we get, by Stoker's theorem,

$$\frac{1}{\sqrt{(1 + \tilde{\alpha})^2 + 2}} \int \int_{\Pi} \mathrm{Curl}\, g \cdot (1, 1 + \tilde{\alpha}, 1)^{\mathrm{T}} \mathrm{d}S = \oint_{\Gamma} \frac{g \cdot f}{|f|} \, \mathrm{d}s.$$

This is in contradiction with the calculation above. The theorem is proved. □

Theorem 6 means the local stable equilibrium (S^*, E^*, I^*) for system (12) is globally stable. Therefore, we obtain

Theorem 7. *The endemic equilibrium of system* (11) *is globally stable when* $R_0 > 1$.

4. An eco-epidemiological system of two noninteraction species where the disease spreads from animals to humans with mass action incidence. Due to the high lethality and virulence of H5N1, its endemic presence, its increasingly large host reservoir, and its significant ongoing mutations, the H5N1 virus is the world's largest current pandemic threat to human. In this section we propose an epidemic model that describes a human infected by avian flu.

Let species (I) stand for poultry infected by the H5N1 virus. The disease transfer among species (I) is similar to the model discussed in Section 2. To keep the model simple we model the incidence rate by a simple mass action.Then the epidemic model for species (I) yields:

$$\begin{cases} \dot{S}_1 = A_1 - dS_1 - \beta_1 S_1 E_1 - \beta_2 S_1 I_1 + \gamma_1 I_1, \\ \dot{E}_1 = \beta_1 S_1 E_1 + \beta_2 S_1 I_1 - (d+\varepsilon) E_1, \\ \dot{I}_1 = \varepsilon E_1 - (\alpha_1 + d + \gamma_1 + \mu + \eta) I_1, \\ \dot{Q}_1 = \eta I_1 - (d+\alpha_2) Q_1. \end{cases} \tag{15}$$

Let species (II) stand for people who get extensive physical contact with infected species (I). Since there is no evidence of efficient human-to-human transmission or of airborne transmission of H5N1 to humans, we assume the disease does not spread among humans. The disease transmission model equation for species (II) is

$$\begin{cases} \dot{S}_2 = A_2 - bS_2 - \beta(E_1 + I_1 + Q_1) S_2 + \gamma_2 I_2, \\ \dot{I}_2 = \beta(E_1 + I_1 + Q_1) S_2 - (\alpha_3 + \gamma_2) I_2, \end{cases} \tag{16}$$

where A_2 is the constant input to species (II), b is the output rate of S_2, γ_2 is the recovery rate, and α_3 is the disease-related death rate.

Thus, we get the following six-dimensional equations:

$$
(17)\qquad \begin{cases} \dot{S}_1 = A_1 - dS_1 - \beta_1 S_1 E_1 - \beta_2 S_1 I_1 + \gamma_1 I_1, \\ \dot{E}_1 = \beta_1 S_1 E_1 + \beta_2 S_1 I_1 - (d+\varepsilon) E_1, \\ \dot{I}_1 = \varepsilon E_1 - (\alpha_1 + d + \gamma_1 + \mu + \eta) I_1, \\ \dot{Q}_1 = \eta I_1 - (d + \alpha_2) Q_1, \\ \dot{S}_2 = A_2 - bS_2 - \beta(E_1 + I_1 + Q_1) S_2 + \gamma_2 I_2, \\ \dot{I}_2 = \beta(E_1 + I_1 + Q_1) S_2 - (\alpha_3 + \gamma_2) I_2. \end{cases}
$$

Based on biological significance, our following discussion is in domain D, where

$$
D = \Big\{ (S_1, E_1, I_1, Q_1, S_2, I_2) \in R_+^6 \mid 0 < S_1 + E_1 + I_1 + Q_1 \leq \frac{A_1}{d}, 0 < S_2 + I_2 \leq \frac{A_2}{b} \Big\}.
$$

It is easy to show that D is the invariable set of system (17). Denote

$$
\widetilde{R}_0 = \frac{A_1 \beta_1}{d(d+\varepsilon)} + \frac{A_1 \varepsilon \beta_2}{d(d+\varepsilon)(\alpha_1 + d + \gamma_1 + \mu + \eta)}.
$$

Then we have

Theorem 8. *If $\widetilde{R}_0 \leq 1$, system* (15) *has a unique disease-free equilibrium $P_0((A_1/d), 0, 0, 0)$ which is globally stable. If $\widetilde{R}_0 > 1$, P_0 is unstable and system* (15) *has a globally stable equilibrium $P^*(S_1^*, E_1^*, I_1^*, Q_1^*)$ where*

$$
(18)\qquad S_1^* = \frac{\omega(d+\varepsilon)}{\varepsilon\beta_2 + \omega\beta_1} I_1^* = \frac{\varepsilon A_1 (1 - (1/\widetilde{R}_0))}{\omega(d+\varepsilon) - \varepsilon\gamma_1},
$$
$$
E_1^* = \frac{\omega}{\varepsilon} I_1^*, \qquad Q_1^* = \frac{\eta}{d + \alpha_2} I_1^*,
$$

where $\omega = \alpha_1 + d + \gamma_1 + \mu + \eta$.

By analyzing the existence of equilibria and the linearized system on equilibria, we can easily obtain

Theorem 9. *If $\widetilde{R}_0 \leq 1$, system (17) has a unique disease-free equilibrium $\widetilde{P}_0((A_1/d), 0, 0, 0, (A_2/b), 0)$, which is locally stable if $\widetilde{R}_0 < 1$. If $\widetilde{R}_0 > 1$ and $\widetilde{P}_0$ is unstable, then system (17) has a local stable equilibrium $\widetilde{P}^*(S_1^*, E_1^*, I_1^*, Q_1^*, S_2^*, I_2^*)$, where*

$$S_2^* = \frac{(A_2 - \alpha_3 I_2^*)}{b}, \qquad I_2^* = \frac{A_2\beta(E_1^* + I_1^* + Q_1^*)}{\alpha_2\beta(E_1^* + I_1^* + Q_1^*) + b(\alpha_2 + \gamma_2)}$$

and S_1^*, E_1^*, I_1^* and Q_1^* are defined by (18).

For the stability of $\widetilde{P}_0$ and $\widetilde{P}^*$, we have

Theorem 10. *If $\widetilde{R}_0 \leq 1$, $\widetilde{P}_0$ is globally stable in domain D.*

Proof. From Theorem 8 we know $P_0((A_1/d), 0, 0, 0)$ is the globally stable equilibrium of subsystem (15) for system (17). Then if $t \to +\infty$, the limiting equation of the last equation of system (17) is

$$\dot{I}_2 = -(\alpha_2 + \gamma_2)I_2,$$

so $I_2(t) = I_2(0)e^{-(\alpha_2+\gamma_2)t} \to 0$, $t \to +\infty$. The limiting equation about S_2 is

$$\dot{S}_2 = A_2 - bS_2,$$

so $S_2 = (A_2/b) + (S_2(0) - (A_2/b))e^{-bt} \to (A_2/b)$, $t \to +\infty$. Applying the theory of the limit system, if $\widetilde{R}_0 \leq 1$, the solutions of system (17) in D approach $\widetilde{P}_0$ as $t \to +\infty$.

Theorem 11. *If $\widetilde{R}_0 > 1$, $\widetilde{P}^*$ is globally stable in domain D.*

Proof. From Theorem 8 we know $P^*(S_1^*, E_1^*, I_1^*, Q_1^*)$ is the globally stable equilibrium of system (15). Then if $t \to +\infty$, the limiting equations of system (17) are

$$\begin{cases} \dot{S}_2 = A_2 - bS_2 - \beta(E_1^* + I_1^* + Q_1^*)S_2 + \gamma_2 I_2 = X(S_2, I_2), \\ \dot{I}_2 = \beta(E_1^* + I_1^* + Q_1^*)S_2 - (\alpha_3 + \gamma_2)I_2 = Y(S_2, I_2). \end{cases} \tag{19}$$

Consider the region $\Omega = \{(S_2, I_2) \in R_+^2 \mid S_2 + I_2 \leq (A_2/b)\}$, which is the positive invariable set of system (19). The equilibrium (S_2^*, I_2^*) in

the region is locally asymptotically stable. Using Dulac's criteria, we have

$$\frac{\partial X}{\partial S_2} + \frac{\partial Y}{\partial I_2} = -b - \beta(E_1^* + I_1^* + Q_1^*) - (\alpha_2 + \gamma_2) < 0,$$

so that there are no periodic solutions in region Ω. Thus, by the Poincaré-Bendixson theory, all solutions starting in Ω approach (S_2^*, I_2^*) as $t \to +\infty$. Hence, by the theory of limit systems, $\widetilde{P}^*$ is globally stable in domain D.

5. Conclusion. For the avian flu model with no input of latent, there is a threshold behavior, with either a disease-free equilibrium or an endemic equilibrium, approached by all solutions. If there is a positive flow of latent into the population, it is not possible to have a disease-free equilibrium. All of these not only embody the importance of strengthening quarantine work to recruitment and treatment for infections, but also show that hunting and isolation are helpful to epidemic control. A simple model of a human infected by the H5N1 virus indicates that the most effective way to control disease spreads among human is to inhibit the influenza virus from spreading among animals.

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College of Science, Jimei University, Xiamen, 361021 P.R. China
Email address: yexingyang@jmu.edu.cn

College of Mathematics and Computer Science, Fujian Normal University, Fuzhou, 350007 P.R. China
Email address: lixup@pub3.fz.fj.cn