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ANALYSIS OF AN SEIRS EPIDEMIC MODEL WITH TIME DELAYS AND PULSE VACCINATION

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ABSTRACT. Pulse vaccination is an important strategy for the elimination of infectious diseases. An SEIRS epidemic model with time delays and pulse vaccination is formulated in this paper. By the comparison theorem in impulsive differential equations, we obtain that the infection-free periodic solution is globally attractive if the pulse vaccination rate is larger than θ^*. Moreover, we show that the disease is uniformly persistent if the pulse vaccination rate is less than θ_* under appropriate conditions. The permanence of the model is investigated analytically.

1. Introduction. Recently epidemic models have been studied by many authors, especially the models with time delays. Cooke and Driessche in literature [**4**] investigated an SEIRS model with two time delays,

$$\begin{cases} dS(t)/dt = bN(t) - bS(t) - (\lambda S(t)I(t)/N(t)) \\ \qquad\qquad +\gamma I(t-\tau)e^{-(b+d)\tau}, \\ E(t) = \int_{t-\omega}^{t} (\lambda S(u)I(u)/N(u))e^{-b(t-u)}\,du, \\ dI(t)/dt = (\lambda S(t-\omega)I(t-\omega)/N(t-\omega))e^{-b\omega} \\ \qquad\qquad -(b+\gamma+d)I(t), \\ R(t) = \int_{t-\tau}^{t} \gamma I(u)e^{-(b+d)(t-u)}\,du, \end{cases} \tag{1.1}$$

where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the population size that are susceptible, exposed, infectious and recovered at time t, respectively, b is the natural birth and death rate of the population, d is the disease-related death rate, λ is the average number of adequate contacts of an

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infectious individual per unit time, γ is the recovery rate of infectious individuals, ω is the latent period of the disease and τ is the immune period of the population. All coefficients are positive constants. They obtain sufficient conditions for global stability of disease free equilibrium of system (1.1). By neglecting the disease-related death rate in the SEIRS model in [**4**], Wang [**19**] presented sufficient conditions for local and global stability of endemic equilibrium.

Infectious diseases have tremendous influence on human life. Every year millions of human beings suffer or die of various infectious diseases. Controlling infectious diseases has been an increasingly complex issue in recent years. Pulse vaccination is an effective and important way to control the transmission of diseases. Some theoretical considerations, practical advantages, and examples of the pulse vaccination strategy are presented in [**1, 5, 6, 13, 17, 18**]. For example, some successes against poliomyelitis and measles have been attributed to the repeated pulse vaccination strategy [**14**]. As indicated in [**8**], models have clearly shown the advantages of a mass campaign approach in rapidly achieving high measles population immunity and interrupting measles virus circulation. Gao et al. [**7**] have developed a model and a pulse vaccination strategy. It reveals an effective strategy for the elimination of infectious diseases. The effectiveness of constant and pulse vaccination policies are compared theoretically and numerically in [**12**].

In this paper, we assume that the impulsive vaccination is applied every T (> 0) years, and θ, $0 < \theta < 1$, denotes the proportion of those vaccinated successfully. Incorporating with pulse vaccination and neglecting disease-related death rate, model (1.1) yields equation (1.2) where $k \in Z_+$, Z_+ denotes the set of positive integer. Of course, $N(t) = S(t) + E(t) + I(t) + R(t) = N(t_0)$ for all $t \geq t_0$. Let $s = S/N$, $e = E/N$, $i = I/N$ and $r = R/N$ denote the fractions of the population that are susceptible, exposed, infectious and recovered, respectively. Hence model (1.2) yields (1.3). Model (1.3) is subjected to the restriction $s(t) + e(t) + i(t) + r(t) = 1$. Note that the variables e and r do not appear in the first and third equations of system (1.3). This allows us to attack (1.3) by studying subsystem (1.4).

(1.2)
$$\begin{cases} \left.\begin{aligned} &dS(t)/dt = bN(t) - bS(t) - (\lambda S(t)I(t)/N(t)) + \gamma I(t-\tau)e^{-b\tau}, \\ &E(t) = \int_{t-\omega}^{t} (\lambda S(u)I(u)/N(u))e^{-b(t-u)}\,du, \\ &dI(t)/dt = (\lambda S(t-\omega)I(t-\omega)/N(t-\omega))e^{-b\omega} - (b+\gamma)I(t), \\ &R(t) = \int_{t-\tau}^{t} \gamma I(u)e^{-b(t-u)}\,du, \end{aligned}\right\} t \neq kT, \\ \left.\begin{aligned} &S(t^+) = (1-\theta)S(t), \\ &E(t^+) = E(t), \\ &I(t^+) = I(t), \\ &R(t^+) = R(t) + \theta S(t), \end{aligned}\right\} t = kT, \end{cases}$$

(1.3)
$$\begin{cases} \left.\begin{aligned} &ds(t)/dt = b - bs(t) - \lambda s(t)i(t) + \gamma i(t-\tau)e^{-b\tau}, \\ &e(t) = \int_{t-\omega}^{t} \lambda s(u)i(u)e^{-b(t-u)}\,du, \\ &di(t)/dt = \lambda s(t-\omega)i(t-\omega)e^{-b\omega} - (b+\gamma)i(t), \\ &r(t) = \int_{t-\tau}^{t} \gamma i(u)e^{-b(t-u)}\,du, \end{aligned}\right\} t \neq kT, \\ \left.\begin{aligned} &s(t^+) = (1-\theta)s(t), \\ &e(t^+) = e(t), \\ &i(t^+) = i(t), \\ &r(t^+) = r(t) + \theta s(t), \end{aligned}\right\} t = kT. \end{cases}$$

(1.4)
$$\begin{cases} \left.\begin{aligned} &ds(t)/dt = b - bs(t) - \lambda s(t)i(t) + \beta i(t-\tau), \\ &di(t)/dt = \lambda\alpha s(t-\omega)i(t-\omega) - (b+\gamma)i(t), \end{aligned}\right\} t \neq kT, \\ \left.\begin{aligned} &s(t^+) = (1-\theta)s(t), \\ &i(t^+) = i(t), \end{aligned}\right\} t = kT, \end{cases}$$

where $\alpha = e^{-b\omega}$, $\beta = \gamma e^{-b\tau}$.

Most of the research on epidemiologic models are established by ODE, delayed ODE or impulsive ODE [**5, 11, 19**]. However, impulsive equations with time delay have seldom been studied by authors. The main purpose of this paper is to establish sufficient conditions that the disease dies out. The second purpose of this paper is to investigate the role of time delays in disease transmission and show that, under appropriate conditions, the disease is uniformly persistent, that is, there is a positive constant q (independent of the choice of the solution) such that $i(t) \geq q$ for all large t.

The organization of this paper is as follows: In the next section, we introduce some notations, give some definitions, and state three lemmas which will be essential to our proofs. In Section 3, the global attractivity conditions for the infection-free periodic solution is presented. Sufficient conditions for the permanence of model (1.4) are obtained in Section 4. In the final section, we try to interpret our mathematical results in terms of their ecological implications. We also point out some future research directions.

2. Notations, definitions and preliminaries. In this section, we introduce some notations and definitions and state three results which will be useful in subsequent sections.

Set $l = \max\{\tau,\ \omega\}$. Let C be the space of continuous functions on $[-l,\ 0]$ with uniform norm. The initial conditions for (1.4) are

$$(\phi_1(\zeta),\ \phi_2(\zeta)) \in C_+ = C([-l,\ 0],\ R_+^2), \quad \phi_i(0) > 0,\ i = 1,\ 2. \tag{2.1}$$

Let $R_+ = [0,\ +\infty)$, $R_+^2 = \{Z \in R^2 : Z \geq 0\}$. Denote $f = (f_1,\ f_2)^T$ as the map defined by the righthand side of the first and second equations of system (1.4). The solution of system (1.4) is a piecewise continuous function $Z : R_+ \to R_+^2$, $Z(t)$ is continuous on $(kT,\ (k+1)T]$, $k \in Z_+$ and $Z(kT^+) = \lim_{t \to kT^+} Z(t)$ exists. Obviously, the smooth properties of f guarantee the global existence and uniqueness of solutions of system (1.4) (see [**2, 10**] for details on fundamental properties of impulsive systems). Since $ds(t)/dt|_{s=0} > 0$ and $di(t)/dt = 0$ whenever $i(t) = 0$, for $t \neq kT$, $k \in Z_+$. Moreover, $s(kT^+) = (1-\theta)s(kT)$, $i(kT^+) = i(kT)$ for $k \in Z_+$. Therefore, we have the following lemma.

Lemma 2.1. *Suppose $Z(t)$ is a solution of system* (1.4) *with initial conditions* (2.1). *Then $Z(t) \geq 0$ for all $t \geq 0$.*

Denote

$$\Omega = \{(s,\ i) \in R^2 \mid s \geq 0,\ i \geq 0,\ s + i \leq 1\}.$$

Using the fact that $s(t) + e(t) + i(t) + r(t) \equiv 1$, it is easy to show that Ω is positively invariant with respect to (1.4) with initial conditions (2.1).

Definition 2.1. System (1.4) is said to be uniformly persistent if there is an $\eta > 0$ (independent of the initial conditions) such that every solution $(s(t),\ i(t))$ with initial conditions (2.1) of system (1.4) satisfies

$$\liminf_{t\to\infty} s(t) \geq \eta, \qquad \liminf_{t\to\infty} i(t) \geq \eta.$$

Definition 2.2. System (1.4) is said to be permanent if there exists a compact region $\Omega_0 \in \operatorname{int}\Omega$ such that every solution of system (1.4) with initial conditions (2.1) will eventually enter and remain in region Ω_0.

To prove our main results we give the following lemmas.

Lemma 2.2 [**7**]. *Let us consider the following impulsive differential equations*

$$\begin{cases} \dot{u}(t) = a - bu(t) & t \neq kT, \\ u(t^+) = (1-\theta)u(t) & t = kT, \end{cases} \tag{2.2}$$

where $a > 0$, $b > 0$, $0 < \theta < 1$. Then there exists a unique positive periodic solution of system (2.2)

$$\tilde{u}_e(t) = \frac{a}{b} + \left(u^* - \frac{a}{b}\right)e^{-b(t-kT)}, \quad kT < t \leq (k+1)T,$$

which is globally asymptotically stable, where

$$u^* = \frac{a}{b}\frac{(1-\theta)(1-e^{-bT})}{1-(1-\theta)e^{-bT}}.$$

Lemma 2.3 [**9, 20**]. *Consider the following equation*

$$\dot{x}(t) = a_1 x(t-\omega) - a_2 x(t),$$

where a_1, a_2, $\omega > 0$; $x(t) > 0$ *for* $-\omega \le t \le 0$. *We have*:

(i) *if* $a_1 < a_2$, *then* $\lim_{t\to\infty} x(t) = 0$;

(ii) *if* $a_1 > a_2$, *then* $\lim_{t\to\infty} x(t) = +\infty$.

The proofs of case (i) and case (ii) are given in Theorem 2.1 [**9**] and Lemma 2.1 [**20**], respectively.

3. Global attractivity of infection-free periodic solution. In this section, we begin the analysis (1.4) by first demonstrating the existence of an infection-free periodic solution, in which infectious individuals are entirely absent from the population permanently, i.e., $i(t) = 0$ for all $t \ge 0$. Under this condition, the growth of susceptible individuals must satisfy:

$$\begin{cases} ds(t)/dt = b - bs(t) & t \ne kT, \\ s(t^+) = (1-\theta)s(t) & t = kT. \end{cases} \tag{3.1}$$

We will show that the fraction of susceptible population s oscillates with period T, in synchronization with the periodic pulse vaccination.

By Lemma 2.2, we know that the periodic solution of system (3.1)

$$\tilde{s}_e(t) = 1 - \frac{\theta}{1-(1-\theta)e^{-bT}}\, e^{-b(t-kT)}, \quad kT < t \le (k+1)T, \tag{3.2}$$

is globally asymptotically stable.

In the section that follows we determine the global attractivity conditions for the infection-free periodic solution $(\tilde{s}_e(t),\ 0)$ of system (1.4).

Theorem 3.1. *The infection-free periodic solution* $(\tilde{s}_e(t),\ 0)$ *of system* (1.4) *is globally attractive provided that* $R^* < 1$, *where*

$$R^* = \frac{\lambda\alpha(1+(\beta/b))(1-e^{-bT})}{(b+\gamma)(1-(1-\theta)e^{-bT})}.$$

Proof. Since $R^* < 1$, we can choose $\varepsilon_1 > 0$ sufficiently small such that

$$\lambda\alpha\left(\frac{(1+(\beta/b))(1-e^{-bT})}{1-(1-\theta)e^{-bT}}+\varepsilon_1\right) < b+\gamma. \tag{3.3}$$

From the first equation of system (1.4), we have $ds(t)/dt < (b+\beta) - bs(t)$. Then we consider the following comparison system with pulses

$$\begin{cases} dx(t)/dt = (b+\beta) - bx(t) & t \neq kT, \\ x(t^+) = (1-\theta)x(t) & t = kT. \end{cases} \tag{3.4}$$

In view of Lemma 2.2, we know that the unique periodic solution of system (3.4),

$$\widetilde{x}_e(t) = \left(1+\frac{\beta}{b}\right)\left[1-\frac{\theta}{1-(1-\theta)e^{-bT}}e^{-b(t-kT)}\right], \quad kT < t \leq (k+1)T,$$

is globally asymptotically stable.

Let $(s(t),\ i(t))$ be the solution of system (1.4) with initial conditions (2.1) and $s(0^+) = s_0 > 0$ and let $x(t)$ be the solution of system (3.4) with initial value $x(0^+) = s_0$. By the comparison theorem in impulsive differential equations [**2, 10**], there exists an integer $k_1 > 0$ such that, for $t > k_1T$,

$$s(t) < \widetilde{x}_e(t) + \varepsilon_1;$$

thus,

$$s(t) < \left(1+\frac{\beta}{b}\right)\frac{1-e^{-bT}}{1-(1-\theta)e^{-bT}} + \varepsilon_1 \triangleq s^M, \quad kT < t \leq (k+1)T, \quad t > k_1T+\omega. \tag{3.5}$$

Further, from the second equation of system (1.4), we know that (3.5) implies

$$\frac{di(t)}{dt} \leq \lambda\alpha s^M i(t-\omega) - (b+\gamma)i(t), \quad t > k_1T+\omega.$$

Consider the following comparison system

$$\frac{dy(t)}{dt} = \lambda\alpha s^M y(t-\omega) - (b+\gamma)y(t), \quad t > k_1T+\omega. \tag{3.6}$$

From (3.3), we have $\lambda\alpha s^M < (b+\gamma)$. According to Lemma 2.3, we have $\lim_{t\to\infty} y(t) = 0$.

Let $(s(t),\ i(t))$ be the solution of system (1.4) with initial conditions (2.1) and let $i(\zeta) = \varphi(\zeta) > 0$, $\zeta \in [-\omega,\ 0]$, and $y(t)$ be the solution of system (3.6) with initial value $y(\zeta) = \varphi(\zeta)$, $\zeta \in [-\omega,\ 0]$. By the comparison theorem, we have $\limsup_{t\to\infty} i(t) \le \limsup_{t\to\infty} y(t) = 0$. Incorporating into the positivity of $i(t)$, we know that $\lim_{t\to\infty} i(t) = 0$. Therefore, there exists an integer $k_2 > k_1$ (where $k_2T > k_1T + \omega$) such that $i(t) < \varepsilon_1$ for all $t > k_2T + \tau$.

For the first equation of system (1.4), we have

$$\frac{ds(t)}{dt} > b - (\lambda\varepsilon_1 + b)s(t), \quad \text{for } t > k_2T + \tau,$$

and

$$\frac{ds(t)}{dt} < (\beta\varepsilon_1 + b) - bs(t), \quad \text{for } t > k_2T + \tau.$$

Consider comparison impulsive differential equations for $t > k_2T + \tau$ and $k > k_2$,

$$\begin{cases} dz_1(t)/dt = b - (\lambda\varepsilon_1 + b)z_1(t) & t \neq kT, \\ z_1(t^+) = (1-\theta)z_1(t) & t = kT, \end{cases} \tag{3.7}$$

and

$$\begin{cases} dz_2(t)/dt = (\beta\varepsilon_1 + b) - bz_2(t) & t \neq kT, \\ z_2(t^+) = (1-\theta)z_2(t) & t = kT. \end{cases} \tag{3.8}$$

By Lemma 2.2, we have that the unique periodic solution of system (3.7),

$$\tilde{z}_{1e}(t) = \frac{b}{\lambda\varepsilon_1 + b} + \left(z_1^* - \frac{b}{\lambda\varepsilon_1 + b}\right)e^{-(\lambda\varepsilon_1 + b)(t-kT)},$$
$$kT < t \le (k+1)T, \quad k > k_2,$$

and the unique periodic solution of system (3.8)

$$\tilde{z}_{2e}(t) = \frac{\beta\varepsilon_1 + b}{b} + \left(z_2^* - \frac{\beta\varepsilon_1 + b}{b}\right)e^{-b(t-kT)},$$
$$kT < t \le (k+1)T,\ k > k_2,$$

are globally asymptotically stable, where

$$z_1^* = \frac{b}{\lambda\varepsilon_1 + b}\frac{(1-\theta)(1-e^{-(\lambda\varepsilon_1+b)T})}{1-(1-\theta)e^{-(\lambda\varepsilon_1+b)T}}$$

and

$$z_2^* = \frac{\beta\varepsilon_1 + b}{b}\frac{(1-\theta)(1-e^{-bT})}{1-(1-\theta)e^{-bT}}.$$

According to the comparison theorem in impulsive differential equations, there exists an integer $k_3 > k_2$ such that

$$\tilde{z}_{1e}(t) - \varepsilon_1 < s(t) < \tilde{z}_{2e}(t) + \varepsilon_1, \quad kT < t \le (k+1)T,\ k > k_3. \tag{3.9}$$

Because ε_1 is sufficiently small, it follows from (3.9) that

$$\tilde{s}_e(t) = 1 - \frac{\theta}{1-(1-\theta)e^{-bT}}e^{-b(t-kT)}, \quad kT < t \le (k+1)T$$

is globally attractive. Therefore, the infection-free solution $(\tilde{s}_e(t),\ 0)$ is globally attractive. The proof is complete. □

Denote

$$\theta^* = \left(\frac{\lambda\alpha}{b+\gamma}\left(1+\frac{\beta}{b}\right) - 1\right)(e^{bT}-1),$$
$$T_* = \frac{1}{b}\ln\left(1 + \frac{\theta(b+\gamma)}{\lambda\alpha(1+(\beta/b)) - (b+\gamma)}\right)$$

and

$$\omega^* = \frac{1}{b}\ln\frac{\lambda(1+(\beta/b))(1-e^{-bT})}{(b+\gamma)(1-(1-\theta)e^{-bT})}.$$

According to Theorem 3.1 we can easily obtain the following results.

Corollary 3.1. *The infection-free periodic solution* $(\tilde{s}_e(t),\ 0)$ *is globally attractive provided that* $\lambda\alpha(1+(\beta/b)) \le b+\gamma$.

Corollary 3.2. *Assume that* $\lambda\alpha(1+(\beta/b)) > b+\gamma$. *Then the infection-free periodic solution* $(\tilde{s}_e(t),\ 0)$ *is globally attractive provided that* $\theta > \theta^*$ *or* $T < T_*$.

Corollary 3.3. *For system* (1.4), *we have*:

(i) *Assume that* $\lambda(1+(\beta/b))(1-e^{-bT}) \le (b+\gamma)(1-(1-\theta)e^{-bT})$. *Then the infection-free periodic solution* $(\tilde{s}_e(t),\ 0)$ *is globally attractive.*

(ii) *Assume that* $\lambda(1+(\beta/b))(1-e^{-bT}) > (b+\gamma)(1-(1-\theta)e^{-bT})$. *Then the infection-free periodic solution* $(\tilde{s}_e(t),\ 0)$ *is globally attractive provided that* $\omega > \omega^*$.

Theorem 3.1 determines the global attractivity of (1.4) in Ω for the case $R^* < 1$. Its epidemiological implication is that the infectious population vanishes in time so the disease dies out. Corollary 3.3 implies that the disease will disappear if the latent period of the disease is large enough.

4. Permanence. In this section we say the disease is endemic if the infectious population persists above a certain positive level for a sufficiently long time.

Denote

$$R_* \triangleq \frac{b(1-\theta)}{b+\beta}R^* = \frac{\lambda\alpha(1-\theta)(1-e^{-bT})}{(b+\gamma)(1-(1-\theta)e^{-bT})},$$

and

$$i^* = \frac{b\alpha(1-\theta)(1-e^{-bT})}{(b+\gamma)(1-(1-\theta)e^{-bT})} - \frac{b}{\lambda}.$$

Theorem 4.1. *Suppose* $R_* > 1$. *Then there is a positive constant* q *such that each positive solution* $(s(t),\ i(t))$ *of system* (1.4) *satisfies*

$$i(t) \ge q, \text{ for } t \text{ large enough.}$$

Proof. Note that the second equation of (1.4) can be rewritten as

(4.1)
$$\begin{aligned}\frac{di(t)}{dt} &= \lambda\alpha s(t)i(t) - (b+\gamma)i(t) - \lambda\alpha(s(t)i(t) - s(t-\omega)i(t-\omega)) \\ &= i(t)[\lambda\alpha s(t) - (b+\gamma)] - \lambda\alpha\frac{d}{dt}\int_{t-\omega}^{t} s(u)i(u)\,du.\end{aligned}$$

Let us consider any positive solution $(s(t),\ i(t))$ of system (1.4). According to this solution, we define

$$V(t) = i(t) + \lambda\alpha\int_{t-\omega}^{t} s(u)i(u)\,du.$$

According to (4.1), we calculate the derivative of V along the solutions of (1.4)

$$\frac{dV(t)}{dt} = i(t)[\lambda\alpha s(t) - (b+\gamma)] = (b+\gamma)i(t)\left(\frac{\lambda\alpha}{b+\gamma}s(t) - 1\right). \tag{4.2}$$

Since $R_* > 1$, we easily see that $i^* > 0$, and there exists a sufficiently small $\varepsilon > 0$ such that

$$\frac{\lambda\alpha}{b+\gamma}\left(\frac{b}{b+\lambda i^*}\frac{(1-\theta)(1-e^{-(b+\lambda i^*)T})}{1-(1-\theta)e^{-(b+\lambda i^*)T}} - \varepsilon\right) > 1. \tag{4.3}$$

We claim that, for any $t_0 > 0$, it is impossible that $i(t) < i^*$ for all $t \geq t_0$. Suppose that the claim is not valid. Then there is a $t_0 > 0$ such that $i(t) < i^*$ for all $t \geq t_0$. It follows from the first equation of (1.4) that, for $t \geq t_0$,

$$\frac{ds(t)}{dt} > -\lambda i^* s(t) + b - bs(t) = b - (b+\lambda i^*)s(t).$$

Consider the following comparison impulsive system for $t \geq t_0$,

$$\begin{cases} du(t)/dt = b - (b+\lambda i^*)u(t) & t \neq kT, \\ u(t^+) = (1-\theta)u(t) & t = kT. \end{cases} \tag{4.4}$$

By Lemma 2.2, we obtain that

$$\widetilde{u}_e(t) = \frac{b}{b+\lambda i^*} + \left(u^* - \frac{b}{b+\lambda i^*}\right)e^{-(b+\lambda i^*)(t-kT)}, \quad kT < t \leq (k+1)T,$$

is the unique positive periodic solution of (4.4), which is globally asymptotically stable, where

$$u^* = \frac{b}{b+\lambda i^*}\frac{(1-\theta)(1-e^{-(b+\lambda i^*)T})}{1-(1-\theta)e^{-(b+\lambda i^*)T}}.$$

By use of the comparison theorem in impulsive differential equations, we know that there exists t_1 $(> t_0+\omega)$ such that the following inequality holds true for $t \geq t_1$

$$s(t) > \widetilde{u}_e(t) - \varepsilon.$$

Thus,

$$s(t) > u^* - \varepsilon \triangleq \sigma \quad \text{for } t \geq t_1. \tag{4.5}$$

From (4.3) we have $(\lambda\alpha/(b+\gamma))\sigma > 1$. By (4.2) and (4.5), we have

$$\frac{dV(t)}{dt} > (b+\gamma)i(t)\left(\frac{\lambda\alpha}{b+\gamma}\sigma - 1\right) \quad \text{for } t \geq t_1. \tag{4.6}$$

Set

$$m = \min_{t\in[t_1,\ t_1+\omega]} i(t).$$

We will show that $i(t) \geq m$ for all $t \geq t_1$. Suppose the contrary. Then there is a $T_0 \geq 0$ such that $i(t) \geq m$ for $t_1 \leq t \leq t_1+\omega+T_0$, $i(t_1+\omega+T_0) = m$ and $(di(t_1+\omega+T_0)/dt) \leq 0$. However, the second equation of system (1.4) and (4.5) imply that

$$\frac{di(t_1+\omega+T_0)}{dt} \geq (\lambda\alpha s(t_1+T_0) - (b+\gamma))m > (b+\gamma)\left(\frac{\lambda\alpha}{b+\gamma}\sigma - 1\right)m > 0.$$

This is a contradiction. Thus, $i(t) \geq m$ for all $t \geq t_1$. As a consequence, (4.6) leads to

$$\frac{dV(t)}{dt} > (b+\gamma)\left(\frac{\lambda\alpha}{b+\gamma}\sigma - 1\right)m \quad \text{for } t \geq t_1,$$

which implies that, as $t \to \infty$, $V(t) \to \infty$. This contradicts $V(t) \leq 1+\lambda\alpha\omega$. Hence, the claim is proved.

By the claim, we are left to consider two cases. First, $i(t) \geq i^*$ for all large t. Second, $i(t)$ oscillates about i^* for all large t. Define

$$q = \min\left\{\frac{i^*}{2},\ q_1\right\} \quad \text{and} \quad q_1 \triangleq i^* e^{-(b+\gamma)\omega}.$$

We hope to show that $i(t) \geq q$ for all large t. The conclusion is evident in the first case. For the second case, let $t^* > 0$ and $\xi > 0$ satisfy

$$i(t^*) = i(t^* + \xi) = i^*,$$

and

$$i(t) < i^* \quad \text{for } t^* < t < t^* + \xi,$$

where t^* is sufficiently large such that

$$s(t) > \sigma \quad \text{for } t^* < t < t^* + \xi.$$

$i(t)$ is uniformly continuous since the positive solutions of (1.4) are ultimately bounded and $i(t)$ is not affected by impulses. Hence, there is a ρ, $(0 < \rho < \omega$, and ρ is independent of the choice of $t^*)$ such that $i(t) > (i^*/2)$ for $t^* \leq t \leq t^* + \rho$.

If $\xi \leq \rho$, there is nothing to prove. Let us consider the case $\rho < \xi$. There two subcases to consider:

(i) If $\rho < \xi \leq \omega$, since $(di(t)/dt) > -(b+\gamma)i(t)$ and $i(t^*) = i^*$, it is obvious that $i(t) \geq q$ for $t^* < t < t^* + \xi$.

(ii) If $\xi > \omega$, since $(di(t)/dt) > -(b+\gamma)i(t)$ and $i(t^*) = i^*$, we obtain $i(t) \geq q$ for $t \in [t^*, t^* + \omega]$. Then, proceeding exactly as in the proof for the above claim, we see that $i(t) \geq q$ for $t \in [t^* + \omega, t^* + \xi]$. Since this kind of interval $[t^*, t^* + \xi]$ is chosen in an arbitrary way (we only need t^* to be large), we conclude that $i(t) \geq q$ for all large t in the second case.

In view of our above discussions, the choices of q is independent of the positive solution, and we have proved that any positive solution of (1.4) satisfies $i(t) \geq q$ for all large t. The proof of Theorem 4.1 is complete. □

Denote

$$\theta_* = \frac{(\lambda\alpha - (b+\gamma))(e^{bT}-1)}{\lambda\alpha(e^{bT}-1)+b+\gamma},$$
$$T^* = \frac{1}{b}\ln\left[1 + \frac{\theta(b+\gamma)}{\lambda\alpha(1-\theta)-(b+\gamma)}\right]$$

and

$$\omega_* = \frac{1}{b}\ln\frac{\lambda(1-\theta)(1-e^{-bT})}{(b+\gamma)(1-(1-\theta)e^{-bT})}.$$

From Theorem 4.1, we also easily obtain the following results.

Corollary 4.1. *Assume that $\lambda\alpha > b+\gamma$. Then the disease is uniformly persistent provided that $\theta < \theta_*$.*

Corollary 4.2. *Assume that $\lambda\alpha(1-\theta) > b+\gamma$. Then the disease is uniformly persistent provided that $T > T^*$.*

Corollary 4.3. *Assume that $\lambda(1-\theta)(1-e^{-bT}) > (b+\gamma)(1-(1-\theta)e^{-bT})$. Then the disease is uniformly persistent provided that $\omega < \omega_*$.*

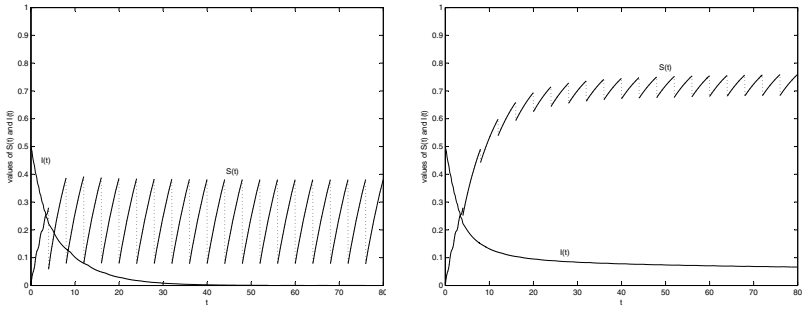
Theorem 4.2. *System* (1.4) *is permanent provided that $R_* > 1$.*

Proof. Denote $(s(t),\ i(t))$ to be any solution of system (1.4). From the first equation of system (1.4), we have

$$\frac{ds(t)}{dt} \geq b - (b+\lambda)s(t).$$

By similar arguments to those in the proof of Theorem 3.1, we have that

$$\lim_{t\to\infty} s(t) \geq p, \tag{4.9}$$



(a) (b)

FIGURE 1. Two figures show the movement paths of S and I as functions of time t. (a) $\theta = 0.8$, $R^* = 0.9843 < 1$. The disease dies out. (b) $\theta = 0.1$, $R_* = 1.0151 > 1$. The disease is permanent. Other parameters are $b = 0.1$, $\lambda = 0.3$, $\omega = 1$, $\tau = 1$, $T = 4$, $\gamma = 0.1$.

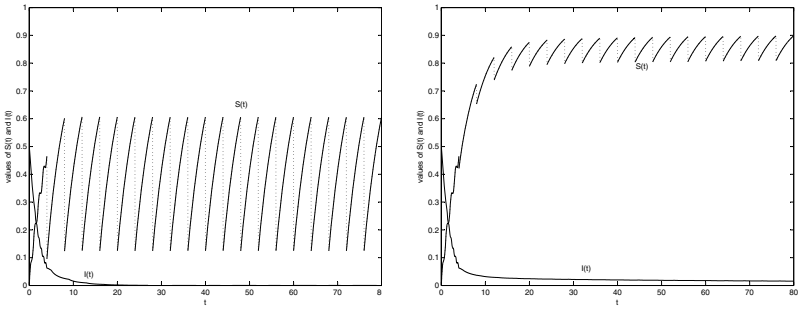
where $p = (b/b+\lambda)((1-\theta)(1-e^{-(b+\lambda)T})/1-(1-\theta)e^{-(b+\lambda)T}) - \varepsilon_0$ ($\varepsilon_0 > 0$ is sufficiently small).

We let $\Omega_0 = \{(s,\ i) \mid p \le s,\ q \le i,\ s+i \le 1\}$. From Theorem 4.1 and inequality (4.9), we know that the set Ω_0 is a global attractor in Ω, and of course, every solution of system (1.4) with initial conditions (2.1) will eventually enter and remain in the region Ω_0. Therefore, system (1.4) is permanent. The proof of Theorem 4.2 is complete. □

Theorem 4.1 determines the permanence of (1.4) in Ω for the case $R_* > 1$. Its epidemiological implication is that the infectious population will persist. Corollaries 4.1, 4.2 and 4.3 imply that: under appropriate conditions, a small vaccination rate or a long period of pulsing or a short latent period of the disease is a sufficient condition for the permanence of the disease, that is, the disease will become endemic.

5. Discussion. We have analyzed the SEIRS epidemic model with pulse vaccination and two time delays. We have shown that $R^* < 1$ implies that the disease will fade out, whereas $R_* > 1$ implies that the disease will be uniformly persistent. Therefore, we can define R^* as the maximal basic reproductive number and R^* can be written as

$$R^* = \Gamma \times \frac{1}{b+\gamma},$$



(a) (b)

FIGURE 2. Two figures show the movement paths of S and I as functions of time t. Parameters are $b = 0.2$, $\lambda = 0.7$, $\omega = 1$, $\tau = 1$, $T = 4$, $\gamma = 0.3$. (a) $\theta = 0.8$, $R^* = 1.5452 > 1$, $R_* = 0.1387 < 1$. The disease dies out. (b) $\theta = 0.1$, $R^* = 2.3612 > 1$, $R_* = 0.9538 < 1$. The disease is permanent.

where $\Gamma \triangleq \lambda\alpha((1 + (\beta/b))(1 - e^{-bT}))/(1 - (1 - \theta)e^{-bT})$, Γ is the maximal number of new infective produced by a typical infective individual per unit time, $1/(b + \gamma)$ is the average time that the infectious individuals stay in the infection class. Similarly, we can define R_* as the minimum basic reproductive number. There is a gap between R_* and R^*. The reason for this gap is that the thresholds are given in concrete terms in this paper. Obviously, from the second and forth equations of system (1.3), we know that the infection-free periodic solution $(\tilde{s}_e(t),\ 0,\ 0,\ 0)$ of system (1.3) is globally attractive if $R_* < 1$, whereas the system (1.3) is permanent if $R_* > 1$.

Let $b = 0.1$, $\lambda = 0.3$, $\omega = 1$, $\tau = 1$, $T = 4$, $\gamma = 0.1$. If $\theta = 0.8$, then $R^* = 0.9843$. According to Theorem 3.1, we know that the disease will disappear. From Figure 1(a), we can also observe the disease will die out. If we choose $\theta = 0.1$, then $R_* = 1.0151$. According to Theorem 4.1, we know that the disease will be permanent, see Figure 1(b). Furthermore, let $b = 0.2$, $\lambda = 0.7$, $\omega = 1$, $\tau = 1$, $T = 4$, $\gamma = 0.3$. If $\theta = 0.8$, then $R^* = 1.5452 > 1$, $R_* = 0.1387 < 1$. If $\theta = 0.1$, then $R^* = 2.3612 > 1$, $R_* = 0.9538 < 1$. Our results cannot solve the two cases. Numerical simulations show that the disease will disappear, see Figure 2(a), or be permanent, see Figure 2(b). This, of course, shows that our results have much room for improvement.

Epidemic models with time delays have received much attention since delays can often be caused by some complicated dynamical behaviors.

Delays in many models can destabilize an equilibrium and thus lead to periodic solutions by Hopf bifurcation [**3, 20**]. It is well known that periodic forcing can drive SIR or SEIR models into behavior which looks chaotic [**15, 16**].

The impulsive model with two time delays (1.4) will be analyzed, in particular paying attention to the following points:

• The global asymptotic stability for SEIRS model with pulse vaccination and two time delays.

• The behavior of the model when an insufficient level of people undergo the vaccination: bifurcation and chaotic solutions.

• Whether periodic or pulse vaccination does a better job than constant vaccination at the same average value.

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