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GLOBAL BEHAVIOR OF A REACTION-DIFFUSION VOLTERRA EQUATION WITH VARIABLE COEFFICIENTS

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ABSTRACT. By using the method of sub- and supersolutions, the technique of monotone iteration and the Lyapunov functional method, we investigated the permanent behavior and global stability of a reaction-diffusion Volterra equation with variable and constant coefficients.

1. Introduction. In [**7**] Volterra proposed a simple model to describe the evolution of a single species population which has the form

$$x'(t) = x(t)\left(a - bx(t) - \int_0^t H(t-s)x(s)\,ds\right), \quad t \geq 0. \tag{1.1}$$

This model describes the growth of a single species whose population density at time t is $x(t)$. Here a and b are positive constants, the term $x(t)(a - bx(t))$ stands for logistic growth and $x(t)\int_0^t H(t-s)x(s)\,ds$ means a hereditary effect, representing competition for resources, which depends on the population's history.

It is worth considering equation (1.1) with diffusion. We assume that the population lives in a bounded domain $\Omega \subset R^n$ and that there is no migration of individuals across the boundary $\partial\Omega$; we further assume that $\partial\Omega$ is a C^2-manifold.

Define the Laplace transformation $\widehat{f}$ of f by

$$\widehat{f}(t) = \int_0^\infty e^{-ts} f(s)\,ds.$$

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In [**9**], Yamada studied global asymptotic stability of equation (1.1) with diffusion

$$u_t = \Delta u + u(a - bu - f * u), \quad x \in \Omega, \quad t \geq 0, \tag{1.2}$$

where

$$f * u = \int_0^t f(t-s)u(x,s)\,ds, \tag{1.3}$$

and obtained the following theorem.

Theorem A. *Let* $\beta = \inf\{\mathrm{Re}\,\widehat{f}(i\eta); \eta \in R^1\}$ *and suppose* $b + \beta > 0$. *Then the solution* u *of* (1.2) *with initial condition*

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1.4}$$

and boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t \geq 0, \tag{1.5}$$

satisfies

$$\lim_{t\to\infty} u(x,t) = \frac{a}{\alpha + b},$$

uniformly for $x \in \Omega$, *where* $\alpha = \int_0^\infty f(s)\,ds$.

In nature, competition exists among species not only for resources, but also for spaces. Based on this consideration, Gourley and Britton [**3**] studied the following integro-differential equation

$$u_t = u + au^2 - bu^3 - (1 + a - b)uf * u + \Delta u, \tag{1.6}$$

where $f * u$ is defined in (1.3) and satisfies

$$f \in C(0,+\infty) \cap L^1(0,+\infty), \tag{1.7}$$

and

$$\int_0^\infty f(t)\,dt = 1. \tag{1.8}$$

Here $b > 0$, and the term $-bu^3$ represents competition for space itself rather than resources.

From the above assumption, they obtained

Theorem B. *Suppose that* f *satisfies* (1.7) *and* (1.8) *and that the coefficients of* (1.6) *satisfy*

$$0 < b < 1 + a; \tag{1.9}$$

furthermore,

$$a < \frac{1}{4}\left[3(b-1) + \sqrt{b^2 + 6b + 1}\right]. \tag{1.10}$$

If $u_0(x) \in C^1(\overline{\Omega})$ *and* (1.5) *hold, then the solution* $u(x,t)$ *of* (1.6) *satisfies*

$$\lim_{t\to\infty} u(x,t) = 1 \tag{1.11}$$

uniformly for $x \in \overline{\Omega}$.

Remark. The condition (1.10) implies

$$a < b.$$

Naturally, we may ask if $[(3(b-1) + \sqrt{b^2 + 6b + 1}]/4 \le a < b$, then does the conclusion in Theorem B still hold true? Furthermore, if all the coefficients in (1.6) are not constants, what may occur? In the present paper, we consider the equation with variable coefficients

$$\begin{cases} u_t - e\Delta u = u(x,t)\big(a(x,t) + b(x,t)u - c(x,t)u^2 - d(x,t) f * u\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in \Omega \times (0,\infty), \\ \partial u/\partial n = 0 \qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = \phi(x) \qquad\qquad\qquad\qquad\quad \text{for } x \in \overline{\Omega}, \end{cases} \tag{1.12}$$

where $f * u$ is defined in (1.3), $e \ge 0$, $0 < a_1 \le a(x,t) \le a_2$, $b_1 \le b(x,t) \le b_2$, $0 < c_1 \le c(x,t) \le c_2$ and $0 < d_1 \le d(x,t) \le d_2$ for

$x \in \overline{\Omega} \times [0, \infty)$. Here a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2 are all constants. In what follows, we always assume the initial function $\phi(x) \in C^1(\overline{\Omega})$.

For existence, uniqueness and regularity of solutions for the initial boundary value problem (1.12), we refer to [**5, 6, 8**].

2. Global behavior of equation (1.12) with constant coefficients. In this section, we assume that, in (1.12), $a_1 = a_2 \equiv a$, $b_1 = b_2 \equiv b$, $c_1 = c_2 \equiv c$ and $d_1 = d_2 \equiv d$, where a, c and d are positive constants and b is a constant. Still we denote the equation by (1.12).

First we give some preliminaries.

For $1 \leq p \leq \infty$, $L^p(\Omega)$ denotes the Banach space of measurable functions u on Ω with the normal norm

$$\|u\|_p = \left[\int_\Omega |u(x)|^p\right]^{1/p} \qquad \text{if } 1 \leq p < \infty,$$
$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty \quad \text{if } p = \infty.$$

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with the usual inner product $(\cdot, \cdot)$. We write $\|\cdot\|$ instead of $\|\cdot\|_2$ so there is no confusion.

The following definition and two lemmas are due to Gopalsamy [**2**].

Definition. A real valued function $K \in L^1_{\text{loc}}(0, +\infty)$ is of positive type, if

$$\int_0^T v(t) \int_0^t v(\xi) K(t - \xi)\, d\xi\, dt \geq 0$$

for every $v \in C(R_+, R)$ and for every $T > 0$. The kernel K is called strongly positive, if there exist numbers $\varepsilon > 0$ and $\alpha > 0$ such that $K(t) - \varepsilon \exp\{-\alpha t\}$ is a positive kernel.

Lemma A. *Let $K : R_+ \to R$ be a bounded function, if*

$$\text{Re}\left(\widehat{K}(\lambda)\right) > 0 \textit{ for } \text{Re}(\lambda) > 0;$$

then K is a positive kernel.

Lemma B. *Assume that K satisfies*

(1) $K \in C[0, +\infty) \cap C^2(0, +\infty)$.

(2) $(-1)^j (d^j/dt^j) K(t) \geq 0$ *for* $t \geq 0$, $j = 0, 1, 2$.

(3) $K(t) \neq$ *constant.*

Then $K(t)$ *is a strongly positive kernel.*

Consider a logistic integrodifferential equation of the form

$$\frac{dx(t)}{dt} = x(t)\left[\alpha - \int_0^t f(s)x(t-s)\,ds\right], \qquad x(0) = x_0 \in (0, +\infty), \quad \alpha \in (0, +\infty). \tag{2.1}$$

Yamada [**9**, Theorem 4.2] obtained a sufficient condition for all solutions of (2.1) to converge. He got the following theorem.

Theorem C. *Assume that*

(1) f *is a strongly positive kernel*;

(2)
$$\int_0^\infty f(t)\,dt = \beta, \quad \frac{\alpha}{\beta}\int_0^\infty tf(t)\,dt < 1.$$

Then every solution of (2.1) *satisfies*

$$\lim_{t\to\infty} x(t) = x^* = \frac{\alpha}{\beta}.$$

It seems that Theorem B can be improved for the special delay kernel; thus, we consider equation (1.12) with constant coefficients again and obtain

Theorem 2.1. *Assume that*

(1) $b \leq c$;

(2) f *is positive when* $c \neq b$ *and strongly positive when* $c = b$;

(3) $f(t) \in L^1(0,\infty)$ *and* $tf(t) \in L^1(0,\infty)$.

Then every solution of (1.12) *satisfies*

$$\lim_{t\to\infty} u(x,t) = u^* \tag{2.2}$$

uniformly for $x \in \overline{\Omega}$*, where* u^* *is the unique positive root of the algebraic equation*

$$c\lambda^2 - (b - d\delta)\lambda - a = 0;$$

here

$$\delta = \int_0^\infty f(s)\,ds. \tag{2.3}$$

Remark. If we choose $f(t) = \exp\{-\alpha t\}$ with $\alpha > 0$, then conditions (2) and (3) in Theorem 2.1 naturally hold true. Thus, for a special delay kernel, Theorem 2.1 improves Theorem B.

Now we give some lemmas which will be useful in the sequel.

Lemma 2.1. *Let* $u \in L^\infty(\Omega)$*. Then*

$$||u||_p \le (||u||_{2(p-\alpha)/(2-\alpha)})^{(p-\alpha)/p}(||u||)^{\alpha/p}, \tag{2.4}$$

for any $p > 2$ *and* $\alpha \le 2$*, especially if* $\alpha = 2$*, then*

$$||u||_p \le (||u||_\infty)^{(p-2)/p}(||u||)^{2/p}. \tag{2.5}$$

Proof. If $\alpha = 2$, then (2.5) is a direct result of (2.4). In the following, we assume that $\alpha < 2$. Then it is easy to see that $p > \alpha$. From the Holder inequality, we have

$$\begin{aligned} ||u||_p &= \left(\int_\Omega u^p\right)^{1/p} = \left(\int_\Omega u^{p-\alpha}u^\alpha\right)^{1/p} \\ &\le \left(\left(\int_\Omega (u^{p-\alpha})^{p'}\right)^{1/((p-\alpha)p')}\right)^{(p-\alpha)/p} \left(\left(\int_\Omega (u^\alpha)^{q'}\right)^{1/(\alpha q')}\right)^{\alpha/p}. \end{aligned}$$

Choose $q' = 2/\alpha > 1$ and $p' = 2/(2-\alpha)$. Then (2.4) follows. □

Lemma 2.2. *Assume that*

$$||u||_\infty \le H, \qquad ||\nabla u||_\infty \le H, \tag{2.6}$$

and

$$\lim_{t\to\infty} ||u(x,t) - \widetilde{u}|| = 0, \qquad \lim_{t\to\infty} ||\nabla u|| = 0, \tag{2.7}$$

where H *and* $\widetilde{u}$ *are positive constants. Then*

$$\lim_{t\to\infty} u(x,t) = \widetilde{u}, \tag{2.8}$$

uniformly for $x \in \overline{\Omega}$.

Proof. From (2.5), (2.6) and (2.7), we can easily obtain

$$\lim_{t\to\infty} ||u(x,t) - \widetilde{u}||_p = 0, \qquad \lim_{t\to\infty} ||\nabla u||_p = 0, \tag{2.9}$$

for $p > 2$. Notice that when $p > n$, $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\Omega)$. Choose $p > \max\{2, n\}$. Then, from (2.9) and the Sobolev imbedding theorem, we have

$$\lim_{t\to\infty} ||u(x,t) - \widetilde{u}||_\infty = 0,$$

which implies that (2.8) holds uniformly for $x \in \overline{\Omega}$. This completes the proof. $\square$

The following lemma is a direct result of the maximal principle.

Lemma 2.3. *For any solution* $u(x,t)$ *of* (1.12), *we have*

$$0 \le u(x,t) \le \max\{||\Phi||_\infty, \gamma\},$$

where γ *is the unique positive solution of equation*

$$a + b\lambda - c\lambda^2 = 0.$$

Lemma 2.4. *The maximal existence interval of solution* $u(x,t)$ *of* (1.12) *is* $[0, +\infty)$ *provided that* $\Phi_0 \in C^1(\overline{\Omega})$.

Proof. The local existence is obvious, from the maximal principle, we have $u(x,t) > 0$ on its maximal existence interval $[0, T)$; thus from equation (1.12), we can obtain

$$u_t \le au + bu^2 - cu^3 + e\Delta u,$$

which implies that $T = \infty$. □

Lemma 2.5. *Assume that L, a_3, b_3 and c_3 are all positive constants, φ, $\psi \in C^1[a_3, +\infty)$ satisfy*

$$\varphi' \le -b_3\psi + c_3\omega, \tag{2.10}$$

where $' = d/dt$, $\omega \in L^1[a_3, +\infty)$, $\psi \ge 0$, $|\psi'| \le L$ and φ is bounded below. Then

$$\lim_{t\to\infty} \psi(t) = 0.$$

Proof. Integrate both sides of (2.10) from a_3 to $t(t \ge a_3)$, we have

$$\varphi(t) - \varphi(a_3) + b_3 \int_a^t \psi(s)ds \le \int_a^t c_3\omega(s)\, ds;$$

thus,

$$b_3 \int_a^t \psi(s)ds \le c_3 \int_a^t \omega(s)\, ds + \varphi(a_3) - \varphi(t) < \infty \quad \text{for } t \ge a_3.$$

Therefore, $\psi \in L^1[a_3, +\infty)$, also in view of $|\psi'| \le L$ and Barbălat lemma [**3**], we reach the conclusion. □

Remark. One can see that this lemma is an extension of Lemma 1 in [**6**].

Proof of Theorem 2.1. Define a Lyapunov function

$$V(t) = \int_\Omega \left(u - u^* - u^* \log \frac{u}{u^*} \right) dx.$$

Then

$$\begin{aligned}
V'(t) &= \int_\Omega (1 - \frac{u^*}{u}) u_t \, dx \\
&= \int_\Omega (u - u^*) \left[a + bu - cu^2 - f * u\right] dx \\
&\quad + \int_\Omega \frac{(u - u^*)}{u} \Delta u \, dx \\
&\le - \int_\Omega (c - b)(u - u^*)^2 \\
&\quad - \int_\Omega \left[(u - u^*) \int_0^t f(t - s) \left[u(x, s) - u^*\right] \, ds \right] dx \\
&\quad + \int_\Omega (u - u^*) \int_t^\infty f(s) \, ds - \int_\Omega \frac{u^*}{u^2} \left|\left|\nabla u\right|\right|^2 \, dx \\
&\le - \int_\Omega (c - b)(u - u^*)^2 \, dx \\
&\quad - \int_\Omega \left[(u - u^*) \int_0^t f(t - s) \left[u(x, s) - u^*\right] \, ds \right] dx \\
&\quad + \int_\Omega \left[u \int_t^\infty f(s) ds \right] dx - \int_\Omega \frac{u^*}{K^2} \left|\left|\nabla u\right|\right|^2 \, dx,
\end{aligned}$$

where $K > \max\{||\Phi||_\infty, \gamma\}$ is a constant. Integrating both sides of the above inequality from 0 to T, we have

$$\begin{aligned}
V(T) &+ \int_0^T \left[\int_\Omega \frac{u^*}{K^2} \left|\nabla u\right|^2 \, dx \right] dt \\
&+ (c - b) \int_0^T \left[\int_\Omega (u - u^*)^2 \, dx \right] dt \\
&\le V(0) - \int_\Omega \left[\int_0^T \left[(u - u^*) \int_0^t f(t - s) \left[u(x, s) - u^*\right] \, ds \right] dt \right] dx \\
&+ K|\Omega| \int_0^T \left[\int_t^\infty f(s) \, ds \right] dt,
\end{aligned}$$

where $|\Omega|$ represents the volume of Ω. In view of (1) and (2) of Theorem 2.1, we obtain that two positive constants β and δ exist such that

$$\frac{u^*}{K^2} \int_0^T \|\nabla u\|^2 \, dt + (c - b) \int_0^T \|u - u^*\|^2 \, dt + \beta \int_0^T \|u - u^*\|^2 \, dt \le \delta.$$

Since conditions (1) and (2) imply that $c-b+\beta>0$, we have

$$\|u-u^*\|^2 \in L^1[0,\infty) \quad \text{and} \quad \|\nabla u\|^2 \in L^1[0,\infty).$$

By Lemma 2.5, we only need to prove that there exists some constant K_2 such that

$$\left|\frac{d}{dt}\|u-u^*\|^2\right| \le K_2 \quad \text{and} \quad \left|\frac{d}{dt}\|\nabla u\|^2\right| \le K_2.$$

Multiplying both sides of (1.12) by $-u$ and then integrating on Ω, we have

$$0 \ge -\frac{1}{2}\frac{d}{dt}\|u\|^2 = \int_\Omega \left[-u^2\left[a+bu-cu^2-df*u\right]\right]\,dx + e\|\nabla u\|^2.$$

By Lemma 2.3, we know that there exists a constant K_1 such that

$$\|\nabla u\|^2 \le K_1. \tag{2.11}$$

Since

$$\begin{aligned}\left|\frac{d}{dt}\|u-u^*\|^2\right| &= \left|\int_\Omega 2(u-u^*)u_t\,dx\right| \\ &= \left|\int_\Omega 2(u-u^*)\left[au+bu^2-cu^3-duf*u+e\Delta u\right]\,dx\right| \\ &\le 2\left(a+bK_3+cK_4+dK_5\right)\|u-u^*\|^2 \\ &\quad + du^*\int_\Omega\left[\int_t^\infty f(t-s)u(x,s)\,ds\right]dx + e\|\nabla u\|^2,\end{aligned}$$

where

$$K_3 = K+u^*, \quad K_4 = K^2+Ku^*+u^{*2}, \quad K_5 = (K+u^*)\int_0^\infty f(s)\,ds;$$

also notice that

$$\|u-u^*\|^2 = \int_\Omega (u-u^*)^2\,dx \le \left(K^2+u^{*2}\right)|\Omega|,$$

and

$$du^* \int_\Omega \left[\int_t^\infty f(t-s)u(x,s)\, ds \right] dx \le du^* K |\Omega| \int_0^\infty f(s)\, ds.$$

We know that there exists a constant K_2 such that

$$\left| \frac{d}{dt} \|u - u^*\|^2 \right| \le K_2.$$

On the other hand,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 &= (u_t, -\Delta u) \\ &\le \left(\nabla u, \nabla \left[au + bu^2 - cu^3 - duf * u\right]\right) - e\|\Delta u\|^2 \\ &\le (a + |b|K)\, \|\nabla u\|^2 - e\|\Delta u\|^2. \end{aligned} \tag{2.12}$$

By virtue of (2.12), we have

$$\frac{d}{dt}\|\nabla u\|^2 \le 2\,(a + |b|K)\, K_1,$$

and

$$\|\Delta u\|^2 \le \frac{1}{e} 2\,(a + |b|K)\, K_1;$$

also from (2.12) we know

$$\begin{aligned} \frac{1}{2}\left|\frac{d}{dt}\|\nabla u\|^2\right| &= |(u_t, -\Delta u)| \\ &\le \left(a + |b|K + 2cK^2 + 2dK \int_0^\infty f(s)\, ds\right) \|\nabla u\|^2 + e\|\Delta u\|^2 \\ &\le \frac{K_2}{2}. \end{aligned}$$

Then, by Lemma 2.2, it remains to prove that there exists a constant H such that $\|\nabla u\|_\infty \le H$.

Let $\{T(t)\}_{t\geq 0}$ be the semi-group generated by the linear equation

$$\begin{cases} \partial/(\partial t)u(x,t) = e\Delta u & x\in\Omega,\ t>0,\\ \partial/(\partial n)u(x,t) = 0 & x\in\partial\Omega,\ t\geq 0. \end{cases}$$

Then $\{T(t)\}_{t\geq 0}$ is an analytic semi-group. Assume that $-A$ is the infinitesimal generator of $\{T(t)\}_{t\geq 0}$. Then $A : \operatorname{Dom} A \subset L^p(\Omega) \to L^p(\Omega)$ is defined by

$$-Au = e\Delta u, \qquad u \in \operatorname{Dom} A,$$

where

$$\operatorname{Dom} A = \left\{ u \in W^{2,p}(\Omega);\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Obviously, A is a linear operator, and A is closed and dense in the space $L^p(\Omega)$. For each $0 \leq \nu \leq 1$, we introduce the fractional power space $X^\nu \equiv D(A^\nu)$ equipped with the graph norm of A^ν, i.e., for any $u \in X^\nu$, $||u||_\nu = ||A^\nu u||$. If we let $p > n$ be fixed, then we have

$$X^\nu \hookrightarrow C^\sigma(\overline{\Omega}) \quad \text{for } 0 \leq \sigma < 2\nu - \frac{n}{p}$$

(the inclusion is continuous), and if ν is chosen so closed to 1 that $2\nu - n/p > 1$, then

$$X^\nu \hookrightarrow \left\{ u \in C^1(\overline{\Omega});\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}. \tag{2.13}$$

Notice that the solution $u(x,t)$ of equation (1.12) is also a solution of the following integral equation

$$u(t) = T(t)\phi + \int_0^t T(t-s)F(s)\,ds, \tag{2.14}$$

where

$$F(s) = u(s)\left(a + bu(s) - cu^2(s) - df * u(s)\right).$$

If $0 < t \leq 1$, then operating A^ν on both sides of (2.14), we have

(2.15)
$$\|A^\nu u(t)\|_p \leq \|A^\nu T(t)\phi\|_p + C_\nu\left(a + |b|\,K + cK^2 + dK\delta\right) K\,|\Omega|^{1/p}\,\frac{1}{1-\nu},$$

where C_ν is a positive constant.

If $t \geq 1$, we use another integral equation

$$u(t) = T(t)u(t-1) + \int_{t-1}^{t} T(t-s)F(s)\,ds. \tag{2.16}$$

Operating A^ν on both sides of (2.16), we get

$$\begin{aligned} \|A^\nu u(t)\|_p &\leq C_\nu ||u||_p + C_\nu \int_{t-1}^{t} \frac{||F||_p}{(t-s)^\nu}\,ds \\ &\leq C_\nu K\,|\Omega|^{1/p} \left[1 + \left(a + |b|\,K + cK^2 + dK\delta\right)\frac{1}{1-\nu}\right]. \end{aligned} \tag{2.17}$$

Equations (2.15) and (2.17) imply that

$$\|A^\nu u(t)\|_p \leq M, \quad \text{for } t > 0.$$

Then from (2.13), we have

$$\|\nabla u\|_\infty \leq H.$$

By Lemma 2.2, we reach the conclusion. □

3. Qualitative analysis of equation (1.12) with variable coefficients. In this section, we consider equation (1.12) with variable coefficients; we adopt the method of successive improvement of sub- and supersolutions due to Redlinger [**5**] for semi-linear parabolic systems with functionals.

The following two theorems show that equation (1.12) is permanent under certain assumptions.

Theorem 3.1. *If*

$$a_1 > d_2 p_2^* \delta,$$

where p_2^* *is the unique positive solution of equation* $a_2 + b_2\lambda - c_1\lambda^2 = 0$ *and* δ *is defined as in* (2.3)*, then*

$$\mu_1 \leq \lim\inf_{t\to\infty} \min_{x\in\overline{\Omega}} u(x,t) \leq \lim\sup_{t\to\infty} \max_{x\in\overline{\Omega}} u(x,t) \leq \mu_2,$$

where μ_1 *and* μ_2 *are given by the following system*

$$(3.1)\qquad \begin{cases} (a_2 - d_1\mu_1\delta) + b_2\mu_2 - c_1\mu_2^2 = 0, \\ (a_1 - d_2\mu_2\delta) + b_1\mu_1 - c_2\mu_1^2 = 0, \\ \mu_i > 0 \qquad \textit{for } i = 1, 2. \end{cases}$$

The following theorem is an extension of Theorem 2.1.

Theorem 3.2. *Assume that*

(1) $b_2 \leq c_1$;

(2) f *is positive when* $c_1 \neq b_2$ *and strongly positive when* $c_1 = b_2$;

(3) $f(t) \in L^1(0,\infty)$ *and* $tf(t) \in L^1(0,\infty)$.

Then every solution $u(x,t)$ *of* (1.12) *satisfies*

$$\eta_1 \leq \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(x,t) \leq \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(x,t) \leq \eta_2,$$

where η_1 *and* η_2 *are, respectively, the unique positive solutions of equation* $a_1 + (b_1 - d_2\delta)\lambda - c_2\lambda^2 = 0$ *and* $a_2 + (b_2 - d_1\delta)\lambda - c_1\lambda^2 = 0$.

Let $D = \Omega \times (0,\infty)$. We now introduce Redlinger's definition of sub- and supersolutions. A pair of sub- and supersolutions for (1.12) is a pair of suitable smooth functions v and w such that

1) $v \leq w$ in $\overline{D}$;

2) the differential inequalities

$$\begin{aligned} v_t &\leq v + av^2 - bv^3 - dvf * \Phi + \Delta v, \\ w_t &\geq w + aw^2 - bw^3 - dwf * \Phi + \Delta w, \end{aligned}$$

are satisfied for all functions $\Phi \in C(\overline{D})$ with $v \leq \Phi \leq w$ in $\overline{D}$;

3) $\nabla v \cdot n \leq 0$ and $\nabla w \cdot n \geq 0$ on $\partial\Omega \times (0,\infty)$;

4) $v(x,0) \leq \phi(x) \leq w(x,0)$ for $x \in \overline{\Omega}$.

The following lemma is from [**3**].

Lemma 3.1. *Let v and w be sub- and supersolutions for* (1.12). *Then there exists a unique regular solution u such that $v \le u \le w$ in $\overline{D}$.*

Consider the following two companion equations

(3.2)
$$\begin{cases} \frac{\partial v}{\partial t} = e\Delta v + v\left(a_1 + b_1 v - c_2 v^2 - d_2 f * v\right) & \text{for } (x,t) \in D, \\ \frac{\partial v}{\partial n} = 0 & \text{for } (x,t) \in \partial\Omega \times (0,\infty), \\ 0 \le v(x,0) \le \inf_{x\in\overline{\Omega}} \phi(x,0) & \text{for } (x,t) \in \overline{D}, \end{cases}$$

and

(3.3)
$$\begin{cases} \frac{\partial w}{\partial t} = e\Delta w + w\left(a_2 + b_2 w - c_1 w^2 - d_1 f * w\right) & \text{for } (x,t) \in D, \\ \frac{\partial w}{\partial n} = 0 & \text{for } (x,t) \in \partial\Omega \\ & \times(0,\infty), \\ w(x,0) \ge \sup_{x\in\overline{\Omega}} \phi(x,0) & \text{for } (x,t) \in \overline{D}. \end{cases}$$

By Lemma 3.1, we have

Lemma 3.2. *If $v(x,t)$, $u(x,t)$ and $w(x,t)$ are, respectively, the unique solution of* (3.2), (1.12) *and* (3.3), *then*

$$0 < v(x,t) \le u(x,t) \le w(x,t), \ \textit{for } (x,t) \in D.$$

The following lemma is a trivial conclusion from the comparison theorem of ODEs.

Lemma 3.3. *Suppose that $p(t)$ is the unique solution of the Cauchy problem*

$$\begin{cases} dp(t)/dt = p(t)(a_2 + b_2 p(t) - c_1 p^2(t)), \\ p(0) = p_0 > 0. \end{cases}$$

Then $p(t)$ satisfies

$$\lim_{t\to\infty} p(t) = p_2^*,$$

and if $p_0 > p_2^*$, then $p(t) \geq p_2^*$ for $t \geq 0$; if $p_0 \leq p_2^*$, then $p(t) \leq p_2^*$ for $t \geq 0$ and if $p_0 = p_2^*$, then $p(t) \equiv p_2^*$ for $t \geq 0$.

Notice that $v \equiv 0$ and $w \equiv K_0$ for $K_0 > \max\{p_2^*, \|\phi\|_\infty\}$ are, respectively, the sub- and supersolutions of (1.12); thus, from Lemma 3.1, there exists a unique regular solution u of (1.12) such that $0 \leq u \leq K_0$, which implies that $\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(x,t)$ and $\limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(x,t)$ both exist.

For simplicity, we denote $u_1 = \liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(x,t)$ and $u_2 = \limsup_{t\to\infty} \max_{x\in\overline{\Omega}} u(x,t)$. Then, by Lemma 3.2, we have $0 \leq u_1 \leq u_2 \leq K_0$. In fact, we can show $0 \leq u_1 \leq u_2 \leq p_2^*$. Let w_1 be the solution of the initial value problem

$$w_1'(t) = w_1\left(a_2 + b_2 w_1(t) - c_1 w_1^2(t)\right), \quad w_1(0) = \|\phi\|_\infty > 0.$$

Then the functions $v \equiv 0$ and $w = w_1$ are sub- and supersolutions for (1.12). Therefore, from Lemma 3.1, there exists a unique regular solution $\widetilde{u}$ of (1.12) such that $0 \leq \widetilde{u} \leq w_1$ in $\overline{D}$. Now we only need to prove $\widetilde{u} = u$, and it is enough to show that $\widetilde{u} \leq K_0$ for all $t \geq 0$. From Lemma 3.3, this is obvious. From $u_2 \leq p_2^*$ and $f \in L^1(\Omega)$, we know that, for any given $\varepsilon > 0$, there exists a T_1 such that $u(x,t) \leq p_2^* + \varepsilon$ and $\int_{T_1}^t f(s)ds < \varepsilon$ for $t \geq T_1$ and $x \in \overline{\Omega}$. By virtue of (1.12), for $t > 2T_1$, we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &\geq e\Delta u + u\bigg(a_1 + b_1 u - c_2 u^2 - d_2 \int_0^t f(t-s)u(x,s)\,ds\bigg) \\
&= e\Delta u + u\bigg(a_1 + b_1 u - c_2 u^2 - d_2 \int_0^{t-T_1} f(t-s)u(x,s)\,ds \\
&\qquad - d_2 \int_{t-T_1}^t f(t-s)u(x,s)\,ds\bigg) \\
&= e\Delta u + u\bigg(a_1 + b_1 u - c_2 u^2 - d_2 \int_{T_1}^t f(s)u(x,t-s)\,ds \\
&\qquad - d_2 \int_{t-T_1}^t f(t-s)u(x,s)\,ds\bigg) \\
&\geq e\Delta u + u\left(a_1 + b_1 u - c_2 u^2 - d_2 K\varepsilon - d_2\left(p_2^* + \varepsilon\right)\delta\right),
\end{aligned}$$

which implies that $u_1 \geq p_1^*$, where p_1^* is the unique positive solution of equation

$$(a_1 - d_2 p_2^* \delta) + b_1\lambda - c_2\lambda^2 = 0.$$

Set $P_1^{(1)} = p_1^*$ and $P_1^{(2)} = p_2^*$. For any $\varepsilon > 0$ small enough, there exists a $T_2 \geq 2T_1$ such that $u(x,t) \geq P_1^{(1)} - \varepsilon > 0$, $\int_{T_2}^t f(s)\,ds < \varepsilon$ and $\int_0^{T_2} f(s)\,ds \geq \delta - \varepsilon > 0$ for $t \geq T_2$ and $x \in \overline{\Omega}$. Thus, for $t \geq 2T_2$, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &\leq e\Delta u + u\Big(a_2 + b_2 u - c_1 u^2 - d_1 \int_0^t f(t-s)u(x,s)\,ds\Big)\\ &= e\Delta u + u\Big(a_2 + b_2 u - c_1 u^2 - d_1 \int_{T_2}^t f(s)u(x,t-s)\,ds\\ &\qquad - d_1 \int_{t-T_2}^t f(t-s)u(x,s)\,ds\Big)\\ &\leq e\Delta u + u\left(a_2 + b_2 u - c_1 u^2 - d_1\left(P_1^{(1)} - \varepsilon\right)(\delta - \varepsilon)\right).\end{aligned}$$

Note that $a_1 > d_2 p_2^* \delta$ implies $a_2 > d_1 P_1^{(1)} \delta$. Then, from Lemma 3.1, we have $u_2 \leq P_2^{(2)}$ where $P_2^{(2)}$ is the unique positive solution of equation

$$\Big(a_2 - d_1 P_1^{(1)} \delta\Big) + b_2 \lambda - c_1 \lambda^2 = 0.$$

After some simple calculations, we know that $P_2^{(2)} < P_1^{(2)}$.

Define two sequences $\left\{P_n^{(1)}\right\}$ and $\left\{P_n^{(2)}\right\}$ as follows:

$$\tag{3.4}\begin{cases}\Big(a_2 - d_1 P_{n-1}^{(1)} \delta\Big) + b_2 P_n^{(2)} - c_1 \Big(P_n^{(2)}\Big)^2 = 0,\\ \Big(a_1 - d_2 P_n^{(2)} \delta\Big) + b_1 P_n^{(1)} - c_2 \Big(P_n^{(1)}\Big)^2 = 0,\\ P_1^{(2)} = p_2^*, \quad P_n^{(i)} > 0 \text{ for } i = 1,2 \text{ and } n = 1,2,\dots.\end{cases}$$

Then we have

Lemma 3.4. *Let* $\left\{P_n^{(1)}\right\}$ *and* $\left\{P_n^{(2)}\right\}$ *be defined as in* (3.4). *Then*

$$\tag{3.5} P_1^{(1)} \leq P_2^{(1)} \leq \cdots \leq P_n^{(1)} \leq \cdots \leq P_n^{(2)} \leq \cdots \leq P_2^{(2)} \leq P_1^{(2)}.$$

Proof. Notice that

$$\begin{aligned} a_2 + b_2 P_1^{(2)} - c_1\Big(P_1^{(2)}\Big)^2 = 0 &= (a_1 - d_2 p_2^* \delta) + b_1 P_1^{(1)} - c_2 \Big(P_1^{(1)}\Big)^2\\ &\leq a_2 + b_2 P_1^{(1)} - c_1 \Big(P_1^{(1)}\Big)^2.\end{aligned}$$

We have

$$P_1^{(2)} \geq P_1^{(1)}.$$

From (3.4), we know

$$\begin{aligned} a_2 + b_2 P_1^{(2)} - c_1 \left(P_1^{(2)}\right)^2 = 0 &= \left(a_2 - d_1 P_1^{(1)} \delta\right) + b_2 P_2^{(2)} - c_1 \left(P_2^{(2)}\right)^2 \\ &\leq a_2 + b_2 P_2^{(2)} - c_1 \left(P_2^{(2)}\right)^2, \end{aligned}$$

which implies that

$$P_2^{(2)} \leq P_1^{(2)}. \tag{3.6}$$

By virtue of (3.6) and (3.4), we obtain

$$\begin{aligned} &\left(a_1 - d_2 P_2^{(2)} \delta\right) + b_1 P_2^{(1)} - c_2 \left(P_2^{(1)}\right)^2 \\ &\quad = 0 \geq \left(a_1 - d_2 P_1^{(2)} \delta\right) + b_1 P_2^{(1)} - c_2 \left(P_2^{(1)}\right)^2 \\ &\quad = b_1 \left(P_2^{(1)} - P_1^{(1)}\right) - c_2 \left(P_2^{(1)} - P_1^{(1)}\right) \left(P_2^{(1)} + P_1^{(1)}\right) \\ &\quad = \left(P_2^{(1)} - P_1^{(1)}\right) \left[b_1 - c_2 \left(P_2^{(1)} + P_1^{(1)}\right)\right]. \end{aligned} \tag{3.7}$$

Note that

$$b_1 P_n^{(1)} - c_2 \left(P_n^{(1)}\right)^2 = -\left(a_1 - d_2 P_n^{(2)} \delta\right) < 0,$$

leads to

$$b_1 - c_2 \left(P_2^{(1)} + P_1^{(1)}\right) < 0.$$

Thus, from (3.7),

$$P_2^{(1)} \geq P_1^{(1)}.$$

By mathematical induction, we complete the proof. □

Lemma 3.4 shows that the limit values of sequences $\left\{P_n^{(1)}\right\}$ and $\left\{P_n^{(2)}\right\}$ both exist, denoted by μ_1 and μ_2 respectively. Then by the constructions of these two sequences, we have

Lemma 3.5. *$P_n^{(1)} \le u_1 \le u_2 \le P_n^{(2)}$, for any $n \ge 0$, where $P_n^{(1)}$ and $P_n^{(2)}$ satisfy* (3.4).

Proof of Theorem 3.1. By Lemmas 3.4 and 3.5, we can easily reach the conclusion.

If all the coefficients in (1.12) are constants, we have

Corollary 3.1. *If $a_1 = a_2 \equiv a$, $b_1 = b_2 \equiv b$, $c_1 = c_2 \equiv c$ and $d_1 = d_2 \equiv d$, then any solution $u(x,t)$ of equation* (1.12) *satisfies* (2.2) *uniformly for $x \in \overline{\Omega}$ provided that $a > d\delta\gamma$, where γ is defined in Lemma* 2.3.

Proof. We only need to show, in this case, $\mu_1 = \mu_2$, where μ_1 and μ_2 satisfy (3.1). Assume that it is false. Then, from (3.1), we obtain

$$-d\delta - b + c\,(\mu_1 + \mu_2) = 0,$$

and

$$2a - d\delta\,(\mu_1 + \mu_2) + b\,(\mu_1 + \mu_2) - c\,(\mu_1 + \mu_2)^2 + 2c\mu_1\mu_2 = 0.$$

Hence,

$$\mu_1\mu_2 = \frac{d^2\delta^2 + bd\delta - ac}{c^2}.$$

Note that $a > d\delta\gamma$ implies

$$ac > d^2\delta^2 + bd\delta.$$

Thus,

$$\mu_1\mu_2 < 0.$$

This is a contradiction, and so we complete the proof. □

Remark. Obviously, Corollary 3.1 improves Theorem B, and the method we used here is different from that of [**3**].

Proof of Theorem 3.2. By Theorem 2.1, under the assumptions of Theorem 3.2, for any solution $v(x,t)$ of (3.2), we have

$$\lim_{t\to\infty} v(x,t) = \eta_1,$$

and for any solution $w(x,t)$ of (3.3),

$$\lim_{t\to\infty} w(x,t) = \eta_2.$$

Then, by Lemma 3.2, we complete the proof. □

Remark. Theorems 3.1 and 3.2 remain true, if we replace the Laplacian operator Δ by a linear, uniform elliptic operator.

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