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COMPACTIFICATION OF MIXED MODULI SPACES IN MORSE–FLOER THEORY

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ABSTRACT. We investigate convergences in spaces which include holomorphic strips and gradient trajectories of a Morse function.

1. Introduction. Let M be a compact manifold and $f : M \to \mathbf{R}$ a Morse function. Let $P = T^*M$ be a cotangent bundle over M, $L_0 = O_M$ a zero section, $H : T^*M \to \mathbf{R}$ a compactly supported Hamiltonian and $L_1 = \phi_1^H(L_0)$ a corresponding Hamiltonian deformation of O_M. Denote by $HM_*(f)$ the Morse homology groups generated by critical points of f and by $HF_*(H)$ the Floer homology groups generated by Hamiltonian paths starting and ending at the zero section. For two Morse functions f^α and f^β, Morse homology groups $HM_*(f^\alpha)$ and $HM_*(f^\beta)$ are isomorphic, and the same is true for two different Hamiltonians H^α and H^β. We denote by

$$T^{\alpha\beta} : HM_*(f^\alpha) \longrightarrow HM_*(f^\beta), \quad S^{\alpha\beta} : HF_*(H^\alpha) \longrightarrow HF_*(H^\beta)$$

the mentioned isomorphisms. (See [**9, 10**] for more details.)

Floer [**1**] proved that Morse and Floer homology groups are isomorphic, provided that f is C^2−small enough, by choosing the Hamiltonian $H_f := f \circ \pi$, where $\pi : T^*M \to M$ is the canonical projection (actually he proved that the sets of generators are in one-to-one correspondence; the same is true for holomorphic discs and gradient trajectories which define the boundary operator on the chain complexes).

The constructions of $T^{\alpha\beta}$ and $S^{\alpha\beta}$ are based on counting the numbers of the solutions of some differential equations which are ordinary in

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Morse case and partial in Floer's (i.e., which are of different types). Therefore, it is not obvious whether the diagram

$$
\begin{array}{ccc}
HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\
\uparrow & & \uparrow \\
HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta)
\end{array}
\tag{1}
$$

commutes.

The isomorphism between Morse and Floer homology groups can be established by counting the number of mixed objects which connect critical points of f and generators of Floer homologies and which are solutions of some differential equations with Lagrangian boundary conditions. (This idea goes back to Piunikhin, Salamon and Schwarz who constructed a similar isomorphism defined by the intersection numbers of spaces of perturbed holomorphic spheres and spaces of gradient trajectories [**6**].) More precisely, let p be a critical point of a Morse function $f : M \to \mathbf{R}$, $H \in C_c^\infty(T^*M)$, X_H a corresponding Hamiltonian vector field and $x : [0,1] \to T^*M$ such that

$$
\dot{x} = X_H(x), \quad x(0),\ x(1) \in O_M. \tag{2}
$$

Denote by $\mathcal{M}(p, f; x, H)$ the set of pairs of maps

$$
\gamma : (-\infty, 0] \longrightarrow M, \quad u : [0, +\infty) \times [0,1] \longrightarrow T^*M
$$

that satisfy

$$
\begin{cases}
(d\gamma/ds) = -\nabla f(\gamma(s)), \\
(\partial u/\partial s) + J((\partial u/\partial t) - X_{\rho_R H}(u)) = 0, \\
u(\partial([0,+\infty) \times [0,1])) \subset O_M, \\
\gamma(-\infty) = p,\ u(+\infty, t) = x(t), \\
\gamma(0) = u(0, (1/2))
\end{cases}
\tag{3}
$$

where $\rho_R : [0, +\infty) \to \mathbf{R}$ is a smooth function such that

$$
\rho_R(s) = \begin{cases} 1 & s \geq R+1, \\ 0 & s \leq R, \end{cases}
$$

and $\rho_R H : \mathbf{R} \times T^*M \to \mathbf{R}$, $\rho_R H(s, x) = \rho_R(s) H(x)$.

Then, for generic f and H, $\mathcal{M}(p,f;x,H)$ is a smooth manifold, compact when its dimension is zero. In that case we define the homomorphism:

$$\psi : CM_*(f) \longrightarrow CF_*(H), \qquad p \longmapsto \sum_{\mu_H(x)+n/2=m_f(p)} n(p,f;x,H)x$$

on the generators, where $n(p,f;x,H)$ is a cardinal number of $\mathcal{M}(p,f;x,H)$. Here $m_f(p)$ is a Morse index of critical point p, $\mu_H(x)$ is a Maslov index of Hamiltonian path x as defined in [**7, 8**] and $n = \dim M$. Similarly, we define the homomorphism $\phi : CF_*(H) \to CM_*(f)$. In [**2**] we showed that chain homomorphisms ϕ and ψ induce homomorphisms Φ and Ψ in homology which are isomorphisms and proved the commutativity of (1) in this case. The main technical tool that we used there was the analysis of compactifications of spaces $\mathcal{M}(p,f;x,H)$ for any p, x. The purpose of this paper is to give the details of this analysis.

2. Convergence of maps with fixed Hamiltonian and domain. Denote by $\mathcal{M}(p,q,f)$ the set of all γ that satisfy:

$$\begin{cases} (d\gamma/ds) + \nabla f(\gamma) = 0, \\ \gamma(-\infty) = p, \ \gamma(+\infty) = q, \end{cases} \tag{4}$$

and by $\mathcal{M}(x,y,H)$ the set of all u which are solutions of:

$$\begin{cases} (\partial u/\partial s) + J((\partial u/\partial t) - X_H(u)) = 0, & \\ u(s,i) \in L_0 & i \in \{0,1\}, \\ u(-\infty,t) = \phi_t^H((\phi_1^H)^{-1})(x), & \\ u(+\infty,t) = \phi_t^H((\phi_1^H)^{-1})(y) & x,y \in L_0 \cap L_1, \end{cases} \tag{5}$$

modulo $\mathbf{R}$ action.

Recall that Hamiltonian paths with ends in O_M (the solutions of (2)) are critical points of the *action functional* defined on $\Omega := \{\alpha : [0,1] \to T^*M \mid \alpha(0), \alpha(1) \in O_M\}$:

$$\mathcal{A}_H : \Omega \longrightarrow \mathbf{R}, \quad \mathcal{A}_H(\gamma) := \int_0^1 \gamma^*\theta - H_t(\gamma(t))\, dt.$$

The perturbed holomorphic strips u, i.e., the solutions of

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0, \tag{6}$$

are negative gradient trajectories of this $\mathcal{A}_H$.

Proposition 1. *Let (γ_n, u_n) be a sequence in $\mathcal{M}(p, f; x, H)$. Provided that (γ_n, u_n) has no $W^{1,2}$-convergent subsequence, there exist*

- *critical points $p = p^0, p^1, \dots, p^m$ of f,*
- *solutions $x^0, x^1, \dots, x^l = x$ of (2),*
- *trajectories $\gamma^i \in \mathcal{M}(p^i, p^{i+1}, f)$, $i = 0, 1, \dots, m-1$,*
- *perturbed holomorphic discs $u^j \in \mathcal{M}(x^j, x^{j+1}, H)$, $j = 0, 1, \dots, l-1$,*
- *sequences $\{t_k^i\}, \{t_k^j\}$ in $\mathbf{R}$,*
- *$(\gamma, u) \in \mathcal{M}(p^m, f; x^0, H)$, and*
- *a subsequence (denoted by (γ_n, u_n) again)*

such that

1. $\gamma_k(\cdot + t_k^i) \overset{C^\infty_{\mathrm{loc}}}{\longrightarrow} \gamma^i$, $i = 0, 1, \dots, m-1$,
2. $u_k(\cdot + t_k^j, \cdot) \overset{C^\infty_{\mathrm{loc}}}{\longrightarrow} u^j$, $j = 0, 1, \dots, l-1$,
3. $(\gamma_k, u_k) \overset{C^\infty_{\mathrm{loc}}}{\longrightarrow} (\gamma, u)$,
4. $1 \leq m + l \leq m_f(p) - (\mu_H(x) + (n/2))$.

The proof will be a consequence of the next two Lemmata.

Lemma 2. *Every sequence in $\mathcal{M}(p, f; x, H)$ has a subsequence which converges with all its derivatives uniformly on compact sets.*

Proof. The sequence γ_n is equicontinuous:

$$d(\gamma_n(s_1), \gamma_n(s_2)) \leq \int_{s_1}^{s_2} |\dot{\gamma_n}(\tau)|\, d\tau \leq \sqrt{s_2 - s_1}\sqrt{\int_{s_1}^{s_2} |\dot{\gamma_n}(\tau)|\, d\tau}$$

$$
\begin{aligned}
&= \sqrt{s_2 - s_1}\sqrt{\int_{s_1}^{s_2} \frac{\partial}{\partial \tau} f(\gamma_n(\tau))\, d\tau} \\
&\leq \sqrt{s_2 - s_1}\sqrt{\max_M f - f(p)}
\end{aligned}
$$

and $\gamma_n(s)$ is bounded for every s because M is compact. So it follows from the Arzela-Ascoli theorem that γ_n has a subsequence (denoted again by γ_n) which converges uniformly on compact sets. Trajectories γ_n are solutions of the gradient equation:

$$
\dot{\gamma_n} = -\nabla f(\gamma_n), \tag{7}
$$

and f is smooth, so γ_n converges together with all derivatives on compact subsets of $(-\infty, 0]$, see [**10**].

Consider now the subsequence (γ_n, u_n). We can assume that the codomain of u_n is a compact set because the set $\cup_n u_n([0, +\infty) \times [0, 1])$ is C^0 bounded in T^*M (see, for example, Section 3.1 in [**4**]). The sequence du_n has uniformly bounded *energy:*

$$
E(u_n) := \int_0^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t} - X_{\rho_R H}(u)\right\|^2 dt\, ds
$$

because we have:

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t} - X_{\rho_R H}(u)\right\|^2 dt\, ds \\
&\qquad = \int_0^{R+1}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t} - X_{\rho_R H}(u)\right\|^2 dt\, ds \\
&\qquad\quad + \int_{R+1}^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t} - X_{\rho_R H}(u)\right\|^2 dt\, ds.
\end{aligned}
$$

The second integral is uniformly bounded because:

$$
\begin{aligned}
&\int_{R+1}^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t}(s,t) - X_{\rho_R H}(u)\right\|^2 dt\, ds \\
&\qquad = \int_{R+1}^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t}(s,t) - X_H(u)\right\|^2 dt\, ds \\
&\qquad \leq \int_0^{+\infty}\int_0^1 \left\|\frac{\partial u_n}{\partial s}\right\|^2 + \left\|\frac{\partial u_n}{\partial t}(s,t) - X_H(u)\right\|^2 dt\, ds \\
&\qquad = \mathcal{A}_H(x(t)) - \mathcal{A}_H(u_n(0,t)) = \mathcal{A}_H(x(t)).
\end{aligned}
$$

The uniform estimate of the first one follows from local regularity theorem B.3.4 in [**3**]: for every solution v of (6) it holds:

$$\|v\|_{W^{1,2}(Q)} \leq c\left(\|\overline{\partial}v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right),$$

where $Q = [0,1]\times[0,R+1]$, $\Omega \supset Q$ is an open subset of $\mathbf{R}^2$, assuming that measure of Ω is finite. But $\|\overline{\partial}v\|_{L^2(\Omega)} = \|\nabla(\rho H)(v)\|_{L^2(\Omega)}$, and it holds that $\|\nabla(\rho H)(v)\|_{L^\infty} \leq c_1$ uniformly (in v) because $\|v\|_{C^0} \leq c_2$ for every solution v of (6). For the same reason we have $\|v\|_{L^2(\Omega)} \leq c_3$.

When the energy of u_n is uniformly bounded, it follows from Gromov compactness that u_n has a subsequence that converges together with all derivatives on compact subsets of $([0,+\infty)\times[0,1])\backslash\{z_1,\dots z_p\}$. If z_j is the interior point of $[0,+\infty)\times[0,1]$, then it is a point in which a bubble can occur [**9**]. In the case of the sequence of holomorphic strips with Lagrangian boundary conditions it is also possible that bubbles appear as holomorphic strips with the same boundary conditions–in the boundary point z_i [**5**]. But in our case neither holomorphic spheres nor strips appear. There are no holomorphic spheres because ω is exact, so we have, for holomorphic $v : S^2 \to T^*M$:

$$\int_{S^2}\|dv\|^2 = \int_{S^2} v^*\omega = \int_{\partial S^2} v^*\theta = 0,$$

and no holomorphic strips with Lagrangian boundary conditions when the Lagrangian manifold L is exact, because $\theta|_L$ is an exact form dF, so:

$$\int_{\Sigma}\|dv\|^2 = \int_{\Sigma} v^*\omega = \int_{\partial\Sigma} v^*\theta = \int_{\partial\Sigma} d(F\circ v) = 0$$

(where $\Sigma = [0,+\infty)\times[0,1]$). This completes the proof. □

Lemma 3. *If the sequence* (γ_n, u_n) C^∞_{loc}*-converges to* $(\gamma, u) \in \mathcal{M}(p, f; x, H)$*, then* (γ_n, u_n) *is also* $W^{1,2}$ *convergent.*

Proof. If the limit (γ, u) is an element of $\mathcal{M}(p, f; x, H)$, then $\gamma_n(s)$ converges uniformly to p, when $s \to -\infty$ and $u_n(s,t)$ converges uniformly to $x(t)$, when $s \to +\infty$. To prove that $\gamma_n(s) \rightrightarrows p$, let us argue by contradiction. Assume that there a sequence of real numbers $s_k \to -\infty$ and a subsequence γ_{n_k} such that $d(p, \gamma_{n_k}(s_k)) > \varepsilon$. Let

U be a neighborhood of p such that $f(x) = f(p) + \sum \pm x_i^2$ in local coordinates in $U \subset B_\varepsilon(p)$. Let s_0 be such that $\gamma(s) \in U$ when $s \le s_0$ (such an s_0 exists because $\gamma(-\infty) = p$) and n_0 such that $\gamma_n(s_0) \in U$, for $n \ge n_0$ (there is such an n_0 because $\gamma_n(s_0) \to \gamma(s_0)$). But we conclude from gradient equations in local coordinates that $\gamma_n(s) \in U$, for all $s \le s_0$. This is in contradiction with our assumption, so $\gamma_n(s) \rightrightarrows p$, when $s \to -\infty$.

In order to prove the uniform convergence of $u_n(s,t)$, take $\varepsilon > 0$ and choose the neighborhood U_ε of $x([0,1])$ such that $|\mathcal{A}_H(\alpha(t)) - \mathcal{A}_H(x(t))| < \varepsilon$ and $\|d\mathcal{A}_H(\alpha(t))\| < \varepsilon$ for every path α contained in U_ε. There exist n_0 and s_0 such that $u_n(s_0,t) \in B_\varepsilon$, for all $t \in [0,1]$, $n \ge n_0$ (because $u(s,t)$ converges toward $x(t)$ uniformly in t, when $s \to \infty$ [**9, 11**] and u_n converges locally uniformly toward u and the set $\{s_0\} \times [0,1]$ is compact). For $s \ge s_0$ (assume $s_0 \ge R+1$) we have:

$$\begin{aligned} |\mathcal{A}_H(u_n(s,t)) - \mathcal{A}_H(x(t))| &= \mathcal{A}_H(u_n(s,t)) - \mathcal{A}_H(x(t)) \\ &\le \mathcal{A}_H(u_n(s_0,t)) - \mathcal{A}_H(x(t)) \\ &= |\mathcal{A}_H(u_n(s_0,t)) - \mathcal{A}_H(x(t))| < \varepsilon \end{aligned}$$

(because $\mathcal{A}_H$ decreases along its gradient trajectories). The sets U_ε form a local base for the critical points of the action functional. By decreasing ε, if necessary, we conclude that $u_n(s,t)$ is contained in an arbitrarily small neighborhood of $x(t)$ for $n \ge n_0$, $s \ge s_0$ (we can assume that the values which $\mathcal{A}_H$ takes in its critical points are different; these critical paths are isolated). So $u_n(s,t) \rightrightarrows x(t)$, when $s \to \infty$ uniformly in t and n.

The uniform estimates:

$$(8) \quad |\gamma_n(s)| \le c_p e^{\varepsilon_p s}, \quad s \le -s_0, \qquad \|u_n(s,t)\| \le c_x e^{-\varepsilon_x s}, \quad s \ge s_0,$$

(c_p, c_x, ε_p, ε_x are constants depending on p and x) follow from the estimates [**9, 10**]:

$$|\gamma(s)| \le c_p e^{\varepsilon_p s}, \quad s \le -s_0, \qquad \|u(s,t)\| \le c_x e^{-\varepsilon_x s}, \quad s \ge s_0$$

and the uniform convergence that we have just proved. But $\gamma_n = \exp(\xi_n)$, $u_n = \exp(\zeta_n)$, and by the assumption $\xi_n \overset{C^\infty_{\mathrm{loc}}}{\rightrightarrows} 0$, $\zeta_n \overset{C^\infty_{\mathrm{loc}}}{\rightrightarrows} 0$, and it follows from (8) that ξ_n and ζ_n converge uniformly everywhere. It is also true for its derivatives, because γ_n and u_n are solutions of (6) and (7); thus, we conclude that (γ_n, u_n) converges in $W^{1,2}$ topology. $\square$

Proof of Proposition 1. Now the rest of the proof is standard: it follows from Lemma 2 that there is a $C^{\infty}_{\mathrm{loc}}$ convergent subsequence, if its limit (γ, u) is an element of $\mathcal{M}(p, f; x, H)$, then it is also $W^{1,2}$-convergent (Lemma 3), so assume that it belongs to $\mathcal{M}(p^m, f; x^0, H)$ for some critical point p^m and Hamiltonian path x^0 (the pair (γ, u) has to belong to some space of this type because γ_n and u_n are solutions of equation (3), and the convergence is together with all derivatives). It holds $\mathcal{A}_H(x^0) > \mathcal{A}_H(x)$ and $f(p^m) < f(p)$ because f and $\mathcal{A}_H$ decrease along their gradient flows. The rest of the proof is the same as in separate cases of gradient trajectories or holomorphic discs, see e.g., [**9, 10**]. $\square$

3. Convergence of maps with variable Hamiltonian or domain. Let p and q be critical points of f such that $m_f(p) = m_f(q)$. In [**2**] we defined, for a fixed $R > 0$:

$$\mathcal{M}_R(p, q, f; H) := \left\{ (\gamma_-, \gamma_+, u) \;\middle|\; \begin{array}{l} \gamma_- : (-\infty, 0] \to M,\ \gamma_+ : [0, +\infty) \to M, \\ u : \mathbf{R} \times [0,1] \to T^*M, \\ (d\gamma_\pm/ds) = -\nabla f(\gamma_\pm),\ (\partial u/\partial s) \\ \qquad + J((\partial u/\partial t) - X_{\rho_R H}(u)) = 0, \\ \gamma_-(-\infty) = p,\ \gamma_+(+\infty) = q, \\ u(\partial(\mathbf{R} \times [0,1])) \subset O_M,\ u(\pm\infty, t) = \gamma_\pm(0) \end{array} \right\},$$

where $\rho_R : \mathbf{R} \to [0,1]$ is a smooth function such that

$$\rho_R = \begin{cases} 1 & |s| \le R \\ 0 & |s| \ge R + 1, \end{cases}$$

and

$$\overline{\mathcal{M}}(p, q, f; H) := \left\{ (R, \gamma_-, \gamma_+, u) \mid (\gamma_-, \gamma_+, u) \in \mathcal{M}_R(p, q, f; H) \right\},$$

for $R > R_0$. The set $\overline{\mathcal{M}}(p, q, f; H)$ is a one-dimensional manifold.

Definition 4. A broken (perturbed) holomorphic strip w is a pair (v_1, v_2) of (perturbed) holomorphic strips such that $v_1(+\infty, t) = v_2(-\infty, t)$. A sequence of perturbed holomorphic strips $u_n : \mathbf{R}\times[0,1] \to T^*M$ is said to converge weakly to a broken trajectory w if there exist a sequence of translations $\phi_n^i : \mathbf{R} \times [0,1] \to \mathbf{R} \times [0,1]$, for $i = 1, 2$ such that $u_n \circ \phi_n^i$ converges to v_i uniformly with all derivatives on compact subset of $\mathbf{R} \times [0,1]$.

In the same way one can define broken gradient trajectory and weak convergence of gradient trajectories, see [**10**].

Proposition 5. *Let $(R_n, \gamma_-^n, \gamma_+^n, u^n)$ be a sequence in $\overline{\mathcal{M}}(p, q, f; H)$. Then it either $W^{1,2}-$ converges toward an element of $\overline{\mathcal{M}}(p, q, f; H)$, or there are four possible limit behaviors*:

1) *There is a subsequence (denoted by $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ again) such that $R_n \to R_0$ and $(\gamma_-^n, \gamma_+^n, u_n)$ converges to $(\gamma_-, \gamma_+, u) \in \mathcal{M}_{R_0}(p, q, f; H)$.*

2) *There is a subsequence of $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ that converges to a broken trajectory in $\mathcal{M}(p, r, f) \times \overline{\mathcal{M}}(r, q, f; H)$. Here (γ_+^n, u^n) converges in $W^{1,2}$ topology, and γ_-^n converges weakly.*

3) *Similarly, there is a subsequence of $(R_n, \gamma_-^n, \gamma_+^n, u_n)$ that converges to a broken trajectory in $\overline{\mathcal{M}}(p, r, f; H) \times \mathcal{M}(r, q, f)$.*

4) *There is a subsequence of R_n, $R_{n_k} \to +\infty$, and there is a subsequence of $(\gamma_-^n, \gamma_+^n, u_n)$ that converges weakly to a broken element of $\mathcal{M}(p, f; x, H) \times \mathcal{M}(x, H; q, f)$.*

Proof. Assume first that R_n is bounded, so there is a compact $K \supset \{R_n\}$. The family ρ_R can be chosen to depend continuously on R, so all estimates in Lemmata 2 and 3 hold for all $R \in K$ uniformly in R too. In the same way as there we conclude that $(\gamma_-^n, \gamma_+^n, u^n)$ has a subsequence that converges locally uniformly, so if it does not converge toward an element of $\overline{\mathcal{M}}(p, q, f; H)$, then either $R_n \to R_0$ (and $(\gamma_-^n, \gamma_+^n, u^n)$ converges in $W^{1,2}$ topology) or $R_n \to R_1 > R_0$ and $(\gamma_-^n, \gamma_+^n, u^n)$ converges to a broken trajectory, denoted by w. Since the dimension of $\overline{\mathcal{M}}(p, q, f; H)$ is one, w can be broken only once. Indeed, if the sequence $(\gamma_-^n, \gamma^n+, u^n)$ degenerates into a trajectory w

which consists of the trajectories $\gamma_1, \gamma_2, \dots, \gamma_k, \gamma_+, u$, where $k \geq 3$, it means that all manifolds $\mathcal{M}(p_i, p_{i+1}, f) \ni \gamma_i$ are nonempty. (Here $p = p_1$, $(\gamma_k, \gamma_+, u) \in \overline{\mathcal{M}}(p_k, q, f; H)$.) But then we would have (since $\dim \mathcal{M}(p, q, f) = m_f(p) - m_f(q) - 1$):

$$\begin{aligned} m_f(p) - m_f(q) &= \Sigma_{i=1}^{k-1}(m_f(p_i) - m_f(p_{i+1})) + (m_f(p_k) - m_f(q)) \\ &\geq \Sigma_{i=1}^{k-1} 1 - 1 = k - 2 > 0, \end{aligned}$$

which contradicts our assumption that $m_f(p) = m_f(q)$. Although the domain of u_n is noncompact, u_n cannot converge to a broken disc because the nonholomorphic part of the domain is compact, and there u_n converges. But there are no solutions of:

$$\begin{cases} u : \mathbf{R} \times [0, 1] \to T^*M, \\ (\partial u/\partial s) + J(\partial u/\partial t) = 0, \\ u(\partial(\mathbf{R} \times [0, 1])) \subset O_M, \end{cases}$$

(except for constants).

This is how the first three cases can happen.

Now let R_n be unbounded, say $R_n \to +\infty$. Define $\hat{u}_n$ and $\check{u}_n$:

$$\hat{u}_n(s, t) := u_n(s - R_n - R_0 - 1, t), \qquad \check{u}_n(s, t) := u_n(s + R_n + R_0 + 1, t).$$

It holds:

$$\overline{\partial}_J \hat{u}_n(s, t) = \begin{cases} 0 & s \in (-\infty, R_0] \cup [2R_n + R_0 + 2, +\infty), \\ X_H(\hat{u}_n(s, t)) & s \in [R_0 + 1, 2R_n + R_0 + 1], \end{cases}$$

and

$$\overline{\partial}_J \check{u}_n(s, t) = \begin{cases} 0 & s \in (-\infty, -2R_n - R_0 - 2] \cup [-R_0, +\infty), \\ X_H(\check{u}_n(s, t)) & s \in [-2R_n - R_0 - 1, -R_0 - 1]. \end{cases}$$

The parts of the domain where $\hat{u}_n$ and $\check{u}_n$ are neither holomorphic nor gradient trajectories of $\mathcal{A}_H$ are compact sets, so all uniform estimates that we proved in Lemmata 2 and 3 hold for $\hat{u}_n$ and $\check{u}_n$. Therefore, $\hat{u}_n$ and $\check{u}_n$ converge locally uniformly with all derivatives toward $\hat{u}$ and $\check{u}$, where $\hat{u}$ and $\check{u}$ are the solutions of:

$$\begin{cases} (\partial\hat{u}/\partial s) + J((\partial\hat{u}/\partial t) - X_{\hat{\rho}_{R_0}H}(\hat{u})) = 0, \\ \hat{u}(\partial(\mathbf{R}\times[0,1])) \subset O_M, \\ \hat{u}(+\infty,t) = x(t), \\ \hat{u}(-\infty,t) = \gamma_-(0), \end{cases}$$
$$\begin{cases} (\partial\check{u}/\partial s) + J((\partial\check{u}/\partial t) - X_{\check{\rho}_{R_0}H}(\check{u})) = 0, \\ \check{u}(\partial(\mathbf{R}\times[0,1])) \subset O_M, \\ \check{u}(-\infty,t) = x(t), \\ \check{u}(+\infty,t) = \gamma_+(0). \end{cases}$$

Here $\hat{\rho}_{R_0}$ and $\check{\rho}_{R_0}$ are smooth functions such that:

$$\hat{\rho}_{R_0}(s) = \begin{cases} 0 & s \le R_0, \\ 1 & s \ge R_0+1, \end{cases}$$
$$\check{\rho}_{R_0}(s) = \begin{cases} 0 & s \ge -R_0, \\ 1 & s \le -R_0-1, \end{cases}$$

and $\gamma_\pm$ the limits of $\gamma_\pm^n$. In general, sequences $\gamma_\pm^n$ can degenerate to a broken trajectory, but in our case that's not possible because of dimension. Indeed, if both the sequence (for example) γ_-^n and the sequence u_n broke, there would exist an element in $\mathcal{M}(p,r,f)$, $\mathcal{M}(r,f;x,H)$ and $\mathcal{M}(x,H;q,f)$, so it would hold (since $\dim\mathcal{M}(r,f;x,H) = m_f(r) - \mu_H(x) - n/2$ and $\dim\mathcal{M}(x,H;q,f) = \mu_H(x) + (n/2) - m_f(q)$):

$$\begin{aligned} 0 &= m_f(p) - m_f(q) \\ &= (m_f(p) - m_f(r)) + \left(m_f(r) - \mu_H(x) - \frac{n}{2}\right) + \left(\mu_H(x) + \frac{n}{2} - m_f(q)\right) \\ &\ge 1 + 0 + 0. \end{aligned}$$

The pair $(\hat{u},\check{u})$ is a broken trajectory in 4) and the proof is finished. □

In [**2**] we also considered, for $\mu_H(x) = \mu_H(y)$ and fixed $\varepsilon > 0$, a zero-dimensional manifold:

$$\mathcal{M}_\varepsilon(x,y,H;f)$$
$$:= \left\{ (u_-,u_+,\gamma) \;\middle|\; \begin{array}{l} u_- : (-\infty,0]\times[0,1]\to T^*M, \\ u_+ : [0,+\infty)\times[0,1]\to T^*M, \\ \gamma : [-\varepsilon,\varepsilon]\to M, \\ (d\gamma/ds) = -\nabla f(\gamma), \\ (\partial u_\pm/\partial s) + J((\partial u_\pm/\partial t) - X_{\rho_R H}(u_\pm)) = 0, \\ u_-(\partial((-\infty,0]\times[0,1])) \subset O_M, \\ u_+(\partial([0,+\infty)\times[0,1])) \subset O_M, \\ u_\pm(0,1/2) = \gamma(\pm\varepsilon), \\ u_-(-\infty,t) = x(t),\ u_+(+\infty,t) = y(t) \end{array} \right\}$$

and a one-dimensional manifold:

$$\underline{\mathcal{M}}(x,y,H;f) := \{(\varepsilon,u_-,u_+,\gamma) \mid \varepsilon > \varepsilon_0, \\ (u_-,u_+,\gamma) \in \mathcal{M}_\varepsilon(x,y,H;f)\}.$$

Proposition 6. *Let $(\varepsilon_n, u_-^n, u_+^n, \gamma_n)$ be a sequence in $\underline{\mathcal{M}}(x,y,H;f)$ and assume that it has no subsequence that converges in $W^{1,2}$ topology. Then there are four possibilities*:

1) *There is a subsequence which converges to an element of $\mathcal{M}_{\varepsilon_0}(x,y,H;f)$*;

2) *There is a subsequence which weakly converges to an element of $\mathcal{M}(x,z,H)\times\underline{\mathcal{M}}(z,y,H;f)$*;

3) *There is a subsequence which weakly converges to an element of $\underline{\mathcal{M}}(x,z,H;f)\times\mathcal{M}(z,y,H)$*;

4) *There is a subsequence which weakly converges to an element of $\mathcal{M}(x,H;p,f)\times\mathcal{M}(p,f;y,H)$.*

Proof. Obviously, all uniform estimates in Lemmata 2 and 3 are true for u_-^n, u_+^n and γ_n, also uniformly in ε, so are the conclusions: the sequences u_-^n, u_+^n and γ_n converge locally uniformly, and the only obstruction to $W^{1,2}$-compactness is the convergence to broken trajectories. If the sequence ε_n is bounded, then the domain of γ_n is compact, so the only possibilities are described in 1), 2) and 3). (Again the sequence (u_-^n, u_+^n, γ_n) can break only once, for the dimensional reason.) If $\varepsilon_n \to +\infty$, then the domains of γ_n are not contained in any compact set–and there we have no global convergence. Then we can consider γ_n as the trajectories with the domain $\mathbf{R}$ (such that $\gamma_n(s) \equiv \text{const}$, for $|s| \geq \varepsilon_n$) and after certain reparametrization, as in subsection 2.4.2 in [**10**], we obtain the last case, concluding the proof. □

4. Gluing. In this section we formulate the converses of Propositions 1, 5 and 6 for the sake of completeness. The proofs are based on the implicit function theorem and the pre-gluing and gluing techniques. Since they all have a local nature, and thus their proofs are the verbatim of proofs of analogous propositions in other versions of Morse-Floer homology, we are going to skip them. We refer the reader to [**10**] for more details about gluing of gradient trajectories and to [**9**] for gluing of holomorphic curves. The next proposition shows that the set of broken trajectories from Proposition 1 is contained in the boundary of $\mathcal{M}(p, f; x, H)$.

Proposition 7. *Let $p = p^0, p^1, \dots, p^m$ be critical points of f, and let $x^0, x^1, \dots, x^l = x$ be Hamiltonian paths which solve* (2). *For any $m + l + 2$-tuple $(\gamma^0, \gamma^1, \dots, \gamma^m, u^0, u^1, \dots, u^l)$ in*

$$\mathcal{M}(p^0, p^1, f) \times \cdots \times \mathcal{M}(p^{m-1}, p^m, f) \times \mathcal{M}(p^m, f; x^0, H) \times \mathcal{M}(x^0, x^1, H) \\ \times \cdots \times \mathcal{M}(x^{l-1}, x^l, H)$$

there exists a sequence (γ_k, u_k) in $\mathcal{M}(p, f; x, H)$ and the sequences $\{t_k^i\}, \{t_k^j\}$ in $\mathbf{R}$ such that

$$\gamma_k(\cdot + t_k^i) \overset{C^\infty_{\mathrm{loc}}}{\Longrightarrow} \gamma^i, \quad i = 0, 1, \dots, m-1,$$
$$u_k(\cdot + t_k^j, \cdot) \overset{C^\infty_{\mathrm{loc}}}{\Longrightarrow} u^j, \quad j = 0, 1, \dots, l-1,$$

and

$$(\gamma_k, u_k) \overset{C^\infty_{\mathrm{loc}}}{\Longrightarrow} (\gamma, u).$$

The following proposition says that every broken object mentioned in Proposition 5 is a limit of some sequence from $\overline{\mathcal{M}}(p,q,f;H)$; hence, the broken objects of this type form the set which is bigger then a boundary of $\overline{\mathcal{M}}(p,q,f;H)$.

Proposition 8. *Let w be a broken trajectory of some of the next three types*:

- $w = (\gamma, \gamma_-, \gamma_+, u) \in \mathcal{M}(p,r,f) \times \overline{\mathcal{M}}(r,q,f;H)$,
- $w = (\gamma_-, \gamma_+, u, \gamma) \in \overline{\mathcal{M}}(p,r,f;H) \times \mathcal{M}(r,q,f)$,
- $w = (\gamma_1, u_1, u_2, \gamma_2) \in \mathcal{M}(p,f;x,H) \times \mathcal{M}(x,H;q,f)$.

Then, there exists a sequence $(R_n, \gamma_-^n, \gamma_+^n, u^n)$ in $\overline{\mathcal{M}}(p,q,f;H)$ that converges weakly to w.

The same is true for the union of broken trajectories from Proposition 6: it is a subset of the boundary of $\underline{\mathcal{M}}(x,y,H;f)$.

Proposition 9. *For any broken trajectory w of some of the next three types*:

- $w = (u, u_-, u_+, \gamma) \in \mathcal{M}(x,z,H) \times \underline{\mathcal{M}}(z,y,H;f)$,
- $w = (u_-, u_+, \gamma, u) \in \underline{\mathcal{M}}(x,z,H;f) \times \mathcal{M}(z,y,H)$,
- $w = (u_1, \gamma_1, \gamma_2, u_2) \in \mathcal{M}(x,H;p,f) \times \mathcal{M}(p,f;y,H)$,

there is a sequence $(\varepsilon_n, u_-^n, u_+^n, \gamma_n)$ in $\underline{\mathcal{M}}(x,y,H;f)$ that converges weakly to w.

In the next section we will summarize the compactness results for the space of mixed objects. It is the direct consequence of Propositions 1 and 7.

5. Compactness and non-compactness. It follows from previous considerations that the boundary of space of mixed trajectories consists of broken mixed trajectories. More precisely, the next theorem holds:

Theorem 10. *The topological boundary of* $\mathcal{M}(p,f;x,H)$ *can be identified with*

$$\bigcup \mathcal{M}(p^0,p^1,f)\times\cdots\times\mathcal{M}(p^{m-1},p^m,f)\times\mathcal{M}(p^m,f;x^0,H)$$
$$\times\mathcal{M}(x^0,x^1,H)\times\cdots\times\mathcal{M}(x^{l-1},x^l,H).$$

Here the union is taken in all integers m and l such that $1\le m+l\le m_f(p)-(\mu_H(x)+(n/2))$, all critical points $p=p^0,p^1,\ldots,p^m$ of f and all Hamiltonian paths $x^0,x^1,\ldots,x^l=x$ such that

$$m_f(p)>m_f(p^1)>\cdots>m_f(p^m)\ge\mu_H(x^0)+\frac{n}{2}>\mu_H(x^1)+\frac{n}{2}$$
$$>\cdots>\mu_H(x)+\frac{n}{2}.$$

The next corollary holds for the dimensional reason.

Corollary 11. *If* $m_f(p)=\mu_H(x)+(n/2)$*, then* $\mathcal{M}(p,f;x,H)$ *is a compact zero-dimensional manifold, i.e., a finite set. If* $m_f(p)=\mu_H(x)+(n/2)+1$*, then* $\mathcal{M}(p,f;x,H)$ *is a one-dimensional manifold with topological boundary*

$$\bigcup_{m_f(q)=m_f(p)-1}\mathcal{M}(p,q,f)\times\mathcal{M}(q,f;x,H)$$
$$\cup\bigcup_{\mu_H(y)=\mu_H(x)+1}\mathcal{M}(p,f;y,H)\times\mathcal{M}(y,x,H).$$

Proof. If $m_f(p)=\mu_H(x)+(n/2)$, then all the components $\mathcal{M}(p^0,p^1,f),\ldots,\mathcal{M}(p^{m-1},p^m,f)$, $\mathcal{M}(p^m,f;x^0,H)$, $\mathcal{M}(x^0,x^1,H),\ldots,\mathcal{M}(x^{l-1},x^l,H)$ of the boundary of $\mathcal{M}(p,f;x,H)$ are the manifolds of the dimension at most -1 because of (9), hence empty sets. For the same reason, when $m_f(p)=\mu_H(x)+(n/2)+1$, then all the mentioned components have the dimension at most -1, except for one, which is zero-dimensional. It could be either $\mathcal{M}(p,q,f)$ (and hence $\mathcal{M}(q,f;x,H)$ also) for some p such that $m_f(q)=m_f(p)-1$, or $\mathcal{M}(p,f;y,H)$ (and $\mathcal{M}(y,x,H)$) for some y such that $\mu_H(y)=\mu_H(x)+1$. $\square$

The first part of Corollary 11 implies that the homomorphism ψ is well defined, and the second that ψ defines a homomorphism on homology groups.

A similar result holds for the manifolds of mixed objects with variable Hamiltonian or domain. From Sections 3 and 4 we conclude that the topological boundary of $\overline{\mathcal{M}}(p,q,f;H)$ can be identified with

$$\begin{aligned}\mathcal{M}_{R_0}(p,q,f;H) \cup & \bigcup_{m_f(r)=m_f(p)-1} \mathcal{M}(p,r,f)\times\overline{\mathcal{M}}(r,q,f;H)\\ \cup & \bigcup_{m_f(r)=m_f(p)} \overline{\mathcal{M}}(p,r,f;H)\times\mathcal{M}(r,q,f)\\ \cup & \bigcup_{\mu_H(x)+(n/2)=m_f(p)} \mathcal{M}(p,f;x,H)\times\mathcal{M}(x,H;q,f)\end{aligned}$$

and the boundary of $\underline{\mathcal{M}}(x,y,H;f)$ with

$$\begin{aligned}\mathcal{M}_{\varepsilon_0}(x,y,H;f) \cup & \bigcup_{\mu_H(z)=\mu_H(x)-1} \mathcal{M}(x,z,H)\times\underline{\mathcal{M}}(z,y,H;f)\\ \cup & \bigcup_{\mu_H(z)=\mu_H(x)} \underline{\mathcal{M}}(x,z,H;f)\times\mathcal{M}(z,y,H)\\ \cup & \bigcup_{m_f(p)=\mu_H(x)+(n/2)} \mathcal{M}(x,H;p,f)\times\mathcal{M}(p,f;y,H).\end{aligned}$$

These established results allow us to prove that $\psi\circ\phi$ and $\phi\circ\psi$ are identities on homology groups (see [**2**] for more details).

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REFERENCES

1. A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geometry **30** (1989), 207–221.

2. J. Katić and D. Milinković, *Piunikin-Salamon-Schwarz isomorphisms for Lagrangian intersections*, Differential Geom. Appl. **22** (2005), 215–227.

3. D. McDuff and D. Salamon, *J-holomorphic Curves and Quantum Cohomology*, AMS, University Lecture Series **6**, 1994.

4. D. Milinković, *On equivalence of two constructions of invariants of Lagrangian submanifolds*, Pacific J. Math. **195** (2000), 371–415.

5. Y.-G. Oh, *Removal of boundary singularities of pseudo-holomorfic curves with Lagrangian boundary conditions*, Comm. Pure Appl. Math. **45** (1992), 121–139.

6. S. Piunikhin, D. Salamon and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology*, in *Contact and symplectic geometry*,Cambridge University Press, Cambridge, 1996.

7. J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993), 827–844.

8. ———, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1–33.

9. D. Salamon, *Lectures on Floer homology*, in *Symplectic geometry and topology*, L. Traynor and Ya. Eliashberg, eds., IAS Park City Math. Series, AMS **7**, 1999.

10. M. Schwarz, *Morse homology*, Birkhäuser, Berlin, 1993.

11. ———, *S^1-cobordism operations in Floer homology*, Ph.D. thesis, ETH, Zürich, 1995.

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