ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 2, 2008

SOME APPLICATIONS OF GASPER'S BIBASIC SUMMATION FORMULA

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ABSTRACT. A new bibasic summation formula for hypergeometric series is found by a special inversion formula and it is applied to derive a class of transformation formulas and summation formulas for basic hypergeometric series only by elementary methods. The ordinary hypergeometric limits of these formulas are also obtained.

1. Introduction. All the notation and terminology is adopted from [**9**]. The (generalized) hypergeometric series is defined by

$$ {}_{r+1}F_r\left[\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n}\, z^n, $$

where the rising factorial $(a)_n$ is given by

$$ (a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad n\ge 1, \quad (a)_0 := 1. $$

The gamma function can be used to extend to rising factorials by defining $(a)_\beta = \lim_{\gamma\to a}\Gamma(\gamma+\beta)/\Gamma(\gamma)$, β arbitrary. A hypergeometric series ${}_{r+1}F_r$ is called very well-poised if $a_i + b_i = 1 + a_0$ for $i = 1,2,\dots,r$, and among the parameters a_i occurs $1 + a_0/2$. We use the standard abbreviation for very well-poised hypergeometric series, ${}_{r+1}V_r(a_0; a_2, a_3, \dots, a_r; z)$

$$:={}_{r+1}F_r\left[\begin{matrix} a_0, 1+a_0/2, a_2, a_3,\dots,a_r \\ a_0/2, 1+a_0-a_2, 1+a_0-a_3,\dots,1+a_0-a_r \end{matrix} ;z\right] $$

2000 AMS *Mathematics subject classification.* Primary 33D15, Secondary 15A09.

Keywords and phrases. Inverse relations, basic hypergeometric series, bibasic summation formulas, transformation formulas.

This work was supported by the cross-subject program of Shandong University under Grant 11010053182023.

Received by the editors on September 8, 2004, and in revised form on December 14, 2005.

DOI:10.1216/RMJ-2008-38-2-703

We shall also use the compact Gasper-Rahman notation

$$(a_1, a_2, \ldots, a_r)_n = (a_1)_n (a_2)_n \cdots (a_r)_n,$$

Given a (fixed) complex number q with $|q| < 1$, the basic hypergeometric series is defined by

$$ {}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, \quad z \right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} ((-1)^n q^{\binom{n}{2}})^{s-r+1} z^n,$$

where, as before, the rising q-factorial $(a; q)_n$ is given by

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \geq 1, \quad (a; q)_0 := 1.$$

The infinite q-factorial $(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^{n-1})$ can be used to extend (finite) q-factorials by defining $(a; q)_\beta := (a; q)_\infty/(aq^\beta; q)_\infty$, β arbitrary. A basic hypergeometric series ${}_{r+1}\phi_r$ is called very well poised if $a_i b_i = qa_0$ for $i = 1, 2, \ldots, r$, and among the parameters a_i occur both $q\sqrt{a_0}$ and $-q\sqrt{a_0}$. We use the standard abbreviation for very well-poised basic hypergeometric series,

$${}_{r+1}W_r(a_0; a_3, a_4, \ldots, a_r; q, z) := {}_{r+1}\phi_r \left[\begin{matrix} a_0, q\sqrt{a_0}, -q\sqrt{a_0}, a_3, a_4, \ldots, a_r \\ \sqrt{a_0}, -\sqrt{a_0}, qa_0/a_3, qa_0/a_4, \ldots, qa_0/a_r \end{matrix} ; q, z \right]$$

We use short notations in the basic hypergeometric context which are analogous to the hypergeometric ones.

All identities in our paper are subject to suitable conditions on the parameters such that the involved hypergeometric or basic hypergeometric series converge. We shall not state these conditions for each identity. The reader should consult [**9**]. And we use some elementary identities in [**9**] to prove the identities in our paper.

Our new bibasic summation formulas are stated in equations (3.4) and (3.5) below.

2. Inverse series relation. In this section, we provide some background information for this paper. Gasper [**7**] showed that some bibasic summation formulas derived by Carlitz [**3**], Al-Salam and Verma [**1**], and William Gosper could be extended to the indefinite bibasic summation formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k = \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}, \quad n=0,1,2,\dots, \tag{2.1}$$

where p and q are independent bases and $a,b,c,$ are arbitrary parameters. And he also derived some summation formulas containing at least two parameters.

When $c=q^{-n}, n=0,1,\dots,$ formula (2.1) reduces to

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \times \frac{(a,b;p)_k(q^{-n},aq^n/b;q)_k}{(q,aq/b;q)_k(apq^n,bpq^{-n};p)_k} q^k = \delta_{n,0}. \tag{2.2}$$

By replacing n,a,b and k by $n-m, ap^mq^m, bp^{-m}q^m$ and $j-m$ in (2.1), we find that

$$\sum_{j=m}^{n} a_{nj}b_{jm} = \delta_{n,m}, \tag{2.3}$$

with

$$a_{nj} = \frac{(-1)^{n+j}(1-aq^jp^j)(1-bq^{-j}p^j)(apq^n,bpq^{-n};p)_{n-1}}{(q;q)_{n-j}(apq^n,bpq^{-n};p)_j(bq^{1-2n}/a;q)_{n-j}},$$

$$b_{jm} = \frac{(ap^mq^m,bq^{-m}p^m;p)_{j-m}}{(q,aq^{1+2m}/b;q)_{j-m}} \left(-\frac{a}{b}q^{1+2m}\right)^{j-m} q^{2\binom{j-m}{2}},$$

where $n=0,1,\dots$ and $\delta_{n,m}$ is the Kronecker delta function.

By using (2.3), it is easy to prove that the system of equations

$$F(n) = \sum_{k=0}^{n} \frac{(-1)^{n+k}(1-aq^kp^k)(1-bq^{-k}p^k)(apq^n,bpq^{-n};p)_{n-1}}{(q;q)_{n-k}(apq^n,bpq^{-n};p)_k(bq^{1-2n}/a;q)_{n-k}} G(k) \tag{2.4}$$

$$n=0,1,2,\dots,$$

is equivalent to the system of equations

$$G(n) = \sum_{k=0}^{n} \frac{(aq^k p^k, bq^{-k} p^k; p)_{n-k}}{(q, aq^{1+2k}/b; q)_{n-k}} (-a/b)^{n-k} q^{(n^2-k^2)} F(k) \tag{2.5}$$

$$n = 0, 1, 2, \dots .$$

If

$$f(n) = \frac{(-1)^n (q;q)_n}{(aq^n/b)^n} G(n),$$

$$g(n) = \frac{(q;q)_n (aq^n/b;q)_n}{(aq^n/b)^n (apq^n, bpq^{-n}; p)_{n-1}} F(n),$$

then

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(apq^k, bpq^{-k}; p)_{n-1} (1 - aq^{2k}/b)}{(aq^k/b; q)_{n+1}} g(k),$$

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1 - aq^k p^k)(1 - bq^{-k} p^k) q^{\binom{n-k}{2}} (aq^n/b; q)_k}{(apq^n, bpq^{-n}; p)_k} f(k),$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

Now we get a pair of inverse series relations including two independent bases:

$$f(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1 - Aq^{2k}/B)(Apq^k, Bpq^{-k}; p)_{n-1}}{(Aq^k/B; q)_{n+1}} g(k), \tag{2.6}$$

(2.7)

$$g(n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1 - Ap^k q^k)(1 - Bp^k q^{-k})(Aq^n/B; q)_k q^{\binom{n-k}{2}}}{(Apq^n, Bpq^{-n}; p)_k} f(k);$$

here it is assumed that $f(0) = 1$.

For the general inverse relations in one dimension, their applications to combinatorial identities and hypergeometric evaluations may be found in [**2, 4–6, 10, 11, 16, 24**].

The multivariable case involving U(n+1) multiple basic (and bibasic) hypergeometric series first arose in [**2, 14, 15, 17**] and then continued, for other root systems, with the works in [**12, 13, 18, 20**–**23**].

In Section 3, 4 and 5, we shall give several applications of the special inverse relation (2.6), (2.7).

3. A bibasic summation formula. In [**8**], Gasper and Rahman showed that (2.1) could be extended to

(3.1)
$$\sum_{k=-m}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(adp/c,bcp/d;p)_k} q^k$$
$$= \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \cdot \left\{ \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n(adp/c,bcp/d;p)_n} q^n - \frac{(c/ad,d/bc;p)_{m+1}(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}(1/a,1/b;p)_{m+1}} \right\},$$

where $n, m = 0, \pm 1, \pm 2, \dots$, and then use this formula to derive some rather general summation and transformation formulas.

Note that when $m = 0, c = q^{-n}$, $n = 0, 1, 2, \dots$, formula (3.1) reduces to

(3.2)
$$\sum_{k=0}^{n} \frac{(1-adp^kq^k)(1-bp^k/dq^k)}{(1-ad)(1-b/d)} \frac{(a,b;p)_k(q^{-n},ad^2q^n/b;q)_kq^k}{(dq,adq/b;q)_k(adpq^n,bp/dq^n;p)_k}$$
$$= \frac{(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)}{(1-ad)(1-d/b)(1-dq^n)(1-adq^n/b)}, \quad n = 0, 1, 2, \dots .$$

Gasper and Rahman have already noted in [**8, 9**] that (3.1) is equivalent to its $m = 0$ special case in (3.2). Equation (3.2) is also a special case of equation (2.22) of [**2**], and Theorems 2.27 and 2.29, due to Macdonald, [**2**].

In order to use (2.6) and (2.7), we find that (3.2) can be written in the form

$$\begin{aligned}(3.3)\quad \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q & \frac{(1-adp^kq^k)(1-bp^k/dq^k)(ad^2q^n/b;q)_k q^{\binom{n-k}{2}}}{(adpq^n, bp/dq^n;p)_k} \\ & \times \frac{(q,q)_k(a,b;p)_k}{(dq,adq/b;q)_k} \\ & = \frac{-b(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)q^{\binom{n}{2}}}{d(1-dq^n)(1-adq^n/b)} \\ & \quad n=0,1,2,\dots .\end{aligned}$$

Let

$$g(n) = \frac{-b(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)q^{\binom{n}{2}}}{d(1-dq^n)(1-adq^n/b)},$$
$$f(n) = \frac{(q,q)_n(a,b;p)_n}{(dq,adq/b;q)_n};$$

then we have

$$f(n) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(1-Aq^{2k}/B)(Apq^k, Bpq^{-k};p)_{n-1}}{(Aq^k/B;q)_{n+1}}\, g(k),$$

where $A = ad, B = b/d$. By using (2.6) and (2.7) to get

$$\begin{aligned}\frac{(q;q)_n(a,b;p)_n}{(dq,adq/b;q)_n} = & \frac{\sum_{k=0}^{n}(-1)^k \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q (ad^2/b;q)_k (adq^k, b/dq^k;p)_n q^{\binom{k+1}{2}}}{(ad^2/b;q)_{n+k+1}} \\ & \times \frac{(1-ad^2q^{2k}/b)(1-d)(1-ad/b)}{(1-dq^k)(1-adq^k/b)}.\end{aligned}$$

That is,

$$\begin{aligned}& \frac{(a,b;p)_n}{(dq,adq/b;q)_n} \\ (3.4)\quad & = \sum_{k=0}^{n} \frac{(-1)^k(1-d)(1-ad/b)(1-ad^2q^{2k}/b)(ad^2/b;q)_k(adq^k,b/dq^k;p)_n q^{\binom{k+1}{2}}}{(1-dq^k)(1-adq^k/b)(q;q)_k(q;q)_{n-k}(ad^2/b;q)_{n+k+1}}, \\ & \quad n=0,1,2,\dots .\end{aligned}$$

It should be noted that when $p = q$ formula (3.4) reduces to Jackson's sum (4.4). Hence, formula (3.4) is a bibasic extension of Jackson's sum (4.4).

4. A unified form of some summation formulas. Formula (3.4) can be rewritten as the form

$$\frac{(q, ad^2/b; q)_n (a, b; p)_n}{(dq, adq/b; q)_n} = \sum_{k=0}^{n} \frac{(1-d)(1-ad/b)(1-ad^2q^{2k}/b)(q^{-n}, ad^2/b; q)_k (adq^k, b/dq^k; p)_n q^{k(n+1)}}{(1-dq^k)(1-adq^k/b)(q; q)_k (ad^2q^n/b; q)_{k+1}}. \tag{4.1}$$

We can use this bibasic sum to derive a very well-poised series formula. Take $q = p^\lambda$ in (4.1), $\lambda = 1, 2, \ldots$, then replace p by q to get

$$\frac{(q^\lambda, ad^2/b; q^\lambda)_n (a, b; q)_n}{(dq^\lambda, adq^\lambda/b; q^\lambda)_n} = \sum_{k=0}^{n} \frac{(1-d)(1-ad/b)(1-ad^2q^{2k\lambda}/b)(q^{-n\lambda}, ad^2/b; q^\lambda)_k (adq^{k\lambda}, b/dq^{k\lambda}; q)_n q^{k(n+1)\lambda}}{(1-dq^{k\lambda})(1-adq^{k\lambda}/b)(q^\lambda; q^\lambda)_k (ad^2q^{n\lambda}/b; q^\lambda)_{k+1}}.$$

So,

$$\frac{(q^\lambda, ad^2/b; q^\lambda)_n (a, b; q)_n}{(dq^\lambda, adq^\lambda/b; q^\lambda)_n (ad, b/d; q)_n} = \sum_{k=0}^{n} \frac{(1-d)(1-ad/b)(1-ad^2q^{2k\lambda}/b)(ad^2/b, q^{-n\lambda}; q^\lambda)_k (adq^n, dq/b; q)_{k\lambda} q^{k\lambda}}{(1-dq^{k\lambda})(1-adq^{k\lambda}/b)(q^\lambda; q^\lambda)_k (ad^2q^{n\lambda}/b; q^\lambda)_{k+1} (ad, d/bq^{n-1}; q)_{k\lambda}},$$

that is,

$$\frac{(q^\lambda, ad^2q^\lambda/b; q^\lambda)_n (a, b; q)_n}{(dq^\lambda, adq^\lambda/b; q^\lambda)_n (ad, b/d; q)_n} = \sum_{k=0}^{n} \frac{(d, ad/b; q^\lambda)_k (dq^\lambda\sqrt{a/b}, -dq^\lambda\sqrt{a/b}; q^\lambda)_k (ad^2/b, q^{-n\lambda}; q^\lambda)_k (adq^n, dq/b; q)_{k\lambda} q^{k\lambda}}{(dq^\lambda, adq^\lambda/b; q^\lambda)_k (d\sqrt{a/b}, -d\sqrt{a/b}; q^\lambda)_k (q^\lambda, ad^2q^{(n+1)\lambda}/b; q^\lambda)_k (ad, d/bq^{n-1}; q)_{k\lambda}}.$$

By using the multiplication formula

$$(a; q)_{k\lambda} = (a, aq, \ldots, aq^{n-1}; q^\lambda)_k,$$

the resulting summation can be written as

$$\begin{aligned} &{}_{2\lambda+6}W_{2\lambda+5}(ad^2/b; d, ad/b, q^{-n\lambda}, dq/b, dq^2/b, \ldots, dq^\lambda/b, \\ &\qquad adq^n, adq^{n+1}, \ldots, adq^{n+\lambda-1}; q^\lambda, q^\lambda) \\ &\qquad = \frac{(q^\lambda, ad^2q^\lambda/b; q^\lambda)_n (a, b; q)_n}{(dq^\lambda, adq^\lambda/b; q^\lambda)_n (ad, b/d; q)_n}, \quad \lambda = 1, 2, \ldots. \end{aligned} \tag{4.2}$$

Note that by replacing a, b, d, in (4.2) by q^a, q^b, q^d, and letting $q \to 1$, we obtain the limit cases

$$\begin{aligned}(4.3)\quad & {}_{2\lambda+6}V_{2\lambda+5}(a+2d-b; d, a+d-b, \\ & \qquad d-b+1, d-b+2, \cdots, d-b+\lambda, -n\lambda; 1) \\ &= \frac{(a)_n(b)_n \prod_{k=1}^{n} k\lambda(a+2d-b+k\lambda)}{(a+d)_n(b-d)_n \prod_{k=1}^{n}(d+k\lambda)(a+2d-b+k\lambda)}, \quad \lambda = 1, 2, \ldots .\end{aligned}$$

Now let us see some examples.

First, let us take $\lambda = 1$ in (4.2). The summation in this special case reads

$$\begin{aligned}(4.4)\quad & {}_8W_7(ad^2/b; d, ad/b, q^{-n}, dq/b, adq^n; q, q) \\ &= \frac{(q, ad^2q/b; q)_n (a, b; q)_n}{(dq, adq/b; q)_n (ad, b/d; q)_n}.\end{aligned}$$

Now observe that by replacing a, b, c, d and e by ad^2/b, dq/b, ad/b, d and adq^n in Jackson's q-analogue of Dougall's ${}_7F_6$ sum, see [**9**], we also can get (4.4). Next, we choose $\lambda = 2$; the summation (4.2) now reads

$$\begin{aligned}(4.5)\quad & {}_{10}W_9(ad^2/b; d, ad/b, q^{-2n}, dq/b, dq^2/b, adq^n, adq^{n+1}; q^2, q^2) \\ &= \frac{(q^2, ad^2q^2/b; q^2)_n (a, b; q)_n}{(dq^2, adq^2/b; q^2)_n (ad, b/d; q)_n}.\end{aligned}$$

Finally, we take $\lambda = 3$; equation (4.2) then reads

$$\begin{aligned}(4.6)\quad & {}_{12}W_{11}(ad^2/b; d, ad/b, q^{-3n}, dq/b, dq^2/b, dq^3/b, \\ & \qquad adq^n, adq^{n+1}, adq^{n+2}; q^3, q^3) \\ &= \frac{(q^3, ad^2q^3/b; q^3)_n (a, b; q)_n}{(dq^3, adq^3/b; q^3)_n (ad, b/d; q)_n}.\end{aligned}$$

5. Another limits case derived from (3.4). Letting $q = p^\lambda$ and then replacing p by q in (3.4), we obtain

$$\begin{aligned}& \frac{(a, b; q)_n}{(dq^\lambda, adq^\lambda/b; q^\lambda)_n} \\ &= \sum_{k=0}^{n} \frac{(-1)^k(1-d)(1-ad/b)(1-ad^2q^{2k\lambda}/b)(ad^2/b;q^\lambda)_k(adq^{k\lambda}, b/dq^{k\lambda};q)_n q^{\lambda\binom{k+1}{2}}}{(1-dq^{k\lambda})(1-adq^{k\lambda}/b)(q^\lambda;q^\lambda)_k(q^\lambda;q^\lambda)_{n-k}(ad^2/b;q^\lambda)_{n+k+1}}.\end{aligned}$$

Let $n \to \infty$; we get after some simplification the following formula

$$\sum_{k=0}^{\infty} \frac{(d, ad/b, ad^2/b; q^\lambda)_k (ad^2 q^\lambda/b; q^\lambda)_{2k} (dq/b; q)_{k\lambda} (-1)^{k(\lambda+1)} (b/d)^{k\lambda} q^{-\binom{\lambda}{2}k^2}}{(q^\lambda, dq^\lambda, adq^\lambda/b; q^\lambda)_k (ad^2/b; q^\lambda)_{2k} (ad; q)_{k\lambda}} = \frac{(q^\lambda, ad^2 q^\lambda/b; q^\lambda)_\infty (a, b; q)_\infty}{(dq^\lambda, adq^\lambda/b; q^\lambda)_\infty (ad, b/d; q)_\infty}, \tag{5.1}$$

when $\lambda = 1$; (5.1) can also be obtained by replacing a, b, c and d by ad^2/b, dq/b, ad/b and d in the sum of a very well-poised ${}_6\phi_5$ series, see [**9**]. Note that, by replacing a, b, d, in (5.1) by q^a, q^b, q^d and letting $q \to 1$, we obtain the limit cases

$$(5.2) \quad {}_{\lambda+5}V_{\lambda+4}(a + 2d - b; d, a + d - b, d - b + 1, d - b + 2, \cdots, d - b + \lambda; 1) = \frac{\Gamma(a + d)\Gamma(b - d)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(1 + d/\lambda)\Gamma(1 + (a + d - b)/\lambda)}{\Gamma(1 + (a + 2d - b)/\lambda)}.$$

Acknowledgments. The author thanks Professor CHU Wenchang for his hospitality and suggestions. We also gratefully acknowledge the valuable and careful comments made by the referees.

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