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OSCILLATION THEOREMS RELATED TO AVERAGING TECHNIQUE FOR SECOND ORDER EMDEN-FOWLER TYPE NEUTRAL DIFFERENTIAL EQUATIONS

ZHITING XU

ABSTRACT. Some oscillation theorems are established by the averaging techniques for the second order Emden-Fowler type neutral delay differential equation

$$(r(t)x'(t))' + q_1(t)|y(t-\sigma_1)|^{\alpha-1}y(t-\sigma_1)$$

$$+q_2(t)|y(t-\sigma_2)|^{\beta-1}y(t-\sigma_2) = 0,\ t \geq t_0,$$

where $x(t) = y(t) + p(t)y(t-\tau)$, τ, σ_1 and σ_2 are nonnegative constants, $0 < \alpha < 1$, $\beta > 1$, and r, p, q_1, $q_2 \in C([t_0,\infty),\mathbf{R})$. These theorems obtained here extend and improve some known results. In particular, two interesting examples that point out the applications of our results are also included.

1. Introduction. In this paper, we study the problem of oscillation of the second order Emden-Fowler type neutral delay differential equation

$$(1.1)\quad (r(t)x'(t))' + q_1(t)|y(t-\sigma_1)|^{\alpha-1}y(t-\sigma_1) + q_2(t)|y(t-\sigma_2)|^{\beta-1}y(t-\sigma_2) = 0, \quad t \geq t_0,$$

where $x(t) = y(t) + p(t)y(t-\tau)$, and the following conditions are assumed to hold:

(A1) τ, σ_1 and σ_2 are nonnegative constants, $0 < \alpha < 1$, $\beta > 1$;

(A2) r, q_1, $q_2 \in C([t_0,\infty),\mathbf{R}^+)$, and $\int^\infty 1/r(s)\,ds = \infty$, $\mathbf{R}^+ = (0,\infty)$;

(A3) $p \in C([t_0,\infty),\mathbf{R})$, and $-1 < p_0 \leq p(t) \leq 1$, p_0 constant.

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Let $\varphi \in C([t_0 - \delta, \mathbf{R})$, where $\delta = \max\{\tau, \sigma_1, \sigma_2\}$, be a given function, and let y_0 be a given constant. Using the method of steps, (1.1) has a unique solution $y \in C([t_0 - \delta, \infty); \mathbf{R})$ in the sense that both $y(t) + p(t)y(t-\tau)$ and $r(t)(y(t) + p(t)y(t-\tau))'$ are continuously differentiable for $t \geq t_0$, $y(t)$ satisfies (1.1) and

$$y(s) = \varphi(s) \quad \text{for } s \in [t_0 - \theta, t_0],$$

and

$$[y(t) + p(t)x(t-\tau)]'_{t_0} = y_0.$$

For further questions concerning existence and uniqueness of solutions of neutral delay differential equations, see [**11**].

We note that second order neutral delay differential equations are used in many fields such as vibrating masses attached to an elastic bar and some variational problems, etc., see, for example [**11**].

Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ with $\sup\{|y(t)| : t \geq T\} > 0$ for any $T \geq t_x$, and satisfy (1.1). As usual, a solution of (1.1) is said to be oscillatory if the set of its zeros is unbounded from above, otherwise it is called nonoscillatory. (1.1) is called oscillatory if all of its solutions are oscillatory. We say that (1.1) satisfies the superlinear condition if $q_1(t) \equiv 0$ and it satisfies the sublinear condition if $q_2(t) \equiv 0$.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions of second order linear and nonlinear neutral delay differential equations (see, for example, [**1**–**3, 5, 7**–**10, 14, 15, 17**–21, 24] and the references therein). For the second order neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f(y(t-\sigma)) = 0. \tag{1.2}$$

To the best of our knowledge, almost all of the known results obtained for (1.2) required the assumption that the function $f(y)$ satisfies $f'(y) \geq k > 0$ or $f(y)/y \geq k > 0$ for $y \neq 0$ (see [**2, 5, 7**–**10, 14, 15, 17, 18, 24**], etc.), which is not applicable for $f(y) = |y|^{\lambda} \operatorname{sgn} y$, the classical Emden-Fowler case. Very recently, the results of Atkinson [**4**] and Belohorec [**6**] for second order ordinary differential equation have been extended to

(1.2) by Wong [**21**] under the assumption that the nonlinear function f satisfies the sublinear condition

$$0 < \int_{0+}^{\varepsilon} \frac{du}{f(u)}, \quad \int_{0-}^{-\varepsilon} \frac{du}{f(u)} < \infty \text{ for all } \varepsilon > 0,$$

as well as the superlinear condition

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{f(u)}, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \text{ for all } \varepsilon > 0.$$

Also it will be of great interest to find some oscillation criteria for the special case for (1.2), even in the Emden-Fowler equation

$$(y(t) + p(t)y(t-\tau))'' + q(t)|y(t-\sigma)|^{\lambda}\operatorname{sgn} y(t-\sigma) = 0, \quad \lambda > 0. \tag{1.3}$$

This problem was posed by Wong [**21**, Remark d]. As an affirmative answer to it, Saker [**19**], Saker and Manojlović [**20**] have established some oscillation criteria for (1.2) and (1.3). However, these results cannot be applied to (1.1). Therefore, in the present paper, by the generalized Riccati technique [**25**] and the averaging technique [**13, 16, 22, 23**], we shall establish several Kamenev-type oscillation criteria for (1.1). Our theorems extend and improve some known results in [**19, 20**]. In particular, two interesting examples that point out the applications of our results are also included.

2. Main results. In this section, we shall establish Kamemev-type oscillation theorems for (1.1) under the cases when $0 \le p(t) \le 1$ and $-1 < p_0 \le p(t) \le 0$, which extend the results in [**13, 16, 22, 23**] to (1.1). It will be convenient to make the following notations in the remainder of this paper. Let $\phi \in C^1([t_0,\infty),\mathbf{R}^+)$ and $\eta \in C^1([t_0,\infty),\mathbf{R})$. Define

$$\mu = \min\left\{\frac{\beta-\alpha}{\beta-1}, \frac{\beta-\alpha}{1-\alpha}\right\}, \quad g(s) = \phi(s)r(s-\sigma_1),$$

$$Q_1(t) = \phi(t)\left\{\mu\left[q_1^{\beta-1}(t)q_2^{1-\alpha}(t)(1-p(t-\sigma_1))^{\alpha(\beta-1)} \times (1-p(t-\sigma_2))^{\beta(1-\alpha)}\right]^{1/\beta-\alpha} + \frac{\eta^2(t)}{r(t-\sigma_1)} - \eta'(t)\right\},$$

$$Q_2(t) = \phi(t)\left\{\mu\ [q_1^{\beta-1}(t)q_2^{1-\alpha}(t)]^{1/\beta-\alpha} + \frac{\eta^2(t)}{r(t-\sigma_1)} - \eta'(t)\right\}.$$

In order to present our theorems, we first introduce, following Philos [**16**], the function class $\Im$ which will be extensively used in the sequel. Namely, let $D_0 = \{(t,s) \in \mathbf{R}^2 : t > s \geq t_0\}$ and $D = \{(t,s) \in \mathbf{R}^2 : t \geq s \geq t_0\}$. We will say that the function $H \in C(D, \mathbf{R})$ belongs to the class $\Im$, denoted by $H \in \Im$, if

(H1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on $(t,s) \in D_0$;

(H2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable;

(H3) For any given functions $\phi \in C^1([1,\infty), \mathbf{R}^+)$ and $\eta \in C^1([1,\infty), \mathbf{R})$, there exists a function $h \in C(D, \mathbf{R})$ such that

$$\frac{\partial}{\partial s}H(t,s) + \left[\frac{\phi'(s)}{\phi(s)} + \frac{2\,\eta(s)}{r(s-\sigma_1)}\right]H(t,s) = -h(t,s)H(t,s) \quad \text{for} \quad (t,s) \in D_0.$$

For $\varphi \in C([t_0,\infty), \mathbf{R})$, we take an operator $A(\,\cdot\,; T, t)$, which is defined in [**22**], in terms of H, as follows

$$A(\varphi; T, t) = \int_T^t H(r,s)\varphi(s)\,ds \quad \text{for} \quad t > T.$$

It is easy to verify that $A(\cdot; T, t)$ is a linear operator.

Theorem 2.1. *Let $H \in \Im$. Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(1) $0 \leq p(t) \leq 1$, *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A\left(Q_1 - \frac{1}{4}gh^2; t_0, t\right) = \infty. \tag{2.1}$$

(2) $-1 < p_0 \leq p(t) \leq 0$, *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A\left(Q_2 - \frac{1}{4}gh^2; t_0, t\right) = \infty. \tag{2.2}$$

Proof. (1) Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) \neq 0$ for $t \geq t_0$. Further, we suppose that there exists a $t_1 > t_0$ such that $y(t) > 0$, $y(t-\tau) > 0$, $y(t-\sigma_1) > 0$, and $y(t-\sigma_2) > 0$ for $t \geq t_1$. Since the substitution $u = -y$ transforms (1.1) into an equation of the same form subject to the assumptions of Theorem 2.1. As an analogous proof of Lemma 1 (1) [**24**], see also [**15**]. Then, for some $T_0 \geq t_1$, we have immediately that

$$x(t) > 0, \quad x'(t) > 0, \quad (r(t)x'(t))' \leq 0 \quad \text{for} \quad t \geq T_0. \tag{2.3}$$

Using (2.3), noting that $x(t) \geq y(t)$, we have

$$y(t) = x(t) - p(t)y(t-\tau) \geq x(t) - p(t)x(t-\sigma) \geq (1-p(t))x(t).$$

Thus, for all $t > T_0$,

$$\begin{aligned} y(t-\sigma_1) &\geq (1-p(t-\sigma_1))x(t-\sigma_1), \\ y(t-\sigma_2) &\geq (1-p(t-\sigma_2))x(t-\sigma_2). \end{aligned}$$

Then (1.1) implies that

$$\begin{aligned} (r(t)x'(t))' &+ q_1(t)(1-p(t-\sigma_1))^{\alpha} x^{\alpha}(t-\sigma_1) \\ &+ q_2(t)(1-p(t-\sigma_2))^{\beta} x^{\beta}(t-\sigma_2) \leq 0, \quad t \geq T_0. \end{aligned} \tag{2.4}$$

Define the function

$$w(t) = \phi(t)\left[\frac{r(t)x'(t)}{x(t-\sigma_1)} + \eta(t)\right]. \tag{2.5}$$

Differentiating (2.5) and using (2.4), we have

(2.6)

$$\begin{aligned} w'(t) &= \frac{\phi'(t)}{\phi(t)}w(t) + \phi(t)\left[\frac{(r(t)x'(t))'}{x(t-\sigma_1)} - \frac{r(t)x'(t)x'(t-\sigma_1)}{x^2(t-\sigma_1)} + \eta'(t)\right] \\ &\leq \frac{\phi'(t)}{\phi(t)}w(t) - \phi(t)\Big[q_1(t)(1-p(t-\sigma_1))^{\alpha}x^{\alpha-1}(t-\sigma_1) \\ &\qquad + q_2(t)(1-p(t-\sigma_2))^{\beta}\frac{x^{\beta}(t-\sigma_2)}{x(t-\sigma_1)}\Big] \\ &\quad - \frac{\phi(t)}{r(t-\sigma_1)}\left(\frac{r(t)x'(t)}{x(t-\sigma_1)}\right)^2 + \phi(t)\eta'(t) \end{aligned}$$

Since $r(t)x'(t) \le r(t-\sigma_1)x'(t-\sigma_1)$. For simplicity, let $\sigma_1 \ge \sigma_2$, (a similar argument holds for $\sigma_1 < \sigma_2$), then $x(t-\sigma_1) \le x(t-\sigma_2)$. Hence, (2.6) yields that

(2.7)
$$w'(t) \le \frac{\phi'(t)}{\phi(t)}w(t) - \phi(t)[\,q_1(t)(1-p(t-\sigma_1))^{\alpha}x^{\beta-1}(t-\sigma_1)] \\ + q_2(t)(1-p(t-\sigma_2))^{\beta}x^{\beta-1}(t-\sigma_1)\,] \\ - \frac{\phi(t)}{r(t-\sigma_1)}\left(\frac{w(t)}{\phi(t)} - \eta(t)\right)^2 + \phi(t)\eta'(t).$$

The Young inequality [**12**, Theorem 61] implies that

$$\frac{\beta-1}{\beta-\alpha}[\,q_1(t)(1-p(t-\sigma_1))^{\alpha}x^{\alpha-1}(t-\sigma_1)] \\ + \frac{1-\alpha}{\beta-\alpha}[q_2(t)(1-p(t-\sigma_2))^{\beta}x^{\beta-1}(t-\sigma_1)] \\ \ge \left[q_1^{\beta-1}(t)q_2^{1-\alpha}(t)\,(1-p(t-\sigma_1))^{\alpha(\beta-1)}(1-p(t-\sigma_2)\,)^{\beta(1-\alpha)}\right]^{1/\beta-\alpha}.$$

Consequently,

$$(2.8)\quad q_1(t)(1-p(t-\sigma_1))^{\alpha}x^{\alpha-1}(t-\sigma_1) + q_2(t) \\ (1-p(t-\sigma_2))^{\beta}x^{\beta-1}(t-\sigma_1) \\ \ge \mu\left[q_1^{\beta-1}(t)q_2^{1-\alpha}(t)\,(1-p(t-\sigma_1))^{\alpha(\beta-1)}(1-p(t-\sigma_2))^{\beta(1-\alpha)}\right]^{1/\beta-\alpha}.$$

Combining (2.7) and (2.8), we have

$$(2.9)\quad w'(t) \le -Q_1(t) + \left[\frac{\phi'(t)}{\phi(t)} + \frac{2\,\eta(t)}{r(t-\sigma_1)}\right]w(t) - \frac{1}{g(t)}w^2(t), \quad t \ge T_0.$$

Apply the operator $A(\,\cdot;T,t)$, $t > T \ge T_0$, to (2.9), and use (H3) to find

$$(2.10)\qquad A(Q_1;T,t) \le H(t,T)w(T) - A(hw + g^{-1}w^2;T,t).$$

Completing squares of w in (2.10) yields

$$A(Q_1;T,t) \le H(t,T)w(T) - A\left(g^{-1}\left(w + \frac{1}{2}gh\right)^2;T,t\right) \\ + \frac{1}{4}A(gh^2;T,t),$$

which can be simplified to

(2.11)
$$A\left(Q_1 - \frac{1}{4}gh^2; T, t\right) + A\left(g^{-1}\left(w + \frac{1}{2}gh\right)^2; T, t\right) \leq H(t,T)w(T).$$

Set $T = T_0$ and in view of (H2), note that the second term is nonnegative, so

$$A\left(Q_1 - \frac{1}{4}gh^2; T_0, t\right) \leq H(t,T_0)w(T_0) \leq H(t,t_0)|w(T_0)|. \tag{2.12}$$

Thus, from (2.12) and (H2), we obtain

(2.13)
$$\begin{aligned} A\left(Q_1 - \frac{1}{4}gh^2; t_0, t\right) &= A\left(Q_1 - \frac{1}{4}gh^2; t_0, T_0\right) \\ &\quad + A\left(Q_1 - \frac{1}{4}gh^2; T_0, t\right) \\ &\leq H(t,t_0)\int_{t_0}^{T_0} |Q_1(s)|\, ds + H(t,t_0)|w(T_0)| \\ &\leq H(t,t_0)\left[\int_{t_0}^{T_0} |Q_1(s)| ds + |w(T_0)|\right]. \end{aligned}$$

Divide (2.13) through by $H(t,t_0)$ and take limsup in it as $t \to \infty$. Condition (2.1) gives a desired contradiction in (2.13). This proves Case (1).

(2) Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) \neq 0$ for $t \geq t_0$. Further, we suppose that there exists a $t_1 > t_0$ such that $y(t) > 0$, $y(t-\tau) > 0$, $y(t-\sigma_1) > 0$, and $y(t-\sigma_2) > 0$ for $t \geq t_1$. As an analogous proof of Lemma 1 (2) [**24**]. Then, for some $T_0 \geq t_1$, we still have that (2.3) holds for $t \geq T_0$. Noting that $y(t) \geq x(t)$, we get

$$y(t-\sigma_1) \geq x(t-\sigma_1), \quad y(t-\sigma_2) \geq x(t-\sigma_2) \quad \text{for} \quad t \geq T_0.$$

Then, (1.1) changes into

$$(a(t)x'(t))' + q_1(t)x^{\alpha}(t-\sigma_1) + q_2(t)x^{\beta}(t-\sigma_2) \leq 0, \quad t \geq T_0. \tag{2.14}$$

Consider the function $w(t)$ defined by (2.5). As the same as the proof of (2.6), and noting (2.3) and (2.14), we can obtain

(2.15)
$$w'(t) \le \frac{\phi'(t)}{\phi(t)} w(t) - \Phi(t)\left[q_1(t)x^{\alpha-1}(t-\sigma_1) + q_2(t)x^{\beta-1}(t-\sigma_1)\right] - \frac{\phi(t)}{r(t-\sigma_1)}\left(\frac{r(t)x'(t)}{r(t-\sigma_1)}\right)^2 + \phi(t)\eta'(t).$$

The Young inequality [**12**, Theorem 61] implies that

$$\frac{\beta-1}{\beta-\alpha}[q_1(t)x^{\alpha-1}(t-\sigma_1)] + \frac{1-\alpha}{\beta-\alpha}[q_2(t)x^{\beta-1}(t-\sigma_1)] \ge [q_1^{\beta-1}(t)q_2^{1-\alpha}(t)]^{1/\beta-\alpha}.$$

Consequently,

$$q_1(t)x^{\alpha-1}(t-\sigma_1) + q_2(t)x^{\beta-1}(t-\sigma_1) \ge \mu\,[q_1^{\beta-1}(t)q_2^{1-\alpha}(t)]^{1/\beta-\alpha}. \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$w'(t) \le -Q_2(t) + \left[\frac{\phi'(t)}{\phi(t)} + \frac{2\,\eta(t)}{r(t-\sigma_1)}\right] w(t) - \frac{1}{g(t)} w^2(t), \quad t \ge T_0.$$

The rest of the proof is similar to that of Case (1) and is thus omitted. Hence, this completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $H \in \Im$. Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(3) $0 \le p(t) \le 1$, *and there exist φ_1, $\varphi_2 \in C([t_0,\infty),\mathbf{R})$ such that for all $T \ge t_0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(Q_1;T,t) \ge \varphi_1(T), \tag{2.17}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(gh^2;T,t) \le \varphi_2(T), \tag{2.18}$$

where φ_1 and φ_2 satisfy

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} A\left(g^{-1}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2; T, t\right) = \infty. \tag{2.19}$$

(4) $-1 < p_0 \le p(t) \le 0$, *and there exist* φ_1, $\varphi_2 \in C([t_0,\infty),\mathbf{R})$ *such that for all* $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(Q_2;T,t) \ge \varphi_1(T), \tag{2.20}$$

and (2.18) *hold, where* φ_1 *and* φ_2 *satisfy* (2.19).

Proof. We only show Case (3). The proof of Case (4) is similar to that of Case (3). Proceeding as the proof of Case 1 of Theorem 2.1, we have that (2.10) and (2.11) hold; again divide (2.11) by $H(t,T)$ and drop the nonnegative second term, and obtain

$$\frac{1}{H(t,T)} A(Q_1;T,t) - \frac{1}{4}\frac{1}{H(t,T)} A(gh^2;T,t) \le w(T), \quad t > T. \tag{2.21}$$

Take limsup in (2.21) as $t \to \infty$, and note from (2.17) and (2.18) that

$$\varphi_1(T) - \frac{1}{4}\,\varphi_2(T) \le w(T),$$

from which it follows that

$$\frac{1}{H(t,T)} A\left(g^{-1}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2;T,t\right) \le \frac{1}{H(t,T)} A(g^{-1}w^2;T,t). \tag{2.22}$$

On the other hand, by (2.10), we have

$$\frac{1}{H(t,T)} A(g^{-1}w^2;T,t) - \frac{1}{H(t,T)} A(|h||w|;T,t) \le w(T) - \frac{1}{H(t,T)} A(Q_1;T,t).$$

Thus, by (2.17),

$$\liminf_{r\to\infty} \left\{ \frac{1}{H(t,T_0)} A(g^{-1}w^2;T_0,t) - \frac{1}{H(t,T)} A(|h||w|;T,t) \right\} \le w(T_0) - \varphi_1(T_0) \le C_0. \tag{2.23}$$

where C_0 is a constant. According to (2.23), there exists a sequence $\{t_j\}_{j=1}^{\infty} \in [t_0, \infty)$ with $\lim_{j\to\infty} t_j = \infty$ such that

$$\frac{1}{H(t_j,T_0)} A\left(g^{-1}w^2; T_0, t_j\right) - \frac{1}{H(t_j,T)} A(|h||w|; T_0, t_j) \leq C_0 + 1. \tag{2.24}$$

Now, we claim that

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} A\left(g^{-1}w^2; T_0, t\right) < \infty. \tag{2.25}$$

If (2.25) does not hold, and noting that (2.24), we get

$$\lim_{j\to\infty} \frac{1}{H(r_j,T_0)} A\left(g^{-1}w^2; T_0, t_j\right) = \infty. \tag{2.26}$$

So (2.24) and (2.26) gives, for j large enough,

$$\frac{A(|h||w|; T_0, t_j)}{A\left(g^{-1}w^2; T_0, t_j\right)} - 1 \geq -\frac{1}{2},$$

that is,

$$A(|h||w|; T_0, t_j) \geq \frac{1}{2} A\left(g^{-1}w^2; T_0, t_j\right). \tag{2.27}$$

The Cauchy-Schwarz inequality follows

$$[A(|h||w|; T_0, t_j)]^2 \leq A(g^{-1}w^2; T, t_j) A(g|h|^2; T, t_j). \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$\frac{1}{H(t_j,T_0)} A\left(g^{-1}w^2; T_0, t_j\right) \leq \frac{4}{H(t_j,T_0)} A(gh^2; T_0, t_j). \tag{2.29}$$

By (2.18), the righthand side of (2.29) is bounded, which contradicts (2.26). Thus (2.25) holds. Hence, by (2.22),

$$\liminf_{r\to\infty} \frac{1}{H(t,T)} A\left(g^{-1}\left(\varphi_1 - \frac{1}{4}\varphi_2\right)_+^2; T, t\right) \leq \liminf_{r\to\infty} \frac{1}{H(t,T)} A\left(g^{-1}w^2; T, t\right) < \infty.$$

which contradicts (2.19). This completes the proof of Case (3). □

Remark 2.1. For the superlinear equation, Theorems 2.1 and 2.2 improve the main results in [**19**]. For the Emden-Fowler type equation (1.3), our results extend and improve the main results in [**18**].

3. Corollaries and examples. As Theorems 2.1 and 2.2 are rather general, it is convenient for applications to derive a number of oscillation criteria with the appropriate choice of the functions H, ϕ and η; here, we will give some corollaries of Theorems 2.1 and 2.2. Finally, we will show the applications of our main results in two interesting examples. We will see that equations (3.18) and (3.19) are oscillatory based on our Corollaries 3.2 and 3.6, though the oscillations cannot be demonstrated by the results of [**1**–**10, 13**–**25**] and other known criteria.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 3.1. *Let* $H \in \Im$. *Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(5) $0 \leq p(t) \leq 1$, *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(Q_1; t_0, t) = \infty, \tag{3.1}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(gh^2; t_0, t) < \infty. \tag{3.2}$$

(6) $-1 < p_0 \leq p(t) \leq 0$, *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(Q_2; t_0, t) = \infty, \tag{3.3}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(gh^2; t_0, t) < \infty. \tag{3.4}$$

Corollary 3.2. *Suppose that one of the following conditions holds. Then* (1.1) *is oscillatory.*

(7) $0 \le p(t) \le 1$, *and*

$$\liminf_{t\to\infty} G(t) \int_t^\infty \theta_1(s)\,ds > \frac{1}{4}. \tag{3.5}$$

(8) $-1 < p_0 \le p(t) \le 0$, *and*

$$\liminf_{t\to\infty} G(t) \int_t^\infty \theta_2(s)\,ds > \frac{1}{4} \tag{3.6}$$

where

$$G(t) = \int_{t_0}^t \frac{1}{r(s-\sigma_1)}\,ds,$$

$$\theta_1(t) = \mu\,\Big[q_1^{\beta-1}(t) q_2^{1-\alpha}(t)(\,1-p(t-\sigma_1)\,)^{\alpha(\beta-1)} \times (1-p(t-\sigma_2))^{\beta(1-\alpha)}\Big]^{1/\beta-\alpha},$$

$$\theta_2(t) = \mu\,\Big[q_1^{\beta-1}(t) q_2^{1-\alpha}(t)\Big]^{1/\beta-\alpha}.$$

Proof. We only show Case (7); the proof of case (8) is analogous to that of case (7). By (3.5), there exist two numbers $T \ge t_0$ and $k > 1/4$ such that

$$G(t)\int_t^\infty \theta_1(s)ds > k \quad \text{for} \quad t > T, \quad \text{and} \quad \lim_{t\to\infty} G(t) = \infty.$$

Let

$$H(t,s) = [\,G(t) - G(s)]^2, \quad \phi(t) = G(t), \quad \text{and} \quad \eta(r) = -\frac{1}{2G(t)}.$$

Then

$$h(r,s) = \frac{2}{r(s-\sigma_1)[G(t)-G(s)]}.$$

Define

$$\Theta_1(t) = \int_t^\infty \theta_1(s)\,ds.$$

Thus, for all $t > T$,

$$\begin{aligned} A(Q_1;T,t) &= \int_T^t [G(t)-G(s)]^2 G(s)\, d\left(-\Theta_1(s)+\frac{1}{G(s)}\right) \\ &= [G(t)-G(T)]^2\left(G(T)\Theta_1(T)-\frac{1}{4}\right) \\ &\quad + \int_T^t \left(G(s)\Theta_1(s)-\frac{1}{4}\right) \\ &\qquad \times \left(\frac{G^2(t)}{G(s)}-4G(t)+3G(s)\right) G'(s)\,ds \\ &\geq \left(k-\frac{1}{4}\right)\int_T^t \left(\frac{G^2(t)}{G(s)}-4G(t)+3G(s)\right) G'(s)\,ds \\ &= \left(k-\frac{1}{4}\right)\left(\ln\frac{G(t)}{G(T)}-\frac{5}{2}\right) \\ &\quad \times G^2(t)+4G(t)G(T)-\frac{3}{2}G^2(T), \end{aligned}$$

and

$$A(gh^2;T,t) = 2[\,G^2(t)-G^2(T)].$$

Hence, it follows from Corollary 3.1 (5) that (1.1) is oscillatory. □

Corollary 3.3. *Let* $\eta \in C([t_0,\infty),\mathbf{R})$. *Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(9) $0 \leq p(t) \leq 1$, *and for some* $\lambda > 1$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^{t} [G(t)-G(s)]^\lambda Q_3(s)\,ds = \infty. \tag{3.7}$$

(10) $-1 < p_0 \leq p(t) \leq 0$, *and for some* $\lambda > 1$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^{t} [G(t)-G(s)]^\lambda Q_4(s)\,ds = \infty \tag{3.8}$$

where

$$\begin{aligned} Q_3(t) = \phi_0(t)\Big\{\mu\, \Big[& q_1^{\beta-1}(t) q_2^{1-\alpha}(t)(1-p(t-\sigma_1))^{\alpha(\beta-1)} \\ & \times(1-p(t-\sigma_2))^{\beta(1-\alpha)}\Big]^{1/\beta-\alpha} + \frac{\eta^2(t)}{r(t-\sigma_1)} - \eta'(t)\Big\}, \end{aligned}$$

$$Q_4(t) = \phi_0(t)\left\{\mu\,[q_1^{\beta-1}(t)q_2^{1-\alpha}(t)]^{1/\beta-\alpha} + \frac{\eta^2(t)}{r(t-\sigma_1)} - \eta'(t)\right\},$$

$$\phi_0(t) = \exp\left[-2\int_{t_0}^{t}\frac{\eta(s)}{r(s-\sigma_1)}\,ds\right], \qquad G(t) = \int_{t_0}^{t}\frac{1}{r(s-\sigma_1)}\,ds.$$

Proof. Let $H(t,s) = [G(t) - G(s)]^\lambda$. Then $h(t,s) = \lambda/(g(s)[G(t) - G(s)])$. Note that

$$A(gh^2; T, t) = \frac{\lambda^2}{\lambda - 1}[G^{\lambda-1}(t) - G^{\lambda-1}(T)].$$

It is easy to show that Corollary 3.3 follows from Corollary 3.1. □

Following the classical ideas of Kamenev [**13**], we define $H(t,s)$ as

$$H(t,s) = (t-s)^\lambda, \quad (t,s) \in D,$$

where $\lambda > 1$. Then, by Theorem 2.1, we have

Corollary 3.4. *Let* $\eta \in C([t_0,\infty),\mathbf{R})$. *Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(11) $0 \le p(t) \le 1$, *and for some* $\lambda > 1$,

$$\limsup_{t\to\infty}\frac{1}{t^\lambda}\int_{t_0}^{t}\left\{(t-s)^\lambda Q_3(s) - \frac{\lambda^2}{4}(t-s)^{\lambda-2}g(s)\right\}ds = \infty. \tag{3.9}$$

(12) $-1 < p_0 \le p(t) \le 0$, *and for some* $\lambda > 1$,

$$\limsup_{t\to\infty}\frac{1}{t^\lambda}\int_{t_0}^{t}\left\{(t-s)^\lambda Q_4(s) - \frac{\lambda^2}{4}(t-s)^{\lambda-2}g(s)\right\}ds = \infty, \tag{3.10}$$

where $Q_3(t)$ *and* $Q_4(t)$ *are defined in Corollary* 3.3.

With the same choice of the function $H(t,s)$ in Corollaries 3.3 and 3.4, more general Kamenev-type oscillation criteria for (1.1) can be obtained by Theorem 3.2. Now, we state it here for completeness.

Corollary 3.5. *Let* $\eta \in C([t_0,\infty),\mathbf{R})$ *and* $\lim_{t\to\infty} G(t) = \infty$. *Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(13) $0 \le p(t) \le 1$, *and there exists* $\varphi \in C([t_0,\infty),\mathbf{R})$ *such that for some* $\lambda > 1$ *and all* $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_T^t [G(t)-G(s)]^\lambda Q_3(s)ds \ge \varphi(T), \tag{3.11}$$

where φ *satisfies*

$$\liminf_{t\to\infty} \frac{1}{G^\lambda(t)} \int_T^t g^{-1}(s)[G(t)-G(s)]^\lambda \varphi^2(s)_+ ds = \infty. \tag{3.12}$$

(14) $-1 < p_0 \le p(t) \le 0$, *and there exists* $\varphi \in C([t_0,\infty),\mathbf{R})$ *such that for some* $\lambda > 1$ *and all* $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_T^t [G(t)-G(s)]^\lambda Q_4(s)\,ds \ge \varphi(T), \tag{3.13}$$

where φ *satisfies* (3.12), $Q_3(t)$, $Q_4(t)$ *and* $G(t)$ *are defined in Corollary* 3.3.

Corollary 3.6. *Let* $\eta \in C([t_0,\infty),\mathbf{R})$. *Then* (1.1) *is oscillatory provided that one of the following conditions holds.*

(15) $0 \le p(t) \le 1$, *and there exist* φ_1, $\varphi_2 \in C([t_0,\infty),\mathbf{R})$ *such that for some* $\lambda > 1$ *and all* $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_T^t (t-s)^\lambda Q_3(s)\,ds \ge \varphi_1(T), \tag{3.14}$$

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_T^t (r-s)^{\lambda-2} g^{-1}(s)\,ds \le \varphi_2(T), \tag{3.15}$$

where φ_1 *and* φ_2 *satisfy*

$$\liminf_{t\to\infty} \frac{1}{t^\lambda} \int_T^t g^{-1}(s)(t-s)^\lambda \left(\varphi_1(s) - \frac{1}{4}\varphi_2(s)\right)_+^2 ds = \infty. \tag{3.16}$$

(16) $-1 < p_0 \le p(t) \le 0$, *and there exist* φ_1, $\varphi_2 \in C([t_0,\infty),\mathbf{R})$ *such that for some* $\lambda > 1$ *and all* $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{t^\lambda}\int_T^t (t-s)^\lambda Q_4(s)\,ds \ge \varphi_1(T), \tag{3.17}$$

and (3.15) *hold, where* φ_1 *and* φ_2 *satisfy* (3.16), $Q_3(t)$ *and* $Q_4(t)$ *are defined in Corollary* 3.3.

Example 3.1. Consider the following Emden-Fowler type neutral delay equation

$$\begin{aligned}\left(y(t) + \frac{p}{\sqrt{t+3}}\,y(t-1)\right)'' &+ q_1(t)|y(t-3)|^{-2n/(2n+1)}y(t-3)\\ &+ q_2(t)|y(t-2)|^{2n/(2n+1)}y(t-2) = 0, \quad t \ge 3,\end{aligned} \tag{3.18}$$

where $r(t) = 1$, $p(t) = p/\sqrt{t+3}$, $p = \pm 1$, n is positive integer, $\alpha = 1/(2n+1)$, $\beta = (4n+1)/(2n+1)$, and q_1, $q_2 \in C([3,\infty),\mathbf{R}^+)$ with $q_1(t)q_2(t) \ge \lambda_1/t^4$, $(\lambda_1 > 0)$. Now, we consider the following two cases.

Case 1. $p = 1$. Noting that $p(t-3) > p(t-2)$, we have

$$\mu = 1, \quad G(t) = t-3, \quad \text{and} \quad \theta_1(t) \ge \frac{\sqrt{\lambda_1}}{t^2}\left(1 - \frac{1}{\sqrt{t}}\right).$$

Then

$$\liminf_{t\to\infty} G(t)\int_t^\infty \theta(s)\,ds \ge \liminf_{t\to\infty}(t-3)\int_t^\infty \frac{\sqrt{\lambda_1}}{s^2}\left(1-\frac{1}{\sqrt{s}}\right)ds = \sqrt{\lambda_1}.$$

Hence, by Corollary 3.2 (7), (3.18) is oscillatory if $\lambda > 1/16$.

Case 2. $p = -1$. we have

$$\mu = 1, \qquad G(t) = t-3, \quad \text{and} \quad \theta_2(t) \ge \frac{\sqrt{\lambda}}{t^2}.$$

Then

$$\liminf_{t\to\infty} G(t)\int_t^\infty \theta_2(s)\,ds \ge \liminf_{t\to\infty}(t-3)\int_t^\infty \frac{\sqrt{\lambda_1}}{s^2}\,ds = \sqrt{\lambda_1}.$$

It follows from Corollary 3.2 (8) that (3.18) is oscillatory if $\lambda_1 > 1/16$.

Example 3.2. Consider the following Emden-Fowler type neutral delay equation

$$\begin{aligned}(3.19)\quad [(t+3)(y(t)+p\,y(t-1))']' + q_1(t)|y(t-3)|^{-2/3}y(t-3)\\ + q_2(t)|y(t-2)|^{2/3}y(t-2) = 0, \quad t \ge 3,\end{aligned}$$

where $r(t) = t+3$, $-1 < p < 1$, $\alpha = 1/3$, $\beta = 5/3$, and $q_1, q_2 \in C([3,\infty), \mathbf{R}^+)$ with $q_1(t)q_2(t) \ge \lambda_2/t$, $(\lambda_2 > 0)$. Let $\eta(t) = 1/2t^2$. Now, we consider the following two cases.

Case 1. $0 \le p < 1$. A direct computation yields

$$\mu = 1, \quad g(t) = 1, \quad \text{and} \quad Q_3(t) \ge \frac{1}{t}\left(\frac{(1-p)\sqrt{\lambda_2}}{\sqrt{t}} + \frac{1}{t^3} + \frac{1}{4t^5}\right).$$

For Corollary 3.6 (15), let $\lambda = 2$. Then, for all $t > T \ge 3$,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t (t-s)^2 Q_3(s)\,ds\\ &\qquad \ge \limsup_{t\to\infty}\frac{1}{t^2}\int_T^t (t-s)^2\left\{\frac{(1-p)\sqrt{\lambda_2}}{s^{3/2}} + \frac{1}{s^3} + \frac{1}{4s^5}\right\} ds\\ &\qquad = \frac{2(1-p)\sqrt{\lambda_2}}{\sqrt{T}} + \frac{1}{2T^2} + \frac{1}{16T^4},\end{aligned}$$

and

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_T^t g^{-1}(s)\,ds = 0.$$

So, set

$$\varphi_1(T) = \frac{2(1-p)\sqrt{\lambda_2}}{\sqrt{T}}, \quad \text{and} \quad \varphi_2(T) = 0.$$

Then

$$\liminf_{t\to\infty} \frac{1}{t^2} \int_T^t g^{-1}(s)(t-s)^2 \left(\varphi_1(s) - \frac{1}{4}\varphi_2(s)\right)_+^2 ds$$

$$= \liminf_{t\to\infty} \frac{4\lambda_2(1-p)^2}{t^2} \int_T^t \frac{(t-s)^2}{s} \, ds = \infty.$$

It follows from Corollary 3.6 (15) that (2.19) is oscillatory.

Case 2. $p = 1$. Then

$$\mu = 1, \quad g(t) = 1, \quad Q_4(t) \geq \frac{1}{t}\left(\frac{\sqrt{\lambda_2}}{\sqrt{t}} + \frac{1}{t^3} + \frac{1}{4t^5}\right).$$

The rest of the proof is similar to that of Case 1. Hence (3.19) is oscillatory by Corollary 3.6 (16).

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School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China
Email address: xztxhyyj@pub.guangzhou.gd.cn