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ORLICZ SPACES OF INTEGRABLE FUNCTIONS WITH RESPECT TO VECTOR-VALUED MEASURES

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ABSTRACT. This paper extend the theory of p-integrable functions with respect to a vector measure studied in [**6**] for $p = 1$ and in [**10**] for $1 < p < \infty$, introducing the same notion in the context of the Orlicz spaces. Some topics of the third section have been treated recently in [**1**] but with a different point of view.

1. Introduction. After the appearance of Orlicz spaces, the theory of L_p-spaces has progressed in several directions. The special structure of Orlicz spaces discovers an amount of new questions that are hidden in the classical theory because of the special properties of L_p-spaces. Moreover, Orlicz spaces can be the first step to consider more abstracts generalizations to the classical theory of Banach spaces. The vigorous growth of the topic is a consequence of the interest of applications to potential theory, interpolation and differential equations among others.

As in classical theory of scalar integration, the spaces of integrable functions with respect to a vector measure has been constructed around L_p-spaces. This paper tries to introduce the integration with respect to vector-valued measures in the setting of Orlicz spaces. As we can see, this extension carries new problems, the solution of which is here not as clear as in classical theory.

In the following, (Ω, Σ, μ) denotes a measure space, where Σ is a σ-algebra and μ a nonnegative measure. For definitions and properties of Orlicz functions and the Orlicz spaces of measurable functions we refer to [**3, 7, 9**]. A nondegenerated Orlicz function Φ is a continuous, nondecreasing and convex function defined in $[0, \infty[$ such that $\Phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. The representation integral of a convex function Φ can be used to obtain interesting properties;

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for example, we use in some occasions in this paper that if $0 < a < 1$ then $\Phi(ax) \leq a\Phi(x)$, for every $x > 0$. In the paper Φ *will be a strictly increasing Orlicz function.* We recall that if $f \in L_\Phi(\mu)$, then there is a $k > 0$ such that $\Phi(|f|/k) \in L_1(\mu)$. The space $L_\phi(\mu)$ endowed with the norm $\|f\|_{L_\Phi(\mu)} := \inf\{k > 0 : \int_\Omega \Phi(|f|/k)\, d\mu \leq 1\}$ is a Banach space. A classical tool in the theory of Orlicz spaces of integrable functions $L_\Phi(\mu)$ is the (maybe infinite) modular $I_{(\Phi,\mu)}(f) := \int_\Omega \Phi(|f|)\, d\mu$.

For topics about vector measures, we refer to [**2, 6**]. For us F always represents a *countably additive* vector measure $F : \Sigma \to X$, where X is a Banach space. From the Bartle-Dunford-Schwartz theorem, the range of a countably additive vector measure defined in a σ-algebra with values in a Banach space X is a relatively weakly compact subset of X, hence this range is a bounded set; then F is a vector measure with bounded semi-variation $\|F\|$ (a bounded vector measure in Σ). The fact that a countably additive vector measure can have unbounded variation $|F|$ is the main reason to use the semi-variation. If $x' \in X'$, $(x'F)$ denotes the scalar measure such that for every $A \in \Sigma$, $(x'F)(A) := \langle F(A), x' \rangle$.

In general the real Σ-measurable function f is said to be F-integrable if it satisfies

1) f is $(x'F)$-integrable for every $x' \in X'$.

2) For every $B \in \Sigma$ there is an element in X usually denoted by $\int_B f dF$ such that for every $x' \in X'$, $\langle x', \int_B f dF \rangle = \int_B f d(x'F)$.

The space of F-integrable functions is denoted by $\mathcal{L}(F)$. The corresponding space of classes of F-integrable functions under the identification of functions which coincide $\|F\|$-almost everywhere is denoted by $L_1(F)$. In $L_1(F)$ we define the norm

$$\|f\|_{L_1(F)} := \sup_{x' \in B_1(X')} \|f\|_{L_1(|x'F|)},$$

where $B_r(Y)$ represents the closed ball of radius r of a Banach space Y. The space $(L_1(F), \|.\|_{L_1(F)})$ is a Banach space. Moreover, the norm

$$|||f|||_{L_1(F)} := \sup_{A \in \Sigma} \left\| \int_A f dF \right\|$$

is equivalent to $\|.\|_{L_1(F)}$, with

$$|||f|||_{L_1(F)} \leq \|f\|_{L_1(F)} \leq 2|||f|||_{L_1(F)}.$$

With respect to the Banach lattices theory we refer to [**8, 11**]. We remark that $L_1(F)$ is a *solid and order continuous Banach lattice*, and this fundamental fact is used in our work on many occasions.

2. Spaces of (Φ, F)-quasi-integrable functions. Let F be a vector-valued measure defined in a measure space (Ω, Σ) with values in a Banach space X, and let Φ be an Orlicz function.

Definition 1. In the space of measurable functions we can define the functional:

$$\|f\|_{(\Phi,F)} := \sup\{\|f\|_{L_\Phi(|x'F|)}, x' \in B_1(X')\}.$$

If $\Phi(|x|) = |x|$, we write $\|.\|_{(1,F)}$ instead of $\|.\|_{L_1(F)}$.

Definition 2. We say that a measurable function is (Φ, F)-quasi-integrable if $\|f\|_{(\Phi,F)} < \infty$.

The set of classes (under the identification of functions which coincide $\|F\|$-almost everywhere) of (Φ, F) quasi-integrable functions is denoted by $QL_\Phi(F)$. It is easy to see that the pair $(QL_\Phi(F), \|.\|_{(\Phi,F)})$ is a normed space.

Theorem 3. $(QL_\phi(F), \|.\|_{(\Phi,F)})$ *is a solid Banach lattice with weak order unit.*

Proof. It is clear that $(QL_\phi(F), \|.\|_{(\Phi,F)})$ is a solid normed lattice with respect to the usual order. Then we have to see that it is complete.

Suppose that $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $(QL_\phi(F), \|.\|_{(\Phi,F)})$. Let n_1 be the smallest index n such that $\|f_n - f_m\|_{(\Phi,F)} \leq 1/2$ for every $m > n$, and in general n_k, $k \in \mathbf{N}$, is the smallest index number $n > n_{k-1}$ such that $\|f_n - f_m\|_{(\Phi,F)} < 1/2^k$ for every $m > n$. The series $|f_{n_1}| + \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|$ is absolutely summable in $L_\Phi(|x'F|)$ for every $x' \in X'$, and as $L_\Phi(|x'F|)$ is complete then the function $g := |f_{n_1}| + \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}| \in L_\Phi(|x'F|)$ for all $x' \in B_1(X')$, and also $g \in QL_\Phi(F)$. Moreover, the set $B := \{\omega :$

$\Phi(g(\omega)) = \infty\} = \{\omega : g(\omega) = \infty\}$ satisfies that $|x'F|(B) = 0$ for every $x' \in B_1(X')$, hence $\|F\|(B) = 0$. For every $\omega \in \Omega \setminus B$, the series $|f_{n_1}(\omega)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(\omega) - f_{n_k}(\omega)|$ converges; hence, the series $f_{n_1}(\omega) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(\omega) - f_{n_k}(\omega))$ also converges in $\Omega \setminus B$. We define $f(\omega) := f_{n_1}(\omega) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(\omega) - f_{n_k}(\omega))$ if $\omega \in \Omega \setminus B$ and $f(\omega) := 0$ if $\omega \in B$. We have that $f \in QL_\Phi(F)$ and $f = \lim_{k\to\infty} f_{n_k}$ pointwise in $\Omega \setminus B$. Furthermore, $|f - f_{n_k}| \leq \sum_{s=k}^{\infty} |f_{n_{s+1}} - f_{n_s}|$ in $\Omega \setminus B$; hence, $\|f - f_{n_k}\|_{(\Phi,F)} \leq \sum_{s=k}^{\infty} \|f_{n_{s+1}} - f_{n_s}\|_{(\Phi,F)}$, and then $\lim_{k\to\infty} \|f - f_{n_k}\|_{(\Phi,F)} = 0$. Hence, $(QL_\Phi(F), \|.\|_{(\Phi,F)})$ is complete. Moreover, it is easy to see that χ_Ω is a weak order unit. $\square$

Definition 4. In $QL_\Phi(F)$ we can define the (maybe infinite) functional:

$$I_{(\Phi,F)}(f) := \sup_{x' \in B_1(X')} I_{(\Phi,|x'F|)}(f).$$

We continue with some relations between $I_{(\Phi,F)}$ and the norm $\|.\|_{(\Phi,F)}$, which generalizes some well-known results in the Orlicz spaces $L_\Phi(\mu)$.

Remark 5. If $f \in QL_\Phi(F)$, there is $k > 0$ such that

$$I_{(\Phi,F)}(f/k) = \sup_{x' \in B_1(X')} \int_\Omega \Phi(|f|/k)\, d(|x'F|) \leq 1,$$

because if not, for every $n \in \mathbf{N}$ there is an $x'_n \in B_1(X')$ such that $\int_\Omega \Phi(|f|/n)\, d(|x'_n F|) \geq 1$, hence $\|f\|_{L_\Phi(|x'_n F|)} \geq n$, which is impossible.

Corollary 6. *$f \in QL_\Phi(F)$ if and only if there is a $k > 0$ such that $I_{(\Phi,F)}(f/k) < \infty$.*

Then we define in $QL_\Phi(F)$ the functional:

$$\delta_{(\Phi,F)}(f) := \inf\{k > 0 : I_{(\Phi,F)}(f/k) \leq 1\}.$$

Proposition 7. *$\delta_{(\Phi,F)}(f) = \|f\|_{(\Phi,F)}$ for every $f \in QL_\Phi(F)$.*

Proof. It is clear that $\|f\|_{(\Phi,F)} \leq 1$ if and only if $I_{(\Phi,F)}(f) \leq 1$. To prove the desired result, we only recall that

$$\begin{aligned}\delta_{(\Phi,F)}(f) &= \inf\{k > 0 : I_{(\Phi,F)}(f/k) \leq 1\}\\ &= \inf\left\{k > 0 : \frac{f}{k} \in B_1(QL_\phi(F))\right\}.\end{aligned}$$

Then $\delta_{(\Phi,F)}(f)$ is the Minkowski's functional of $B_1(QL_\phi(F))$, hence $\delta_{(\Phi,F)}(f) = \|f\|_{(\Phi,F)}$. □

3. Spaces of (Φ, F)-integrable functions.

Definition 8. We say that a measurable function f (identifying functions which coincide $\|F\|$-almost everywhere) is (Φ, F)-integrable if there is an $a > 0$ such that $\Phi(|f|/a) \in L_1(F)$.

The set of classes of (Φ, F)-integrable functions is denoted by $L_\Phi(F)$.

Proposition 9. $L_\Phi(F) \subseteq L_1(F)$.

Proof. For every support line of Φ, $y = mx - n$, $m > 0$, $n \geq 0$, we know that for every $x \geq 0$, $\Phi(x) \geq mx - n$. Then for every $f \in L_\Phi(F)$ such that $\Phi(|f|/a) \in L_1(F)$ for some $a > 0$, as $\Phi(|f|/a) \geq m/a|f| - n\chi_\Omega$, we have that $|f| \leq a/m(\Phi(|f|/a) + n\chi_\Omega)$ and $a/m(\Phi(|f|/a) + n\chi_\Omega) \in L_1(F)$. As $L_1(F)$ is solid then $f \in L_1(F)$. □

The fact that $L_1(F)$ is a solid normed space can be used to prove as in the scalar case,

Corollary 10. $(L_\Phi(F), \|.\|_{(\Phi,F)})$ *is a solid normed lattice.*

Then it is natural to ask if $(L_\Phi(F), \|.\|_{(\Phi,F)})$ is a Banach space. In order to explain the problem, we put in consideration of the reader the following arguments. Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(L_\Phi(F), \|.\|_{(\Phi,F)})$, hence there is $f \in QL_\Phi(F)$ such that f is the $\|.\|_{(\Phi,F)}$-limit of $\{f_n\}_{n=1}^\infty$. The question is: Does f belong to

$L_\Phi(F)$? The fact that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence, implies that given $1 > \varepsilon > 0$ there is an n_0 such that for every $n_1, n_2 > n_0$,

$$\sup_{x' \in B_1(X')} \int_\Omega \Phi\left(\frac{|f_{n_1} - f_{n_2}|}{\varepsilon}\right) d(|x'F|) \le 1,$$

and as Φ is convex, $\Phi(|f_{n_1} - f_{n_2}|) \le \varepsilon \Phi(|f_{n_1} - f_{n_2}|/\varepsilon)$; hence, the norm convergence implies the modular convergence, that is,

$$\sup_{x' \in B_1(X')} \int_\Omega \Phi(|f_{n_1} - f_{n_2}|) d(|x'F|) \le \varepsilon. \tag{1}$$

Suppose for example that $\Phi(|f_n|) \in L_1(F)$ for every $n \in \mathbf{N}$. Then it is reasonable to think that if $f \in L_\Phi(F)$ then, for every $A \in \Sigma$, $\int_A \Phi(|f|)\, dF \in X$ is the limit in X of the sequence $(\int_A \Phi(|f_n|)\, dF)_{n=1}^\infty$. But in this case $(\int_A \Phi(|f_n|)\, dF)_{n=1}^\infty$ must be a Cauchy sequence in X for every $A \in \Sigma$; hence, given $\varepsilon > 0$ it will be m_0 such that, for every $m_1, m_2 > m_0$,

$$\sup_{x' \in B_1(X')} \int_\Omega |\Phi(|f_{m_1}|) - \Phi(|f_{m_2}|)|\, d(|x'F|) < \varepsilon \tag{2}$$

If we compare, the condition (2) is stronger than condition (1). Moreover, the condition (2) implies in some sense that we have certain control in the way the Orlicz function increases, and in this setting to solve the problem of the completeness of $L_\Phi(F)$ we consider the class of Orlicz functions having the Δ_2 property.

Definition 11. We say that an Orlicz function Φ has the Δ_2 property if $\Phi(2t)/\Phi(t)$ is bounded for every $t \ge 0$.

As in the corresponding spaces of Orlicz $L_\Phi(\mu)$ where μ is a scalar measure, it is easy to see that if Φ satisfies the Δ_2 property and $f \in L_\Phi(F)$, then $\Phi(k|f|) \in L_1(F)$, for every $k > 0$. Then $I_{(\Phi,F)}$ is always finite in $QL_\Phi(F)$. Moreover, in this case, the modular convergence in $QL_\Phi(F)$ is equivalent to the norm convergence.

Proposition 12. *Suppose that Φ satisfies the Δ_2 property, and let X be a Banach space without copies of c_0. Then $QL_\Phi(F) = L_\Phi(F)$.*

Proof. Given an $f \in QL_\Phi(F)$, if $A_n := \{\omega \in \Omega : \Phi(|f|(\omega)) \leq n\}$, then $(\Phi(|f|\chi_{A_n}))_{n=1}^\infty$ is a sequence of bounded functions (hence it is contained in $L_\Phi(F)$) which converges point-wise to $\Phi(|f|)$. We recall that $\Phi(|f|\chi_{A_n}) = \Phi(|f|)\chi_{A_n}$. Define $g_1 := \Phi(|f|\chi_{A_1})$, $g_n := \Phi(|f|\chi_{A_n}) - \Phi(|f|\chi_{A_{n-1}})$, $n > 1$. Clearly $(g_n)_{n=1}^\infty \subset L_\Phi(F)$, and for every $x' \in X'$ and every $B \in \Sigma$, $\sum_{n=1}^\infty |\langle \int_B g_n dF, x'\rangle| = \sum_{n=1}^\infty | \int_B g_n d(x'F) | \leq \sum_{n=1}^\infty \int_B g_n d(|x'F|) = \int_B \Phi(|f|) d(|x'F|) < \infty$, for all $x' \in X'$. Hence, from the Bessaga-Pełczyński's characterization of the Banach spaces without copies of c_0, there is an $\int_B \Phi(|f|)\, dF \in X$ such that $\langle \int_B \Phi(|f|)\, dF, x'\rangle = \sum_{n=1}^\infty \int_B g_n\, d(x'F) = \int_B \Phi(|f|)\, d(x'F)$. □

In order to study the completeness of $L_\Phi(F)$ without additional assumptions about X, we have the following technical proposition.

Proposition 13. *Suppose that Φ satisfies the Δ_2 property. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that if h and g are in the unit ball of $L_\Phi(F)$ and $\|h - g\|_{(\Phi,F)} < \delta$, then $\sup_{x' \in B_1(X')} \int_\Omega |\Phi(|h|) - \Phi(|g|)|\, d(|x'F|) < \varepsilon$.*

Proof. Let $k > 0$ be such that $\Phi(2x) \leq k\Phi(x)$, for every $x \in \mathbf{R}$. Let $g, u \in L_\Phi(F)$ be such that $\|g\|_{(\Phi,F)} \leq 1$ and $\|u\|_{(\Phi,F)} \leq 1/2$. Then, for every $x' \in B_1(X')$, $I_{(\Phi,|x'F|)}(g) \leq 1$, $I_{(\Phi,|x'F|)}(2g) \leq k$ and $I_{(\Phi,|x'F|)}(2u) \leq 1$. Moreover,

$$\begin{aligned} I_{(\Phi,|x'F|)}(g+u) &= I_{(\Phi,|x'F|)}\left(\frac{1}{2}2g + \frac{1}{2}2u\right) \\ &\leq \frac{1}{2} I_{(\Phi,|x'F|)}(2g) + \frac{1}{2} I_{(\Phi,|x'F|)}(2u) \\ &\leq \frac{1}{2}(k+1). \end{aligned}$$

Fix $g \in L_\Phi(F)$ such that $\|g\|_{(\Phi,F)} \leq 1$ and $x' \in B_1(X')$. We define

$$\tau_{(g,x')}(u) := I_{(\Phi,|x'F|)}(g+u) - I_{(\Phi,|x'F|)}(g),$$

for every $u \in L_\Phi(F)$. It is clear that if $\|u\|_{(\Phi,F)} \leq 1/2$ then $|\tau_{(g,x')}(u)| \leq (k+3)/2$.

The map $\tau_{(g,x')}$ is convex in $L_\Phi(F)$. In fact, if $0 \le a, b \le 1$, $a+b=1$, and $u, v \in L_\Phi(F)$, as $I_{(\Phi,|x'F|)}$ is a convex function in $L_\Phi(F)$, we have

$$\begin{aligned}
a\tau_{(g,x')}(u) + b\tau_{(g,x')}(v)& \\
&= a(I_{(\Phi,|x'F|)}(g+u) - I_{(\Phi,|x'F|)}(g)) \\
&\quad + b(I_{(\Phi,|x'F|)}(g+v) - I_{(\Phi,|x'F|)}(g)) \\
&= aI_{(\Phi,|x'F|)}(g+u) + bI_{(\Phi,|x'F|)}(g+v) - I_{(\Phi,|x'F|)}(g) \\
&\ge I_{(\Phi,|x'F|)}(a(g+u) + b(g+v)) - I_{(\Phi,|x'F|)}(g) \\
&= I_{(\Phi,|x'F|)}(g + au + bv) - I_{(\Phi,|x'F|)}(g) \\
&= \tau_{(g,x')}(au+bv).
\end{aligned}$$

Given $0 < \varepsilon < 1$, we take $f \in L_\Phi(F)$ such that $\|f\|_{(\Phi,F)} \le \varepsilon/(3(k+3))$, or equivalently $\|3(k+3)/2\varepsilon f\|_{(\Phi,F)} \le 1/2$. Then, from the convexity of $\tau_{(g,x')}$, as $2\varepsilon/3(k+3) < 1$, we have

$$\tau_{(g,x')}(f) \le \frac{2\varepsilon}{3(k+3)}\tau_{(g,x')}\left(\frac{3(k+3)}{2\varepsilon}f\right) \le \frac{2\varepsilon}{3(k+3)} \cdot \frac{k+3}{2} = \frac{\varepsilon}{3}.$$

But also

$$\begin{aligned}
0 &= \tau_{(g,x')}(0) \\
&= \tau_{(g,x')}\left(\frac{1}{1+(2\varepsilon/3(k+3))}f + \frac{2\varepsilon/3(k+3)}{1+(2\varepsilon/3(k+3))}\left(-\frac{3(k+3)}{2\varepsilon}f\right)\right) \\
&\le \frac{1}{1+(2\varepsilon/3(k+3))}\tau_{(g,x')}(f) \\
&\quad + \frac{2\varepsilon/3(k+3)}{1+(2\varepsilon/3(k+3))}\tau_{(g,x')}\left(-\frac{3(k+3)}{2\varepsilon}f\right),
\end{aligned}$$

hence

$$0 \le \tau_{(g,x')}(f) + \frac{2\varepsilon}{3(k+3)}\tau_{(g,x')}\left(-\frac{3(k+3)}{2\varepsilon}f\right) \le \tau_{(g,x')}(f) + \frac{\varepsilon}{3}.$$

Then $\tau_{(g,x')}(f) \ge -\varepsilon/3$, hence $|\tau_{(g,x')}(f)| \le \varepsilon/3$. Then we have that

$$|I_{(\Phi,|x'F|)}(h) - I_{(\Phi,|x'F|)}(g)| \le \frac{\varepsilon}{3}$$

for all $x' \in B_1(X')$, and for every pair of functions g and h in the unit ball of $L_\Phi(F)$ such that $\|h-g\|_{(\Phi,F)} \le \varepsilon/3(k+3)$. In consequence,

$$\begin{aligned}\sup_{x' \in B_1(X')} &|I_{(\Phi,|x'F|)}(h) - I_{(\Phi,|x'F|)}(g)| \\ &= \sup_{x' \in B_1(X')} \left| \int_\Omega (\Phi(|h|) - \Phi(|g|))\, d(|x'F|) \right| \\ &\le \frac{\varepsilon}{3} < \frac{\varepsilon}{2}.\end{aligned}$$

Let $A := \{\omega \in \Omega : |h(\omega)| \ge |g(\omega)|\}$. Then,

$$\begin{aligned}\sup_{x' \in B_1(X')} &\int_\Omega |\Phi(|h|) - \Phi(|g|)|\, d(|x'F|) \\ &\le \sup_{x' \in B_1(X')} \int_A (\Phi(|h|) - \Phi(|g|))\, d(|x'F|) \\ &\quad + \sup_{x' \in B_1(X')} \int_{\Omega\setminus A} (\Phi(|g|) - \Phi(|h|))\, d(|x'F|) \\ &= \sup_{x' \in B_1(X')} \left| \int_\Omega (\Phi(|f|\chi_A) - \Phi(|g|\chi_A))\, d(|x'F|) \right| \\ &\quad + \sup_{x' \in B_1(X')} \left| \int_\Omega (\Phi(|g|\chi_{\Omega\setminus A}) - \Phi(|h|\chi_{\Omega\setminus A}))\, d(|x'F|) \right| \\ &< \varepsilon. \qquad \square\end{aligned}$$

Then we can extend this result to the result of [**5**, Lemma 5].

Corollary 14. *Let Φ be an Orlicz function with the Δ_2 property. Then $I_{(\Phi,F)}$ is uniformly continuous in the closed unit ball of $L_\Phi(F)$.*

Proof. For every $\varepsilon > 0$, there is a $\delta > 0$ such that $\|f-g\|_{(\Phi,F)} < \delta$, which implies $|I_{(\Phi,F)}(f) - I_{(\Phi,F)}(g)| \le \sup_{x' \in B_1(X')} |I_{(\Phi,|x'F|)}(f) - I_{(\Phi,|x'F|)}(g)| < \varepsilon$ for every f, g in the unit ball of $L_\Phi(F)$. $\square$

Proposition 15. *Let Φ be an Orlicz function with the Δ_2 property. Then $L_\Phi(F)$ is complete.*

Proof. It is an immediate consequence of Proposition 13 and the comments we had previously to the introduction of the Δ_2 property in this setting, because if we take a Cauchy sequence in $L_\Phi(F)$ without loss of generality we can suppose that it is contained in the unit ball of $L_\Phi(F)$. $\square$

Remark 16. We don't know if the Δ_2 property is necessary for the completeness of $L_\Phi(F)$. Fortunately, if $L_\Phi(F)$ is not complete, the completion of $L_\Phi(F)$ coincides with its closure in $QL_\Phi(F)$, denoted $\overline{L_\Phi(F)}$.

If $f \in L_\Phi(F)$, as the set of simple functions is dense in $L_1(F)$, then there is a Cauchy positive sequence $(S_n)_{n=1}^\infty$ of simple functions which converges to $\Phi(|f|)$ in the topology of $L_1(F)$. Moreover, the convexity of Φ implies that $\Phi(a)+\Phi(b) \leq \Phi(a+b)$ (hence $\Phi(|a-b|) \leq |\Phi(a)-\Phi(b)|$) and that $\Phi^{-1}(a) + \Phi^{-1}(b) \geq \Phi^{-1}(a+b)$ (hence $\Phi^{-1}(|a-b|) \geq |\Phi^{-1}(a) - \Phi^{-1}(b)|$) for every $a, b \geq 0$. Then we have

Proposition 17. *If Φ satisfies the Δ_2 property, the set of simple functions is dense in $L_\Phi(F)$.*

Proof. Let $f \in L_\Phi(F)^+$, and let $(S_n)_{n=1}^\infty$ be a sequence of positive simple functions converging to $\Phi(f)$ in $L_1(F)$. Given $\varepsilon > 0$, suppose that $S_n = \sum_{i=1}^{k(n)} a_i^n \chi_{A_i^n}$ satisfies that $\sup_{x' \in B_1(X')} \int_\Omega |\Phi(f) - S_n| \, d(|x'F|) \leq \varepsilon$. The simple function $R_n := \sum_{i=1}^{k(n)} \Phi^{-1}(a_i^n) \chi_{A_i^n}$ satisfies that $\Phi(R_n(\omega)) = S_n(\omega)$. Then $\sup_{x' \in B_1(X')} \int_\Omega |\Phi(f) - \Phi(R_n)| \, d(|x'F|) \leq \varepsilon$. But $\Phi(|f - R_n|) \leq |\Phi(f) - \Phi(R_n)| = |\Phi(f) - S_n|$. Then $\sup_{x' \in B_1(X')} \int_\Omega \Phi(| \ f - R_n|) \, d(|x'F|) \leq \varepsilon$. Moreover, let n_0 be such that if $n, m \geq n_0$, hence

$$\|S_n - S_m\|_{L_1(F)} = \sum_{i=1}^{k(n)} \sum_{j=1}^{k(m)} |a_i^n - a_j^m| \, \|F\|(A_i^n \cap A_j^m) \leq \varepsilon.$$

Then,

$$\begin{aligned} I_{(\Phi,F)}(R_n - R_m) &= \sum_{i=1}^{k(n)} \sum_{j=1}^{k(m)} \Phi(|\Phi^{-1}(a_i^n) - \Phi^{-1}(a_j^m)|)\|F\|(A_i^n \cap A_j^m) \\ &\leq \sum_{i=1}^{k(n)} \sum_{j=1}^{k(m)} \Phi(\Phi^{-1}(|a_i^n - a_j^m|))\|F\|(A_i^n \cap A_j^m) \leq \varepsilon, \end{aligned}$$

hence $(R_n)_{n=1}^{\infty}$ is a Cauchy sequence of simple functions in $L_\Phi(F)$ which converges to f. Then The set of simple functions is dense in $L_\Phi(F)^+$. The result follows because for every $f \in L_\Phi(F)$, $f = f^+ - f^-$. □

Remark 18. Reasoning as in Remark 5 and Proposition 7, if Φ satisfies the Δ_2 property then the functional $|||.|||_{(\Phi,F)}$ in $L_\Phi(F)$ is such that

$$|||f|||_{(\Phi,F)} := \inf\{k > 0 : |||\Phi(|f|/k)|||_{(1,F)} \leq 1\}$$

is a norm in $L_\Phi(F)$.

Remark 19. Propositions 15 and 17 have been obtained in [**1**] using Köthe function space arguments. Our results complete this information with new facts, showing the role of the Δ_2 property and giving an approximation to the problem using more specific Orlicz theory techniques.

Proposition 20. *Suppose that Φ satisfies the Δ_2 property. Then the norm $|||.|||_{(\Phi,F)}$ in $L_\Phi(F)$ is equivalent to $\|.\|_{(\Phi,F)}$.*

Proof. For every $f \in L_\Phi(F)$, $f \neq 0$,

$$\begin{aligned} &\sup_{B\in\Sigma} \| \int \Phi(|f|/\delta_{(\phi,F)}(f))\, dF\| \\ &\qquad = |||\Phi(|f|/\delta_{(\Phi,F)}(f))|||_{(1,F)} \\ &\qquad \leq \|\Phi(|f|/\delta_{(\phi,F)}(f))\|_{(1,F)} \\ &\qquad = \sup_{x'\in B_1(X')} \int_\Omega \Phi(|f|/\delta_{(\phi,F)}(f))\, d(|x'F|) \end{aligned}$$

$$\leq \sup_{x' \in B_1(X')} \int_\Omega \Phi(|f|/\|f\|_{(\Phi,F)})\, d(|x'F|)$$
$$\leq \sup_{x' \in B_1(X')} \int_\Omega \Phi(|f|/\|f\|_{L_\Phi(|x'F|)}))\, d(|x'F|) = 1,$$

putting $|f|/\|f\|_{L_1(|x'F|)} = 0$ if $\|f\|_{L_1(|x'F|)} = 0$. Then

$$\begin{aligned}\delta_{(\Phi,F)}(f) &\geq \inf\Big\{k>0 : \sup_{B\in\Sigma} \|\int_B \Phi(|f|/k)dF\| \leq 1\Big\}\\ &= \inf\{k>0 : |||\Phi(|f|/k)|||_{(1,F)} \leq 1\}\\ &\geq \inf\Big\{k>0 : \Big\|\frac{1}{2}\Phi(|f|/k)\Big\|_{(1,F)} \leq 1\Big\}.\end{aligned}$$

But as Φ satisfies the Δ_2 property, there is a $\delta > 0$ such that $\Phi((1+\delta)x)/2 \leq \Phi(x)$ for every $x > 0$, see [**9**, Theorem 2.3.3]. Then

$$\begin{aligned}\inf\Big\{k>0 : \Big\|\frac{1}{2}\Phi(|f|/k)\Big\|_{(1,F)}\Big\}\\ &= \inf\Big\{k>0 : \Big\|\frac{1}{2}\Phi\Big(\frac{(1+\delta)|f|}{(1+\delta)k}\Big)\Big\|_{(1,F)}\Big\}\\ &= \frac{1}{1+\delta}\inf\Big\{s>0 : \Big\|\frac{1}{2}\Phi((1+\delta)|f|/s)\Big\| \leq 1\Big\}\\ &\geq \frac{1}{1+\delta}\inf\{s>0 : \|\Phi(|f|/s)\|_{(1,F)} \leq 1\}\\ &= \frac{1}{1+\delta}\delta_{(\phi,F)}(f).\end{aligned}$$

Then $\delta_{(\phi,F)}(f) \geq |||f|||_{(\Phi,F)} \geq 1/(1+\delta)\delta_{(\phi,F)}(f)$, and the result follows from Proposition 7. □

The representation of many of the more common Banach lattices as suitable spaces of functions has been successful. We mention mainly Kakutani's representation of abstract L_p and M-spaces by spaces of functions $L_p(\mu)$ and $C(K)$, respectively. Kakutani's method has been adapted to other situations, for example in the representation of the order continuous Banach lattices with weak order unit, see [**8**, 1.b].

Theorem 21. *Suppose that Φ satisfies the Δ_2 property. Then $L_\Phi(F)$ is an order continuous Banach lattice with weak order unit.*

Proof. Let $(f_n)_{n=1}^\infty$ be an increasing positive sequence in $L_\Phi(F)$ such that there is a $g \in L_\Phi(F)$ with $f_n \le g$. From [**6**, Corollary II.4.2] as $(\Phi(f_n))_{n=1}^\infty$ is a monotone sequence in $L_1(F)$ bounded by $\Phi(g) \in L_1(F)$, then $(\Phi(f_n))_{n=1}^\infty$ converges $\|F\|$-almost everywhere and in the $L_1(F)$-norm, to a function $h \in L_1(F)$. If we define $f := \Phi^{-1}(h)$, we have

$$\begin{aligned}\sup_{x'\in B_1(X')} \int_\Omega |\Phi(f_n) - h|\, d(|x'F|) \\ &= \sup_{x'\in B_1(X')} \int_\Omega |\Phi(f_n) - \Phi(f)|\, d(|x'F|) \\ &\ge \sup_{x'\in B_1(X')} \int_\Omega \Phi(|f_n - f|)\, d(|x'F|).\end{aligned}$$

Then $(f_n)_{n=1}^\infty$ norm-converges to f in $L_\Phi(F)$. As a consequence from the characterization of the order continuity in Banach lattices, see [**11**, II.5.12] or [**8**, Proposition 1.a.8] $L_\Phi(F)$ has order continuous norm.

Moreover, it is clear that χ_Ω is a weak order unit in $L_\Phi(F)$. $\square$

Using the result of [**8**, Theorem 1.b.14], we have the following representation.

Corollary 22. *Suppose that Φ satisfies the Δ_2 property. There is a probability measure space $(\mathcal{O}, \mathcal{S}, \nu)$, an ideal $\mathcal{I}$ of $L_1(\nu)$ and a lattice norm $\nu_\Phi(.)$ in $\mathcal{I}$ such that $L_\Phi(F)$ is order isometric to $(\mathcal{I}, \nu_\Phi)$.*

From the Bartle-Dunford-Schwartz theorem [**2**, Corollary I.2.6], there is a positive finite measure λ_F on (Ω, Σ) such that $\lambda_F(A) \le \|F\|(A)$ for all $A \in \Sigma$ and that $\lambda_F(A) = 0$ implies $\|F\|(A) = 0$.

Definition 23. A Köthe function space X with respect to the finite measure space $(\Omega, \Sigma, \lambda)$ is a Banach space of (λ-almost everywhere classes) of λ-integrable real functions such that it contains the characteristic functions of the sets of Σ, and if f is $\lambda-$measurable and $g \in X$ such that $|f| \le |g|$ then $f \in X$ with $\|f\| \le \|g\|$.

With the measure λ_F and the former definition we have the following.

Proposition 24. *Suppose that Φ satisfies the Δ_2 property. Then $L_\Phi(F)$ is a Köthe function space with respect to $(\Omega, \Sigma, \lambda_F)$.*

Proof. If f is λ_F-measurable and $g \in L_\Phi(F)$ is such that $|f| \leq |g|$, as $\Phi(|f|) \leq \Phi(|g|)$ and $L_1(F)$ is a Köthe space with respect to $(\Omega, \Sigma, \lambda_F)$, then $\Phi(|f|) \in L_1(F)$; hence, $f \in L_\Phi(F)$. □

4. Ultraproducts. The results obtained in the previous sections can be used to analyze some new aspects of the spaces $L_\Phi(F)$, which can be considered as extensions of the scalar case. An example is the construction of ultraproducts, a well-known topic in Orlicz spaces of measurable functions with respect to scalar measures. Our motivation is that this extension is not a straightforward modification of known things, because we have to start studying ultraproducts of vector-valued measures.

Concerning ultraproducts of Banach spaces, the standard paper is [**4**], and we refer to it for concrete definitions. We only set the notation we will use. Let D be a nonempty index set and $\mathcal{U}$ an ultrafilter in D. Given a family $\{X_d,\ d \in D\}$ of Banach spaces, $(X_d)_\mathcal{U}$ denotes the corresponding ultraproduct Banach space. We recall that $(X'_d)_\mathcal{U}$ is contained in $((X_d)_\mathcal{U})'$ as an isometric subspace, and that $(X_d)_\mathcal{U}$ is reflexive if and only if $((X_d)_\mathcal{U})' = (X'_d)_\mathcal{U}$.

Let $\{A_d, d \in D\}$ be a family of sets. We denote by $(A_d)_\mathcal{U}$ the set of all classes of $\Pi_{d\in D} A_d$, under the equivalence

$$(a_d)_{d\in D} \mathcal{R} (b_d)_{d\in D} \longleftrightarrow \{d \in D : a_d = b_d\} \in \mathcal{U}.$$

Let $\{(\Omega_d, \Sigma_d), d \in D\}$ be a family of measure spaces. We define the Boolean algebra

$$\Sigma_{\mathcal{U},0} := \{(A_d)_\mathcal{U}, A_d \in \Sigma_d, d \in D\},$$

and we denote by $\Sigma_\mathcal{U}$ the σ-algebra generated by $\Sigma_{\mathcal{U},0}$.

Let $\{X_d, d \in D\}$ be a family of Banach spaces, and let $\{F_d, d \in D\}$ be a family of countably additive vector-valued measures $F_d : \Sigma_d \to X_d$ for all $d \in D$, such that $\sup_{d\in D} \|F_d\|(\Omega_d) < \infty$.

Definition 25. We define the ultraproduct vector measure $F_0 : \Sigma_{\mathcal{U},0} \to (X_d)_{\mathcal{U}}$ such that

$$F_0((A_d)_{\mathcal{U}}) := (F_d(A_d))_{\mathcal{U}}.$$

Obviously, the hypothesis $\sup_{d\in D} \|F_d\|(\Omega_d) < \infty$ implies that F_0 has finite semi-variation in $\Sigma_{\mathcal{U},0}$.

Theorem 26. *If $(X_d)_{\mathcal{U}}$ does not contain copies of c_0, then the ultraproduct vector measure $(F_d)_{\mathcal{U}} : \Sigma_{\mathcal{U},0} \to (X_d)_{\mathcal{U}}$ can be uniquely extended to a countably additive measure $F : \Sigma_{\mathcal{U}} \to (X_d)_{\mathcal{U}}$.*

Proof.

Step 1. First we will see that $|(x'_d)_{\mathcal{U}} F_0| \le (|x'_d F_d|)_{\mathcal{U}}$ in $\Sigma_{\mathcal{U},0}$, for every $(x'_d)_{\mathcal{U}} \in (X'_d)_{\mathcal{U}}$. Let $A := (A_d)_{\mathcal{U}}$ be a subset of $\Sigma_{\mathcal{U},0}$, and let $\{E^1, E^2, \dots, E^n\} \subset \Sigma_{\mathcal{U},0}$ be a partition of A in $\Sigma_{\mathcal{U},0}$. Without lost of generality, we can suppose that if $E^i = (E^i_d)_{\mathcal{U}}$, then $E^i_d \cap E^j_d = \varnothing$ if $i \ne j$, $i,j = 1,\dots,n$, for all $d \in D$; hence, $\{E^1_d, \dots, E^n_d\} \subset \Sigma_d$ is a partition of A_d for every $d \in D$. Then we have

$$\begin{aligned}\sum_{i=1}^n |\langle F_0(E^i), (x'_d)_{\mathcal{U}}\rangle| &\le \lim_{\mathcal{U}} \sum_{i=1}^n |\langle F_d(E^i_d), x'_d\rangle| \\ &\le \lim_{\mathcal{U}} |x'_d F_d|(A_d) = (|x'_d F_d|)_{\mathcal{U}}(A).\end{aligned}$$

Hence, $(|(x'_d)_{\mathcal{U}} F_0|(A) \le (|x'_d F_d|)_{\mathcal{U}}(A)$.

Step 2. For every $x' \in ((X_d)_{\mathcal{U}})'$, $|x' F_0|$ is strongly additive in $\Sigma_{\mathcal{U},0}$. In fact, from the ultraproduct version of local duality, $((X_d)_{\mathcal{U}})'$ is finitely representable in $(X'_d)_{\mathcal{U}}$, hence there are an index set S and an untrafilter $\mathcal{S}$ on S such that $((X_d)_{\mathcal{U}})'$ is isometric to a subspace of $(X'_{(d,s)})_{\mathcal{U}\times\mathcal{S}}$, where for every $d \in D$, $X_{(d,s)} = X_d$, for all $s \in S$. Moreover, this subspace is norm-1 complemented. We denote by

$$I : (X_d)_{\mathcal{U}} \hookrightarrow (X_{(d,s)})_{\mathcal{U}\times\mathcal{S}}$$

the canonical isometric embedding, and by

$$J : ((X_d)_{\mathcal{U}})' \longrightarrow (X'_{(d,s)})_{\mathcal{U}\times\mathcal{S}}$$

the above-mentioned isometric mapping. If $(x_d)_{\mathcal{U}} \in (X_d)_{\mathcal{U}}$ and $x' \in ((X_d)_{\mathcal{U}})'$ with $J(x') = (x'_{(d,s)})_{\mathcal{U}\times\mathcal{S}}$, then

$$\langle (x_d)_{\mathcal{U}}, x' \rangle = \langle I((x_d)_{\mathcal{U}}), J(x') \rangle = \lim_{\mathcal{U}\times\mathcal{S}} \langle x_{(d,s)}, x'_{(d,s)} \rangle,$$

where $x_{(d,s)} = x_d$, for all $s \in S$.

We can identify F_0 with $(F_{(d,s)})_{\mathcal{U}\times\mathcal{S}}$, where for every $d \in D$, $F_{(d,s)} = F_d$, for all $s \in S$. Hence using Step 1, $|x'F_0| \leq ((|x'_{(d,s)}F_{(d,s)}|)_{\mathcal{U}\times\mathcal{S}})$. The measure $(|x'_{(d,s)}F_{(d,s)}|)_{\mathcal{U}\times\mathcal{S}}$ is strongly additive in $\Sigma_{0,\mathcal{U}\times\mathcal{S}}$, because it is the ultraproduct of strongly additive positive measures with $\sup_{(d,s)\in D\times S} |x'_{(d,s)}F_{(d,s)}| \leq \sup_{(d,s)\in D\times S} \|x'_{(d,s)}\| \ \|F_{(d,s)}\|(\Omega_{(d,s)}) < \infty$, where for every $d \in D$, $\Omega_{(d,s)} = \Omega_d$ and $\Sigma_{(d,s)} = \Sigma_d$, for all $s \in S$. Then $|x'F_0|$ is strongly additive in $\Sigma_{\mathcal{U},0}$.

Step 3. We will see that F_0 is strongly additive in $\Sigma_{\mathcal{U},0}$. Let $\{A^n, n \in \mathbf{N}\}$ be a disjoint sequence in $\Sigma_{\mathcal{U},0}$. As $|x'F_0|$ is strongly additive in $\Sigma_{\mathcal{U},0}$ for every $x' \in ((X_d)_{\mathcal{U}})'$, then $\sum_n |\langle x', F_0(A^n)\rangle| \leq \sum_{n=1}^{\infty} |x'F_0|(A^n) < \infty$. Hence, from the Bessaga-Pełczyński's characterization of the Banach spaces without copies of c_0, the series $\sum_{n=1}^{\infty} F_0(A^n)$ converges in $(X_d)_{\mathcal{U}}$.

Then from the Carathéodory-Hahn-Kluvánek extension theorem, see [**2**, Theorem I-5-2], F_0 has a unique countably additive extension to $\Sigma_{\mathcal{U}}$. □

We denote this extension by F. Our aim is the study of the relationships between $L_\Phi(F)$ and $(L_\Phi(F_d))_{\mathcal{U}}$.

Theorem 27. *Let Φ be an Orlicz function with the Δ_2 property. Suppose that $(X_d)_{\mathcal{U}}$ is reflexive. Then $L_\Phi(F)$ is isomorphic to a subspace of $(L_\Phi(F_d))_{\mathcal{U}}$ via a positive isomorphism.*

Proof. The space of the simple functions $f = \sum_{n=1}^{m} c_n \chi_{A^n}$, $A^n = (A_d^n)_{\mathcal{U}} \in \Sigma_{\mathcal{U},0}$ for $n = 1, \dots, m$, is dense in $L_\Phi(F)$. We put $f_d := \sum_{n=1}^{m} c_n \chi_{A_d^n}$. Then we define the mapping between the subspace of the simple functions of $\Sigma_{\mathcal{U},0}$ in $L_\Phi(F)$ and $(L_\Phi(F_d))_{\mathcal{U}}$ such that

$$T(f) := (f_d)_{\mathcal{U}} = \left(\sum_{n=1}^{m} c_n \chi_{A_d^n} \right)_{\mathcal{U}} \in (L_\Phi(F_d))_{\mathcal{U}}.$$

Suppose that f is positive. For every $x' = (x'_d)_{\mathcal{U}} \in (X'_d)_{\mathcal{U}} = ((X_d)_{\mathcal{U}})'$ and for every $B = (B_d)_{\mathcal{U}} \in \Sigma_{\mathcal{U},0}$,

$$\begin{aligned}
\left\langle \int_B \Phi(f)\, dF, x' \right\rangle &= \int_B \Phi(f)\, d(x'F) \\
&= \sum_{n=1}^{m} \Phi(c_n)(x'F)(A^n \cap B) \\
&= \left(\sum_{n=1}^{m} \Phi(c_n)(x'F_d)(A_d^n \cap B_d) \right)_{\mathcal{U}} \\
&= \left(\int_{B_d} \sum_{n=1}^{m} \Phi(c_n) \chi_{A_d^n}\, d(x'_d F_d) \right)_{\mathcal{U}} \\
&= \left(\left\langle \int_{B_d} \Phi(f_d)\, dF_d, x'_d \right\rangle \right)_{\mathcal{U}} \\
&= \left\langle \left(\int_{B_d} \Phi(f_d)\, dF_d \right)_{\mathcal{U}}, x' \right\rangle.
\end{aligned}$$

Then $\int_B \Phi(f)\, dF = (\int_{B_d} \Phi(f_d)\, dF_d)_{\mathcal{U}}$.

For every $\varepsilon > 0$, as $|||\Phi(f_d)|||_{(1,F_d)} = \sup_{A \in \Sigma_d} \| \int_A \Phi(f_d)\, dF_d \|$, for every $d \in D$, there is an $A_d \in \Sigma_d$ such that

$$|||\Phi(f_d)|||_{(1,F_d)} \le \left\| \int_{A_d} \Phi(f_d) dF_d \right\| + \varepsilon.$$

Then

$$\begin{aligned}
\|(\Phi(f_d))_{\mathcal{U}}\|_{((L_1(F_d),|||\cdot|||))_{\mathcal{U}}} &\le \left\| \left(\int_{A_d} \Phi(f_d), dF_d \right)_{\mathcal{U}} \right\| + \varepsilon \\
&= \left\| \int_{(A_d)_{\mathcal{U}}} \Phi(f) dF \right\| + \varepsilon \\
&\le |||\Phi(f)|||_{(1,F)} + \varepsilon;
\end{aligned}$$

hence, $\|(\Phi(f_d))_{\mathcal{U}}\|_{((L_1(F_d),|||\cdot|||))_{\mathcal{U}}} \le |||\Phi(f)|||_{(1,F)}$. As

$$|||\Phi(f/|||f|||_{(\Phi,F)})|||_{(1,F)} \le 1,$$

then

$$\|(\Phi(f_d/|||f|||_{(\Phi,F)}))_{\mathcal{U}}\|_{((L_1(F_d),|||\cdot|||))_{\mathcal{U}}} \le 1,$$

hence given $\varepsilon > 0$, there is a $D \in \mathcal{U}$ such that for every $d \in D$, $|||\Phi(f_d/|||f|||_{(\Phi,F)})|||_{(1,F_d)} \leq 1+\varepsilon$. Hence,

$$\left|\left|\left|\frac{1}{1+\varepsilon}\Phi(f_d/|||f|||_{(\Phi,F)})\right|\right|\right|_{(1,F_d)} \leq 1,$$

and also

$$|||\Phi(f_d/((1+\varepsilon)|||f|||_{(\Phi,F)}))|||_{(1,F_d)} \leq 1.$$

Then

$$\|(f_d)_{\mathcal{U}}\|_{((L_\Phi(F_d),|||\cdot|||_{(\Phi,F_d)}))_{\mathcal{U}}} \leq |||f|||_{(\Phi,F)}.$$

But $\|\Phi(f)\|_{(1,F)} = \sup_{x'\in B_1(((X_d)_{\mathcal{U}})')} \int_\Omega \Phi(f)d(|x'F|)$. Then, given $\varepsilon > 0$, there is an $x' \in B_1(((X_d)_{\mathcal{U}})')$, $x' = (x'_d)_{\mathcal{U}}$, such that

$$\begin{aligned}
\|\Phi(f)\|_{(1,F)} &\leq \int_\Omega \Phi(f)\,d(|x'F|) + \varepsilon \\
&\leq \int_{(\Omega_d)_{\mathcal{U}}} \Phi(f)\,d((|x'_dF_d|)_{\mathcal{U}}) + \varepsilon \\
&= \sum_{n=1}^{m} \Phi(c_n)(|x'_dF_d|)_{\mathcal{U}}(A^n) + \varepsilon \\
&= \left(\sum_{n=1}^{m} \Phi(c_n)\ \ |x'_dF_d|(A^n_d)\right)_{\mathcal{U}} + \varepsilon \\
&= \lim_{\mathcal{U}} \int_{\Omega_d} \Phi(f_d)\,d(|x'_dF_d|) + \varepsilon.
\end{aligned}$$

As $x' \in B_1(((X_d)_{\mathcal{U}})')$, there is a $D_0 \in \mathcal{U}$ such that $\|x'_d\| \leq 1+\varepsilon$ for every $d \in D_0$. Then $x'_d/(1+\varepsilon) \in B_1(X'_d)$ for every $d \in D_0$. Using this fact, we have

$$\begin{aligned}
\|\Phi(f)\|_{(1,F)} &\leq \lim_{\mathcal{U}} \int_{\Omega_d} \Phi(f_d)\,d(|x'_dF_d|) + \varepsilon \\
&\leq (1+\varepsilon)\|(\Phi(f_d))_{\mathcal{U}}\|_{((L_1(F_d),\|\cdot\|))_{\mathcal{U}}} + \varepsilon.
\end{aligned}$$

Hence, $\|\Phi(f)\|_{(1,F)} \leq \|(\Phi(f_d))_{\mathcal{U}}\|_{((L_1(F_d),\|\cdot\|))_{\mathcal{U}}}$.

Suppose that $\delta = (\delta_{(\Phi,F_d)}(f_d))_{\mathcal{U}} = \|(f_d)_{\mathcal{U}}\|_{((L_\Phi(F_d),\|\cdot\|_{(\Phi,F_d)}))}$. Given $\varepsilon > 0$, there is a $D \in \mathcal{U}$ such that $\delta_{(\Phi,F_d)}(f_d) \leq \delta + \varepsilon$. Then, for every

$d \in D$, $\|\Phi(f_d/(\delta+\varepsilon))\|_{(1,F_d)} \leq 1$, hence $\|\Phi(f/(\delta+\varepsilon))\|_{(1,F)} \leq 1$, and then $\delta_{(\Phi,F)}(f) \leq \delta+\varepsilon$. As ε is arbitrary, $\delta_{(\Phi,F)}(f) \leq \delta$, or $\|f\|_{(\Phi,F)} \leq \|(f_d)_{\mathcal{U}}\|_{((L_\Phi(F_d),\|\cdot\|_{(\Phi,F_d)}))}$.

For general step functions, the same result follows by putting $f = f^+ - f^-$.

Then T can be extended to $L_\Phi(F)$ and the extension $\widehat{T}$ is injective; hence, $\widehat{T}(L_\Phi(F))$ is a subspace of $(L_\Phi(F_d))_{\mathcal{U}}$ which is isomorphic to $L_\Phi(F)$. Moreover, for construction T is positive; then $\widehat{T}$ satisfies the same property. This concludes the proof. □

Remark 28. From the former proof it is clear that if $(X_d)_{\mathcal{U}}$ is not reflexive then $L_\Phi(F)$ is isomorphic to a subspace of $(L_\Phi(F_{d,s}))_{\mathcal{U}\times\mathcal{S}}$ via a positive isomorphism, where for every $d \in D$, $F_{d,s} = F_d$ for every $s \in S$.

REFERENCES

1. O. Delgado, *Banach function spaces of L_1 of a vector measure and related Orlicz spaces*, Indag. Math. (new series) **15** (2004), 485–495.

2. J. Diestel and J.J. Uhl, *Vector measures*, Math Surveys **15**, American Mathematical Society, Providence, 1977.

3. R. Haydom, M. Levy and Y. Raynaud, *Randomly normed spaces*, Hermann, Paris 1991.

4. S. Heinrich, *Ultraproducts in Banach spaces theory*, J. reine Angew. Math. **313** (1980), 72–104.

5. A. Kaminska, *On uniform convexity in Orlicz spaces*, Indag. Math. **44** (1982), 27–38.

6. I. Kluvánek and G. Knowles, *Vector measures and control systems*, North Holland Math. Studies **20**, North Holland, Amsterdam, 1975.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces* I, Springer Verlag, Berlin, 1977.

8. ———, *Classical Banach spaces* II, Springer Verlag, Berlin, 1979.

9. M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, Inc., New York, 1991.

10. E. Sánchez Pérez, *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces*, Illinois J. Math. **45** (2001), 907–923.

11. H.H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, 1974.

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