ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 2, 2008

ON A RIEMANNIAN INVARIANT OF CHEN TYPE

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ABSTRACT. In [**6**] we proved Chen's inequality regarded as a problem of constrained maximum. In this paper we introduce a Riemannian invariant obtained from Chen's invariant, replacing the sectional curvature by the Ricci curvature of k-order. This invariant can be estimated, in the case of submanifolds M in space forms $\widetilde{M}(c)$, varying with c and the mean curvature of M in $\widetilde{M}(c)$.

1. Introduction. We consider a Riemannian manifold (M, g) of dimension n, and we fix the point $x \in M$. The scalar curvature is defined by

$$\tau = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j),$$

where R is the Riemann curvature tensor of (M, g) and $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame in T_xM.

Let L be a vector subspace of dimension $k \in [2, n]$ in T_xM. If $X \in L$ is a unit vector, and $\{e'_1, e'_2, \dots, e'_k\}$ is an orthonormal frame in L, with $e'_1 = X$, we shall denote

$$\mathrm{Ric}_L(X) = \sum_{j=2}^{k} k(e'_1 \wedge e'_j),$$

where $k(e'_1 \wedge e'_j)$ is the sectional curvature given by $\mathrm{Sp}\,\{e'_1, e'_j\}$.

Using the Ricci curvature of k-order at the point $x \in M$,

$$\theta_k(x) = \frac{1}{k-1} \min_{\substack{L,\ \dim L = k \\ X \in L,\ \|X\| = 1}} \mathrm{Ric}_L(X),$$

we define the invariant

$$\delta_k(M) : M \to R,$$
$$\delta_k(M) = \tau - \theta_k.$$

Received by the editors on November 28, 2005.
DOI:10.1216/RMJ-2008-38-2-567

For $k = 2$ we have $\delta_k(M) = \tau - \min(K) = \delta_M$, where K is the sectional curvature and δ_M is the Chen's invariant.

2. Optimizations on Riemannian manifolds. Let $(N, \widetilde{g})$ be a Riemannian manifold, let (M, g) be a Riemannian submanifold of N, and let $f : N \to R$ a differentiable function. To these ingredients we attach the optimum problem

$$\min_{x \in M} f(x). \tag{2.1}$$

Let's remember the result obtained in [**6**].

Theorem 2.1. *If $x_0 \in M$ is an optimal solution of the problem* (2.1)*, then*

i) $(\operatorname{grad} f)(x_0) \in T^{\perp}_{x_0} M$,

ii) *the bilinear form*

$$\alpha : T_{x_0}M \times T_{x_0}M \longrightarrow R,$$
$$\alpha(X,Y) = \mathrm{Hess}_f(X,Y) + \widetilde{g}(h(X,Y), (\text{grad f})(x_0))$$

is positive semi-definite, where h is the second fundamental form of the submanifold M in N.

Remark. The bilinear form α is nothing else but $\mathrm{Hess}_{f|M}(x_0)$.

3. The inequality satisfied by the Riemannian invariant $\delta_k(M)$. Chen showed in [**1**] that Chen's invariant δ_M of a Riemannian submanifold in a real space form $\widetilde{M}(c)$ satisfies the inequality

$$\delta_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\,\|H\|^2 + (n+1)c\right\},$$

where H is the mean curvature vector of submanifold M in $\widetilde{M}(c)$ and $n \geq 3$ is the dimension of M. The equality is attained at the point $x \in M$ if and only if there is an orthonormal frame $\{e_1, \ldots, e_n\}$ in

T_xM and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ in which the Weingarten operators take the following form

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdot & 0 \\ 0 & h_{22}^{n+1} & 0 & \cdot & 0 \\ 0 & 0 & h_{33}^{n+1} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & h_{nn}^{n+1} \end{pmatrix},$$

with $h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \dots = h_{nn}^{n+1}$ and

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdot & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 \end{pmatrix}, \text{ for all } r \in [n+2, m].$$

The invariant $\delta_k(M)$ satisfies the same inequality. Indeed, obviously one has $\min(K) \leq \theta_k$, which implies $\delta_k(M) \leq \delta_M$. Therefore

$$\delta_k(M) \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right\}.$$

We give another proof of this inequality for two reasons: to obtain the equality case and because this proof is useful in order to obtain a stronger inequality in Lagrangian case.

Theorem 3.1. *Consider $(\widetilde{M}(c), \widetilde{g})$ a real space form of dimension m, $M \subset \widetilde{M}(c)$ a submanifold of dimension $n \geq 3$, and $k \in [3, n]$. Then*

$$\delta_k(M) \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right\},$$

the equality occurring at the point x if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in T_xM and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ for which the Weingarten operators take the form

$$A_r = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & a^r & 0 & \cdot & 0 \\ 0 & 0 & a^r & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a^r \end{pmatrix}, \textit{ for all } r \in [n+1, m].$$

Proof. Let us consider the point $x \in M$, $\{e_1, e_2, \dots, e_n\}$ an orthonormal frame in T_xM and $\{e_{n+1}, e_{n+2}, \dots, e_m\}$ an orthonormal frame in $T_x^{\perp}M$.

If $L = \mathrm{Sp}\,\{e_1, e_2, \dots, e_k\}$, then

$$\mathrm{Ric}_L(e_1) = \sum_{i=2}^{k} R(e_1, e_i, e_1, e_i). \tag{3.1}$$

From Gauss's equation we obtain the following relations

$$\tau = \frac{n(n-1)}{2}c + \sum_{r=n+1}^{m} \sum_{1 \le i < j \le n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \tag{3.2}$$

and

$$(k-1)c = \sum_{i=2}^{k} R(e_1, e_i, e_1, e_i) - \sum_{r=n+1}^{m} \sum_{i=2}^{k} (h_{11}^r h_{ii}^r - (h_{1i}^r)^2). \tag{3.3}$$

From (3.1) and (3.3), it follows

$$\frac{\mathrm{Ric}_L(e_1)}{k-1} = c + \frac{1}{k-1} \sum_{r=n+1}^{m} \sum_{i=2}^{k} (h_{11}^r h_{ii}^r - (h_{1i}^r)^2). \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$\begin{aligned} \tau - \frac{\mathrm{Ric}_L(e_1)}{k-1} &= \frac{(n+1)(n-2)}{2}c + \sum_{r=n+1}^{m} \sum_{1 \le i < j \le n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \\ &\quad - \frac{1}{k-1} \sum_{r=n+1}^{m} \sum_{i=2}^{k} (h_{11}^r h_{ii}^r - (h_{1i}^r)^2) \\ &= \frac{(n+1)(n-2)}{2}c \end{aligned} \tag{3.5}$$

$$+\sum_{r=n+1}^{m}\left(\sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{1}{k-1}h_{11}^r\sum_{i=2}^{k} h_{ii}^r\right)$$
$$-\frac{k-2}{k}\sum_{i=2}^{k}(h_{1i}^r)^2 - \sum_{i=k+1}^{n}(h_{1i}^r)^2 - \sum_{2\le i<j\le n}(h_{ij}^r)^2.$$

As $k \geq 3$, by using (3.5), one gets

$$\tau - \frac{\mathrm{Ric}_L(e_1)}{k-1} \le \frac{(n+1)(n-2)}{2}c + \sum_{r=n+1}^{m}\left(\sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{1}{k-1}h_{11}^r\sum_{i=2}^{k} h_{ii}^r\right). \tag{3.6}$$

For $r \in [n+1, m]$, let us consider the quadratic form

$$f_r : R^n \to R,$$
$$f_r(h_{11}^r, h_{22}^r, \dots, h_{nn}^r) = \sum_{1\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{1}{k-1}h_{11}^r\sum_{i=2}^{k} h_{ii}^r \tag{3.7}$$

and the constrained extremum problem

$$\max f_r \quad \text{subject to} \quad P\colon\ h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r, \tag{3.8}$$

where k^r is a real constant.

The partial derivatives of the function f_r are

$$\frac{\partial f_r}{\partial h_{11}^r} = \sum_{i=2}^{n} h_{ii}^r - \frac{1}{k-1}\sum_{i=2}^{k} h_{ii}^r, \tag{3.9}$$

$$\frac{\partial f_r}{\partial h_{jj}^r} = \sum_{i\in\overline{1,n}\setminus\{j\}} h_{ii}^r - \frac{1}{k-1}h_{11}^r, \quad j \in [2,k], \tag{3.10}$$

$$\frac{\partial f_r}{\partial h_{ll}^r} = \sum_{i\in\overline{1,n}\setminus\{l\}} h_{ii}^r, \quad l \in [k+1,n]. \tag{3.11}$$

For an optimal solution $(h^r_{11}, h^r_{22}, \dots, h^r_{nn})$ of the problem in question, the vector $(\mathrm{grad})\,(f_1)$ is normal at P, that is, it is collinear with the vector $(1, 1, \dots, 1)$.

From (3.9), (3.10), (3.11), it follows that a critical point of the considered problem has the form

$$(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) = (0, a^r, a^r, \dots, a^r). \tag{3.12}$$

As $\sum_{j=1}^n h^r_{jj} = k^r$, by using (3.12), we obtain $(n-1)a^r = k^r$, therefore

$$a^r = \frac{k^r}{n-1}. \tag{3.13}$$

Let $p \in P$ be an arbitrary point.

The 2-form $\alpha : T_pP \times T_pP \to R$ has the expression

$$\alpha(X, Y) = \mathrm{Hess}_{f_r}(X, Y) + \langle h'(X, Y), (\mathrm{grad}\,\mathrm{f_r})(p)\rangle\,,$$

where h' is the second fundamental form of P in R^n and $\langle\ ,\ \rangle$ is the standard inner-product of R^n.

In the standard frame of R^n, the Hessian of f_r has the matrix

$$\begin{pmatrix}
0 & (k-2/k-1) & (k-2/k-1) & \cdot & (k-2/k-1) & 1 & \cdot & 1 \\
(k-2/k-1) & 0 & 1 & \cdot & 1 & 1 & \cdot & 1 \\
(k-2/k-1) & 1 & 0 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(k-2/k-1) & 1 & 1 & \cdot & 0 & 1 & \cdot & 1 \\
1 & 1 & 1 & \cdot & 1 & 0 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 0
\end{pmatrix}.$$

As P is totally geodesic in R^n, considering a vector X tangent at p to P, that is, verifying the relation $\sum_{i=1}^n X^i = 0$, we have

$$\begin{aligned}
\alpha(X, X) = -\frac{1}{k-1}\Big[&(X^1 + X^2)^2 + (X^1 + X^3)^2 + \cdots \\
&+ (X^1 + X^k)^2 + (k-2)(X^2)^2 + (k-2)(X^3)^2 + \cdots \\
&+ (k-2)(X^k)^2 + (k-1)\sum_{i=k+1}^n (X^i)^2\Big] \le 0.
\end{aligned}$$

So $\mathrm{Hess}_{f|P}$ is negative definite.

Consequently $(0, a^r, \dots, a^r)$, with $a^r = k^r/(n-1)$, is a global maximum point; therefore,

(3.14)
$$f_r \leq \frac{(n-1)(n-2)}{2}(a^r)^2 = \frac{(n-2)}{2(n-1)}(k^r)^2 = \frac{(n-2)}{2(n-1)}n^2(H^r)^2.$$

From (3.6) and (3.14), it follows that

$$\begin{aligned}\tau - \mathrm{Ric}_L(e_1)/k - 1 &\leq [(n+1)(n-2)/2]c \\ &\quad + \sum_{r=n+1}^{m} [(n-2)/2(n-1)]n^2(H^r)^2 \\ &= [(n+1)(n-2)/2]c + [n^2(n-2)/2(n-1)]\,\|H\|^2 \\ &= (n-2/2)\{(n^2/n-1)\,\|H\|^2 + (n+1)c\};\end{aligned}$$

therefore,

$$\delta_k(M) \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\,\|H\|^2 + (n+1)c\right\}. \tag{3.15}$$

In (3.15) we have equality if and only if the same thing occurs in the inequality (3.6) and, in addition, (3.12) occurs. Therefore, in (3.15) we have equality if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in T_xM and an orthonormal frame $\{e_{n+1}, \dots, e_m\}$ in $T_x^\perp M$ for which the Weingarten operators have the following form

$$A_r = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & a^r & 0 & \cdot & 0 \\ 0 & 0 & a^r & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a^r \end{pmatrix}, \text{ for all } r \in [n+1, m].$$

4. The Lagrangian case. Let $(\widetilde{M}, \widetilde{g}, J)$ be a Kähler manifold of real dimension $2m$. A submanifold M of dimension n of $(\widetilde{M}, \widetilde{g}, J)$ is called a totally real submanifold if for any point x in M the relation $J(T_xM) \subset T_x^\perp M$ holds.

If, in addition, $n = m$, then M is called Lagrangian submanifold. For a Lagrangian submanifold, the relation $J(T_xM) = T_x^\perp M$ occurs.

A Kähler manifold with constant holomorphic sectional curvature is called a complex space form and is denoted by $\widetilde{M}(c)$. The Riemann curvature tensor $\widetilde{R}$ of $\widetilde{M}(c)$ satisfies the relation

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c}{4}\{&\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y + \widetilde{g}(JY,Z)JX - \widetilde{g}(JX,Z)JY \\ &+ 2\widetilde{g}(X,JY)JZ\}.\end{aligned}$$

Remark. i) If M is a totally real submanifold of real dimension n in a complex space form $\widetilde{M}(c)$ of real dimension $2m$, then

$$A_{JY}X = -Jh(X,Y) = A_{JX}Y,$$

where X and Y are two arbitrary vector fields on M.

ii) Let $m = n$ (M is Lagrangian in $\widetilde{M}(c)$). If we consider the point $x \in M$, the orthonormal frames $\{e_1, \dots, e_n\}$ in T_xM and $\{Je_1, \dots, Je_n\}$ in $T_x^{\perp}M$, then

$$h^i_{jk} = h^j_{ik}, \text{ for all } i, j, k \in [1, n],$$

where h^i_{jk} is the component after Je_i of the vector $h(e_j, e_k)$.

Theorem 4.1. *Let M be a totally real submanifold of dimension n, $n \geq 3$, in complex space form $\widetilde{M}(c)$ of real dimension $2m$. Then*

$$\delta_k(M) \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1)\frac{c}{4}\right\},$$

the equality occurring if and only if there is an orthonormal frame $\{e_1, \dots, e_n\}$ in T_xM and an orthonormal frame $\{e_{n+1}, \dots, e_{2m}\}$ in $T_x^{\perp}M$ for which the Weingarten operators take the form

$$A_r = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 \\ 0 & a^r & 0 & \cdot & 0 \\ 0 & 0 & a^r & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a^r \end{pmatrix}, \text{ for all } r \in [n+1, 2m].$$

Proof. Similar with the proof of Theorem 3.1. □

If $k = n$, and M is a Lagrangian submanifold in the complex space form $\widetilde{M}(c)$, the previous result can be improved.

Theorem 4.2. *Let M be a Lagrangian submanifold in complex space form $\widetilde{M}(c)$ of real dimension $2n$, $n \geq 3$. Then*

$$\delta_n(M) \leq \frac{(n+1)(n-2)}{8}c + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)} \|H\|^2 .$$

Proof. Let us consider the point $x \in M$, the vector $X \in T_xM$ and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal frame in T_xM, with $e_1 = X$. The fact that M is a Lagrangian submanifold imply that $\{Je_1, Je_2, \ldots, Je_n\}$ is an orthonormal frame in $T_x^{\perp}M$.

If $L = T_xM$, we shall denote $\mathrm{Ric}\,(X) = \mathrm{Ric}_L(X)$.

With an argument similar to those in the previous theorem, we obtain

(4.1)
$$\begin{aligned}
\tau - \frac{\mathrm{Ric}\,(X)}{n-1} &= \frac{(n+1)(n-2)}{8}c + \sum_{r=1}^{n} \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2) \\
&\quad - \frac{1}{n-1} \sum_{r=1}^{n} \sum_{j=2}^{n} (h_{11}^r h_{jj}^r - (h_{1j}^r)^2) \\
&\leq \frac{(n+1)(n-2)}{8}c \\
&\quad + \sum_{r=1}^{n} \left(\sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r - \frac{1}{n-1} \sum_{r=1}^{n} \sum_{j=2}^{n} h_{11}^r h_{jj}^r \right) \\
&\quad - \sum_{1 \leq i < j \leq n} (h_{ij}^i)^2 - \sum_{1 \leq i < j \leq n} (h_{ij}^j)^2 \\
&\quad + \frac{1}{n-1} \left(\sum_{j=2}^{n} (h_{1j}^1)^2 + \sum_{j=2}^{n} (h_{1j}^j)^2 \right).
\end{aligned}$$

Using the symmetry in the three indexes of h^k_{ij}, one gets

(4.2)
$$\tau - \frac{\mathrm{Ric}(X)}{n-1} \le \frac{(n+1)(n-2)}{8}c + \sum_{r=1}^{n} \sum_{1\le i<j\le n} h^r_{ii}h^r_{jj} - \frac{1}{n-1}\sum_{r=1}^{n}\sum_{j=2}^{n} h^r_{11}h^r_{jj} - \sum_{1\le i\ne j\le n}(h^i_{jj})^2 + \frac{1}{n-1}\left(\sum_{j=2}^{n}(h^j_{11})^2 + \sum_{j=2}^{n}(h^1_{jj})^2\right). \qquad \square$$

Let us consider the quadratic forms $f_1,\ f_r : R^n \to R$, $r \in [2,n]$, defined respectively by

$$f_1(h^1_{11}, h^1_{22}, \dots, h^1_{nn}) = \sum_{1\le i<j\le n} h^1_{ii}h^1_{jj} - \frac{1}{n-1}\sum_{j=2}^{n} h^1_{11}h^1_{jj} - \sum_{j=2}^{n}(h^1_{jj})^2 + \frac{1}{n-1}\sum_{j=2}^{n}(h^1_{jj})^2,$$

$$f_r(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) = \sum_{1\le i<j\le n} h^r_{ii}h^r_{jj} - \frac{1}{n-1}\sum_{j=2}^{n} h^r_{11}h^r_{jj} - \sum_{\substack{1\le j\le n\\ j\ne r}}(h^r_{jj})^2 + \frac{1}{n-1}(h^r_{11})^2.$$

We start with the problem

$$\max f_1$$
$$\text{subject to} \quad P : h^1_{11} + h^1_{22} + \cdots + h^1_{nn} = k^1,$$

where k^1 is a real constant.

The first two partial derivatives of the quadratic form f_1 are

$$\frac{\partial f_1}{\partial h^1_{11}} = \sum_{2\le j\le n} h^1_{jj} - \frac{1}{n-1}\sum_{j=2}^{n} h^1_{jj}, \tag{4.3}$$

$$\frac{\partial f_1}{\partial h^1_{22}} = \sum_{\substack{1\le j\le n\\ j\ne 2}} h^1_{jj} - \frac{1}{n-1}h^1_{11} - 2h^1_{22} + \frac{2}{n-1}h^1_{22}. \tag{4.4}$$

As for an optimal solution $(h^1_{11}, h^1_{22}, \dots, h^1_{nn})$ of the problem in question, the vector $\operatorname{grad}(f_1)$ is collinear with the vector $(1, 1, \dots, 1)$, we obtain

$$\begin{aligned} (4.5) \qquad & \sum_{1\le j\le n} h^1_{jj} - h^1_{11} - \frac{1}{n-1}\sum_{j=2}^{n} h^1_{jj} \\ & = \sum_{1\le j\le n} h^1_{jj} - h^1_{22} - \frac{1}{n-1}h^1_{11} - 2h^1_{22} + \frac{2}{n-1}h^1_{22}; \end{aligned}$$

therefore,

$$(4.6) \qquad \frac{n-2}{n-1}h^1_{11} = \frac{3n-5}{n-1}h^1_{22} - \frac{1}{n-1}\sum_{j=2}^{n} h^1_{jj}.$$

Similarly, we obtain

$$(4.7) \qquad \frac{n-2}{n-1}h^1_{11} = \frac{3n-5}{n-1}h^1_{ii} - \frac{1}{n-1}\sum_{j=2}^{n} h^1_{jj}, \text{ for all } i \in [2, n],$$

whence

$$(4.8) \qquad h^1_{22} = h^1_{33} = \cdots = h^1_{nn} = a^1.$$

The relations (4.6) and (4.8) imply

$$(4.9) \qquad \frac{n-2}{n-1}h^1_{11} = \frac{3n-5}{n-1}a^1 - a^1, \quad \therefore (n-2)h^1_{11} = (2n-4)a^1,$$

whence

$$(4.10) \qquad h^1_{11} = 2a^1.$$

As $h^1_{11}+h^1_{22}+h^1_{33}+\cdots+h^1_{nn} = k^1$, by using (4.8) and (4.10), we obtain

$$(4.11) \qquad 2a^1 + (n-1)a^1 = k^1,$$

therefore

$$a^1 = \frac{k^1}{n+1}. \tag{4.12}$$

As f_1 is obtained from the function studied in Theorem 3.1 by subtracting some square terms, $f_1 \mid P$, will have the Hessian negative definite. Consequently, the point $(h^1_{11}, h^1_{22}, \dots, h^1_{nn})$ given by the relations (4.8), (4.10) and (4.12) is a maximum point, and hence

(4.13)

$$f_1 \le 2a^1(n-1)a^1 + C^2_{n-1}(a^1)^2 - \frac{1}{n-1}2a^1(n-1)a^1 - (n-1)(a^1)^2 + \frac{1}{n-1}(n-1)(a^1)^2 = \frac{(a^1)^2}{2}(n^2-n-2) = \frac{(a^1)^2}{2}(n+1)(n-2).$$

From (4.12) and (4.13), one gets

$$f_1 \le \frac{(k^1)^2}{2(n+1)}(n-2) = \frac{(n-2)n^2}{2(n+1)}(H^1)^2. \tag{4.14}$$

Further on, we shall consider the problem

$$\max f_2$$
$$\text{subject to} \quad P : h^2_{11} + h^2_{22} + \cdots + h^2_{nn} = k^2,$$

where k^2 is a real constant.

The first three partial derivatives of the quadratic form f_2 are

$$\frac{\partial f_2}{\partial h^2_{11}} = \sum_{j=2}^{n} h^2_{jj} - \frac{1}{n-1}\sum_{j=2}^{n} h^2_{jj} - 2h^2_{11} + \frac{2}{n-1}h^2_{11}, \tag{4.15}$$

$$\frac{\partial f_2}{\partial h^2_{22}} = \sum_{\substack{1\le j\le n\\ j\ne 2}} h^2_{jj} - \frac{1}{n-1}h^2_{11}, \tag{4.16}$$

$$\frac{\partial f_2}{\partial h^2_{33}} = \sum_{\substack{1\le j\le n\\ j\ne 3}} h^2_{jj} - \frac{1}{n-1}h^2_{11} - 2h^2_{33}. \tag{4.17}$$

For a solution $(h_{11}^2, h_{22}^2, \dots, h_{nn}^2)$ of the problem in question, the vector grad(f_2) is collinear with $(1, 1, \dots, 1)$.

Consequently

$$\sum_{j=1}^{n} h_{jj}^2 - h_{22}^2 - \frac{1}{n-1} h_{11}^2 = \sum_{j=1}^{n} h_{jj}^2 - h_{33}^2 - \frac{1}{n-1} h_{11}^2 - 2h_{33}^2,$$

therefore

$$h_{22}^2 = 3h_{33}^2. \tag{4.18}$$

Similarly, we obtain

$$h_{22}^2 = 3h_{jj}^2 = 3a^2, \text{ for all } j \in [3, n]. \tag{4.19}$$

From (4.15), (4.16) and (4.19) we obtain

$$3h_{11}^2 - \frac{3}{n-1} h_{11}^2 = 3a^2 - \frac{1}{n-1}(3a^2 + (n-2)a^2), \tag{4.20}$$

hence

$$h_{11}^2 = \frac{2a^2}{3}. \tag{4.21}$$

We shall denote $a^2 = 3b^2$. The relations (4.19) and (4.21) becomes

$$h_{11}^2 = 2b^2, \tag{4.22}$$

$$h_{22}^2 = 9b^2, \tag{4.23}$$

$$h_{33}^2 = \cdots = h_{nn}^2 = 3b^2. \tag{4.24}$$

As $h_{11}^2 + h_{22}^2 + \cdots + h_{nn}^2 = k^2$, we obtain $2b^2 + 9b^2 + (n-2)3b^2 = k^2$; therefore,

$$b^2 = \frac{k^2}{3n+5}. \tag{4.25}$$

With an argument similar to those in the previous problem we obtain that the point $(h_{11}^2, h_{22}^2, \dots, h_{nn}^2)$ given by the relations (4.22)–(4.25) is a maximum point. Hence,

$$\begin{aligned}
f_2 &\le 2b^2(9b^2 + (n-2)3b^2) + 9b^2(n-2)3b^2 + C_{n-2}^2(3b^2)^2 \\
&\quad - \frac{1}{n-1}2b^2(9b^2 + (n-2)3b^2) - (2b^2)^2 - (n-2)(3b^2)^2 \\
&\quad + \frac{1}{n-1}(2b^2)^2 \\
&= \frac{(b^2)^2}{2(n-1)}(9n^3 - 6n^2 - 29n + 10) \\
&= \frac{(b^2)^2}{2(n-1)}(3n+5)(3n-1)(n-2).
\end{aligned} \tag{4.26}$$

From (4.25) and (4.26) we obtain

$$f_2 \le \frac{(k^2)^2(3n-1)(n-2)}{2(3n+5)(n-1)} = \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}(H^2)^2.$$

Similarly, one gets

$$f_r \le \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}(H^r)^2, \text{ for all } r \in [2, n]. \tag{4.27}$$

As $n - 2/n + 1 \le [(3n-1)(n-2)]/[(3n+5)(n-1)]$, for all $n \ge 3$, from (4.14) and (4.27), we obtain

$$f_r \le \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}(H^r)^2, \text{ for all } r \in [1, n]. \tag{4.28}$$

From (4.2) and (4.28), one gets

$$\begin{aligned}
\tau - \frac{\mathrm{Ric}(X)}{n-1} &\le \frac{(n+1)(n-2)}{8}c + \sum_{r=1}^{n} \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}(H^r)^2 \\
&= \frac{(n+1)(n-2)}{8}c + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)}\|H\|^2;
\end{aligned} \tag{4.29}$$

therefore,

$$\delta_n(M) \leq \frac{(n+1)(n-2)}{8}c + \frac{(3n-1)(n-2)n^2}{2(3n+5)(n-1)} \|H\|^2 . \tag{4.30}$$

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