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ON A MODEL FOR PHASE TRANSITIONS WITH VECTOR HYSTERESIS EFFECT

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ABSTRACT. The paper deals with a system of nonlinear PDEs which describes a phase transition model with vector hysteresis and diffusion effect. Existence of solutions for the system under consideration is proved by the method of Yosida approximation, L^∞-estimates and energy type inequalities in L^2.

1. Introduction. The present paper deals with a system of nonlinear PDEs which is a model of a class of phase transitions where the hysteresis and diffusive effects are taken into account:

$$a\mathbf{w}_t - \kappa\Delta\mathbf{w} + \partial\mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u) \quad \text{in} \quad Q, \tag{1}$$

$$\mathbf{c}\cdot\mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u) \quad \text{in} \quad Q. \tag{2}$$

Here N and m are positive integers, $\mathbf{w} = (w_1, \dots, w_m)$, $T > 0$, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q = (0,T) \times \Omega$; a, κ, $\mathbf{c} = (c_1, \dots, c_m)$, d are given constants; $\mathbf{F} : R^m \times R \to R^m$, $h : R^m \times R \to R$, $f_{i*}, {f_i}^* : R \to R$, $i = 1, \dots, m$, are given functions. We assume that $f_{i*}, {f_i}^* \in C^2(R)$, $f_{i*} \le {f_i}^*$ on R and there exist constants $k_i > 0$ such that $f_{i*} = {f_i}^*$ on $(-\infty, -k_i] \cup [k_i, \infty)$, $i = 1, \dots, m$.

For each $u \in R$ we denote by $\partial I_u^{(i)}(\cdot)$ the subdifferential of the indicator function $I_u^{(i)}(\cdot)$ of the interval $[f_{i*}(u), f_i^*(u)]$, $i = 1, \dots, m$, namely,

$$I_u^{(i)}(w_i) = \begin{cases} 0 & \text{if } f_{i*}(u) \le w_i \le f_i^*(u) \\ +\infty & \text{otherwise} \end{cases}$$

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and

$$\partial I_u^{(i)}(w_i) = \begin{cases} \varnothing & \text{if } w_i > f_i^*(u) \text{ or } w_i < f_{i*}(u) \\ [0,+\infty) & \text{if } w_i = f_i^*(u) > f_{i*}(u) \\ \{0\} & \text{if } f_{i*}(u) < w_i < f_i^*(u) \\ (-\infty, 0] & \text{if } w_i = f_{i*}(u) < f_i^*(u) \\ R & \text{if } w_i = f_i^*(u) = f_{i*}(u). \end{cases}$$

Define $K(u) = \{\mathbf{w} \in R^m : \ f_{i*}(u) \le w_i \le {f_i}^*(u), i = 1, \dots, m\}$.

We denote by $I_{K(u)}(\cdot)$ the indicator function of the set $K(u)$ and $\partial \mathbf{I}_{K(u)}(\cdot)$ denotes the subdifferential of $I_{K(u)}(\cdot)$. The subdifferential $\partial \mathbf{I}_{K(u)}(\mathbf{w})$ is a set-valued mapping and in our statement of the problem $\partial \mathbf{I}_{K(u)}(\mathbf{w}) = \{0\}$ if $\mathbf{w} \in \operatorname{int} K$, and $\partial \mathbf{I}_{K(u)}(\mathbf{w})$ coincides with the cone of normals to K at the point $\mathbf{w}$ if $\mathbf{w} \in \partial K$. In our statement of the problem it is easy to see that

$$\partial \mathbf{I}_{K(u)}(\mathbf{w}) = (\partial I_u^{(1)}(w_1), \dots, \partial I_u^{(m)}(w_m)).$$

In this paper we study the system (1), (2), together with the following boundary and initial conditions

$$\frac{\partial \mathbf{w}}{\partial \nu} = 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Sigma = (0,T) \times \partial\Omega, \tag{3}$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \quad u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \tag{4}$$

where ν is the unit outward normal vector on $\partial\Omega$, $\mathbf{w}_0$, u_0 are given initial data.

The system (1), (2) is a model for solid-liquid phase transition of a multi-component substance where we take into account the hysteresis effect in the evolution of the interface. Equations (1) and (2) correspond respectively to the kinetics of the vector order parameter $\mathbf{w}$ and the balance of the internal energy; u is the relative temperature of the physical system under consideration. The righthand sides of the equations of system (1), (2) describe possible nonlinearities respectively in the kinetics of the order parameter and the external energy supply.

The hysteresis effect is described by the term $\partial \mathbf{I}_{K(u)}(\mathbf{w})$ in differential inclusion (1). It is known that some types of hysteresis operators can be represented by ordinary (or partial) differential inclusion containing subdifferential of the indicator function of a closed set (whose shape could possibly depend on the unknown variables). Let us note that this characterization of hysteresis operators was used for analysis of many nonlinear phenomena, for example, a real-time control problems, see [**9**], solid-liquid phase transitions, see [**7, 19**], shape memory alloys, see [**1**], filtration problems, see [**14**]. Very recently this approach has been used to study the phenomena of hysteresis in processes in population dynamics, see [**2, 21**].

Differential inclusion (1) describes the relaxation dynamics of the vector order parameter. The relation assigning to a function $u(t)$ the solution $\mathbf{w}(t)$ of differential inclusion (1) corresponds to generalized vector play hysteresis operator which is often used to describe solid-liquid phase transitions with supercooling effect and martensite-austenite phase transitions in shape memory allows. Let us note that models with hysteresis are the object of active recent investigations, see papers [**8, 10, 12, 13**] as well as the monographs [**5, 11, 18, 22**].

Various special cases of the system (1), (2) have been already studied. In [**7**], Colli et al. studied the following system

$$\begin{aligned} aw_t - \kappa\Delta w + \partial I_u(w) \ni F(w,u) \quad &\text{in} \quad Q,\\ cw_t + du_t - \Delta u = g(x,t) \quad &\text{in} \quad Q \end{aligned}$$

as a model for the Stefan problem with phase relaxation and temperature dependent constraint for the scalar order parameter. Later, Kubo in [**14**] studied filtration problems with hysteresis described by similar systems with convective term (we refer the reader also to the papers [**8**–10], the monograph [**5**] as well as the references therein).

Very recently, in [**21**], Ôtani studied the following nonlinear parabolic system with hysteresis effect

$$w_t - \nabla\cdot(\nabla w + \vec{\lambda}(w)) + \partial I_U(w) \ni F(w,U) \quad \text{in} \quad Q,$$

$$u_{it} - \nabla\cdot(\nabla u_i + \vec{\mu}_i(u_i)) = h_i(w,U) \quad \text{in} \quad Q, \quad i = 1,\dots,m,$$

which is a model for population interaction with hysteresis effect of $1+m$ biological species with densities (w,U), $U = (u_1,\dots,u_m)$. To this end in [**21**], a further extension of the recently proposed L^∞-energy method is developed. It should be noted that the L^∞-energy method (proposed in [**20, 21**] and the references therein) was found to be an effective tool applicable to various types of parabolic equations and systems including doubly nonlinear parabolic equations, porous medium equations, strongly nonlinear parabolic equations governed by the ∞-Laplacian, complicated parabolic systems from applied sciences, etc.

In mathematical aspects the present paper has been influenced mainly by the papers [**7, 21**]. Using the L^∞-energy method we will obtain results for boundedness and existence of solutions of the system (1)–(4). As concerns for uniqueness, the result presented here is based on the method of L^1-semigroups proposed in [**8**], and later developed in [**7**].

2. Preliminary notes. Denote by H the Hilbert space $L^2(\Omega)$ with the usual scalar product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, and by $\mathbf{H}$ the product space $H \times \cdots \times H$ (m-times). Denote by V the Sobolev space $H^1(\Omega)$ equipped with the norm $|u|_V = (u,u)_V^{1/2}$, where $(u,v)_V = (u,v)_H + a(u,v)$, $a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx$, $u, v \in V$, and by $\mathbf{V}$ the product space $V \times \cdots \times V$ (m-times).

We give the definition of solutions in a weak (variational) sense for the system (1)–(4).

Definition 2.1. Let $\kappa > 0$. A pair of functions $\{\mathbf{w}, u\}$, ($\mathbf{w} = (w_1, \dots, w_m)$) is called a solution of the system (1)–(4) if:

(i) w_i, $u \in L^\infty(0,T; V \cap L^\infty(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap W^{1,2}(0,T;H)$, $i = 1, \dots, m$.

(ii) $a\mathbf{w}_t - \kappa\Delta\mathbf{w} + \partial\mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u)$ in $\mathbf{H}$, almost everywhere in $(0,T)$.

(iii) $\mathbf{c} \cdot \mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u)$ in H, almost everywhere in $(0,T)$.

(iv) $(\partial w_i / \partial \nu) = 0$, $(\partial u / \partial \nu) = 0$ in $L^2(\partial\Omega)$, almost everywhere in $(0,T)$, $i = 1, \dots, m$.

(v) $\mathbf{w}(0) = \mathbf{w}_0$, $u(0) = u_0$.

For simplicity of notation we will denote in the sequel by $\mathbf{w}'$ and u' the time-derivatives $\mathbf{w}_t$ and u_t of $\mathbf{w}$ and u, respectively.

Note that the inclusion (ii) is equivalent to the following conditions:

(ii)-(a) $\mathbf{w} \in K(u)$ almost everywhere in Q.

(ii)-(b) $(a\mathbf{w}'(t) - \kappa\Delta\mathbf{w}(t) - \mathbf{F}(\mathbf{w}(t), u(t)), \mathbf{w}(t) - \mathbf{z}) \leq 0$ for all $\mathbf{z} \in \mathbf{H}$ with $\mathbf{z} \in K(u(t))$ almost everywhere in Ω for almost every $t \in (0, T)$.

Throughout the paper we suppose that the following assumptions hold:

H1. $a > 0$, $c_i \neq 0$, $d > 0$ are given constants, $i = 1, \ldots, m$.

H2. $f_{i*}, f_i{}^* \in C^2(R)$ are such that $f_{i*} \leq f_i{}^*$ on R and there exist constants $k_i > 0$ such that $f_{i*}(u) = f_i{}^*(u) = r_i u + s_i$ on $(-\infty, -k_i]$ and $f_{i*}(u) = f_i{}^*(u) = p_i u + q_i$ on $[k_i, \infty)$, where r_i, s_i, p_i, q_i are given constants, $i = 1, \ldots, m$. Moreover, if $c_i > 0$, $c_i < 0$, then f_{i*}, f_i^* are assumed to be nondecreasing (nonincreasing) functions on R, $i = 1, \ldots, m$.

H3. $w_{0i}, u_0 \in L^\infty(\Omega) \cap V$, $i = 1, \ldots, m$ and $\mathbf{w}_0 \in K(u_0)$ almost everywhere in Ω, $\mathbf{w}_0 = (w_{01}, \ldots, w_{0m})$.

H4. $\mathbf{F}$ and h are locally Lipschitz continuous functions from $R^m \times R$ i nto R^m and R, respectively.

H5. There exist positive constants $C_\mathbf{F}$ and C_h such that $F_i(\mathbf{w}, u)w_i \leq C_\mathbf{F}(|w_i|^2 + |u|^2 + 1)$, $i = 1, \ldots, m$, and $h(\mathbf{w}, u)u \leq C_h(|\mathbf{w}|^2 + |u|^2 + 1)$, $\mathbf{w} \in R^m$, $u \in R$.

3. Auxiliary problems. Let $M > 0$ be a constant large enough which is to be fixed later in the sequel. For simplicity of the notation we will denote in the sequel by C, C_α, various positive constants whose value can vary from line to line (and possibly depend on the index quantity α).

Consider the following cut-off function:

$$\chi_M(r) = \begin{cases} -M & \text{if } r < -M \\ r & \text{if } -M \leq r \leq M \\ M & \text{if } r > M \end{cases}$$

and define the auxiliary functions

$$\begin{aligned} \chi_M(w_i)(t,x) &= \chi_M(w_i(t,x)), \quad (t,x) \in Q, \quad i = 1, \ldots, m, \\ \chi_M(\mathbf{w})(t,x) &= (\chi_M(w_1)(t,x), \ldots, \chi_M(w_m)(t,x)), \quad (t,x) \in Q, \\ \chi_M(u)(t,x) &= \chi_M(u(t,x)), \quad (t,x) \in Q. \end{aligned}$$

In this section we introduce an approximate system with approximation parameters M and $\mu > 0$. To this end for $(\mathbf{w}, u) \in R^m \times R$ we denote

$$\begin{aligned}\partial \mathbf{I}^{\mu}_{K_M(u)}(\mathbf{w}) &= (\partial {I^{\mu}_{u,M}}^{(1)}(w_1), \dots , \partial {I^{\mu}_{u,M}}^{(m)}(w_m)) \\ &= \frac{1}{\mu}\Big([w_1 - f^*_{1,M}(u)]^+ - [f_{1*,M}(u) - w_1]^+, \dots , \\ &\qquad [w_m - f^*_{m,M}(u)]^+ - [f_{m*,M}(u) - w_m]^+ \Big),\end{aligned}$$

and

$$\begin{aligned}\mathbf{J}_{u,M}\mathbf{w} &= (J^{(1)}_{u,M} w_1, \dots , J^{(m)}_{u,M} w_m) \\ &= (\max\{\min\{w_1, f^*_{1,M}(u)\}, f_{1*,M}(u)\}, \dots , \\ &\qquad \max\{\min\{w_m, f^*_{m,M}(u)\}, f_{m*,M}(u)\}),\end{aligned}$$

where $f^*_{i,M}(u) = f^*_i(\chi_M(u))$, $f_{i*,M}(u) = f_{i*}(\chi_M(u))$, $i = 1, \dots , m$. Moreover, denote

$$\begin{aligned}\mathbf{J}_u\mathbf{w} &= (J^{(1)}_u w_1, \dots , J^{(m)}_u w_m) \\ &= (\max\{\min\{w_1, f^*_1(u)\}, f_{1*}(u)\}, \dots , \\ &\qquad \max\{\min\{w_m, f^*_m(u)\}, f_{m*}(u)\}).\end{aligned}$$

Note that $\partial \mathbf{I}^{\mu}_{K_M(u)}$ is the Yosida regularization of the subdifferential graph of the indicator function $I_{K_M(u)}$ of the set $K_M(u) = \{\mathbf{w} \in R^m : f_{i*,M}(u) \le w_i \le f^*_{i,M}(u),\ i = 1, ..., m\}$.

Consider the following approximate system of PDEs

$$a\mathbf{w}' - \kappa\Delta\mathbf{w} + \partial \mathbf{I}^{\mu}_{K_M(u)}(\mathbf{w}) = \mathbf{F}_M(\mathbf{w}, u) \quad \text{in} \quad Q, \tag{5}$$

$$\mathbf{c} \cdot (\mathbf{J}_u\mathbf{w})' + du' - \Delta u = h_{M,\mathbf{J}}(\mathbf{w}, u) \quad \text{in} \quad Q, \tag{6}$$

$$\frac{\partial \mathbf{w}}{\partial \nu} = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \tag{7}$$

$$\mathbf{w}(0, x) = \mathbf{w}_0(x), \quad u(0, x) = u_0(x) \quad \text{in} \quad \Omega, \tag{8}$$

where

$$\mathbf{F}_M(\mathbf{w}, u) = \mathbf{F}(\chi_M(\mathbf{w}), \chi_M(u)), \qquad h_{M,\mathbf{J}}(\mathbf{w}, u) = h(\mathbf{J}_{u,M}\mathbf{w}, \chi_M(u)).$$

4. Main results.

4.1 System (1)–(4) with $\kappa > 0$.

Theorem 4.1. *Suppose that assumptions* H1–H5 *are satisfied. Then there exists a constant* $\kappa_0 > 0$ *such that for each* $0 < \kappa < \kappa_0$ *the system* (1)–(4) *possesses at least one solution.*

Proof. Consider the approximate system (5)–(8). By [**6**] it follows that there exists a unique solution $\{\mathbf{w}_{\mu,M}, u_{\mu,M}\}$ of the system (5)–(8). For simplicity of the notation the solution of the approximate system (5)–(8) will be denoted by $\{\mathbf{w}, u\}$ in the sequel.

We will prove estimates for $\mathbf{w}, u$ which are independent of M and μ. Denote $k_0 = \max\{k_1, \dots, k_m\}$. Without loss of generality it can supposed that $M \geq k_0 + 1$.

First we prove the estimate for u. To this end we follow [**21**]. Multiplying equation (6) by $|u - k_0|^{r-2}[u - k_0]^+$ (where $[u - k_0]^+$ is the positive part of $u - k_0$) and integrating over Ω we have that

$$\begin{aligned} \text{(9)} \qquad & \mathbf{c} \cdot \int_\Omega (\mathbf{J}_u\mathbf{w})'|u - k_0|^{r-2}[u - k_0]^+ \, dx \\ & + d \int_\Omega u'|u - k_0|^{r-2}[u - k_0]^+ \, dx \\ & - \int_\Omega \Delta u |u - k_0|^{r-2}[u - k_0]^+ \, dx \\ = & \int_\Omega h_{M,\mathbf{J}}(\mathbf{w}, u)|u - k_0|^{r-2}[u - k_0]^+ \, dx \\ & \text{a.e. in} \quad (0, T). \end{aligned}$$

Integration by parts gives that

$$-\int_\Omega \Delta u |u - k_0|^{r-2}[u - k_0]^+ \, dx = (r - 1) \int_\Omega |\nabla u|^2 |[u - k_0]^+|^{r-2} \, dx.$$

Since $J_u^{(i)} w_i = p_i u + q_i$ almost everywhere on the set $\{[u-k_0]^+ > 0\}$, $i = 1, \dots, m$, we have for the lefthand side (LHS) of (9) that

$$\text{LHS} \geq \frac{\sum_{i=1}^m c_i p_i + d}{r} \frac{d}{dt} |[u-k_0]^+|^r_{L^r(\Omega)} \quad \text{a.e. on} \quad (0,T). \tag{10}$$

Note that

$$\begin{aligned} &\int_\Omega h_{M,\mathbf{J}}(\mathbf{w}, u)|u-k_0|^{r-2}[u-k_0]^+ \, dx \qquad (11)\\ &= \int_\Omega h(\mathbf{J}_{u,M}\mathbf{w}, \chi_M(u))\chi_M(u)|u-k_0|^{r-2} \frac{[u-k_0]^+}{\chi_M(u)} \, dx \\ &\leq C_h \int_\Omega (|\mathbf{J}_{u,M}\mathbf{w}|^2 + |\chi_M(u)|^2 + 1)|u-k_0|^{r-2} \frac{[u-k_0]^+}{\chi_M(u)} \, dx \\ &\leq C_h \int_\Omega \left(\sum_{i=1}^m |f_i^*(\chi_M(u))|^2 + |\chi_M(u)|^2 + 1 \right) |u-k_0|^{r-2} \frac{[u-k_0]^+}{\chi_M(u)} \, dx \\ &\leq C_{h,f} \int_\Omega (|\chi_M(u)|^2 + 1)|u-k_0|^{r-2} \frac{[u-k_0]^+}{\chi_M(u)} \, dx \\ &\leq C_{h,f} \int_\Omega |u||[u-k_0]^+|^{r-1} \, dx + C \int_{\{\Omega:\ u \geq k_0\}} |[u-k_0]^+|^{r-1} \, dx \\ &\leq C_{h,f,k_0,\Omega} |[u-k_0]^+|^{r-1}_{L^r(\Omega)} (|[u-k_0]^+|_{L^r(\Omega)} + 1). \end{aligned}$$

Rearranging the terms of (9) in view of estimates (10), (11), and dividing both sides by $|[u-k_0]^+|^{r-1}_{L^r(\Omega)}$, we get that

$$\frac{d}{dt} |[u-k_0]^+|_{L^r(\Omega)} \leq C_* (|[u-k_0]^+|_{L^r(\Omega)} + 1), \tag{12}$$

where $C_* = C_{h,f,k_0,\Omega,\mathbf{c},\mathbf{p},d}$. Integrating (12) with respect to t on $[0,T]$ and letting r tend to ∞, we deduce that

$$|[u(t)-k_0]^+|_{L^\infty(\Omega)} \leq |[u_0-k_0]^+|_{L^\infty(\Omega)} + C_* \int_0^t (|[u(\tau)-k_0]^+|_{L^\infty(\Omega)} + 1) \, d\tau$$

and therefore $|[u(t)-k_0]^+|_{L^\infty(\Omega)} \leq C_1$, $t \in [0,T]$, where $C_1 = C_{h,f,k_0,\Omega,\mathbf{c},\mathbf{p},d,T,u_0}$. Analogously it can be proved that $|[-u(t)-k_0]^+|_{L^\infty(\Omega)} \leq C_1$, $t \in [0,T]$, and thus we obtain the required estimate

$$|u(t)|_{L^\infty(\Omega)} \leq C_1, \quad t \in [0,T]. \tag{13}$$

Suppose further that $M \geq \max\{k_0 + 1, C_1\}$. To prove the estimates for $\mathbf{w}$ we follow the approach of [**21**] and consider the auxiliary function

$$(14)\qquad u_{i*}(t) = \max\left(\sup_{x\in\Omega,\ 0\leq s\leq t} f_{i*,M}(u(s,x)), 1\right), \quad i = 1, \dots, m.$$

Multiplying the ith component of (5) by $|w_i(s,x) - u_{i*}(t)|^{r-2}[w_i(s,x) - u_{i*}(t)]^+$ and integrating over Ω, we have that

$$\begin{aligned}(15)\quad & a\frac{1}{r}\frac{d}{ds}|[w_i(s) - u_{i*}(t)]^+|^r_{L^r(\Omega)} + \kappa(r-1) \\ & \times \int_\Omega |\nabla w_i(s)|^2 |[w_i(s) - u_{i*}(t)]^+|^{r-2}\, dx \\ & + \int_\Omega \partial {I^\mu_{u,M}}^{(i)}(w_i(s))|w_i(s) - u_{i*}(t)|^{r-2}[w_i(s) - u_{i*}(t)]^+\, dx \; (:= I_1) \\ \leq & \int_\Omega F_{i,M}(\mathbf{w}(s), u(s))|w_i(s) - u_{i*}(t)|^{r-2}[w_i(s) - u_{i*}(t)]^+ dx \;\; (:= I_2).\end{aligned}$$

Note that, cf. [**21**],

$$\begin{aligned}(16)\quad I_1 = {} & \frac{1}{\mu}\int_\Omega [w_i(s) - f^*_{i,M}(u(s))]^+ |w_i(s) - u_{i*}(t)|^{r-2} \\ & \times [w_i(s) - u_{i*}(t)]^+\, dx - \frac{1}{\mu}\int_\Omega [f_{i*,M}(u(s)) - w_i(s)]^+ \\ & \qquad \times |w_i(s) - u_{i*}(t)|^{r-2}[w_i(s) - u_{i*}(t)]^+ dx.\end{aligned}$$

The first term on the righthand side of (16) is nonnegative, while to estimate the second term note that if $[w_i(s) - u_{i*}(t)]^+ > 0$ then $w_i(s) > u_{i*}(t) \geq f_{i*,M}(u(s))$ and thus $[f_{i*,M}(u(s)) - w_i(s)]^+ = 0$. Therefore,

$$(17)\qquad I_1 \geq 0.$$

We have also that

(18)
$$
\begin{aligned}
I_2 = \int_\Omega & F_{i,M}(\mathbf{w}(s), u(s))\chi_M(w_i(s))|w_i(s) - u_{i*}(t)|^{r-2} \\
&\qquad\qquad \times \frac{[w_i(s) - u_{i*}(t)]^+}{\chi_M(w_i(s))}\, dx \\
&\le C_{\mathbf{F}} \int_\Omega (|\chi_M(w_i(s))|^2 + |\chi_M(u(s))|^2 + 1)|w_i(s) - u_{i*}(t)|^{r-2} \\
&\qquad\qquad \times \frac{[w_i(s) - u_{i*}(t)]^+}{\chi_M(w_i(s))}\, dx \\
&\le C_{\mathbf{F}} \int_\Omega (|\chi_M(w_i(s))|^2 + |u(s)|^2 + 1)|w_i(s) - u_{i*}(t)|^{r-2} \\
&\qquad\qquad \times \frac{[w_i(s) - u_{i*}(t)]^+}{\chi_M(w_i(s))}\, dx \\
&\le C^* \int_\Omega (|\chi_M(w_i(s))|^2 + 1)|w_i(s) - u_{i*}(t)|^{r-2} \frac{[w_i(s) - u_{i*}(t)]^+}{\chi_M(w_i(s))}\, dx \\
&\le C^* \int_\Omega |w_i(s)||[w_i(s) - u_{i*}(t)]^+|^{r-1}\, dx \\
&\quad + C^* \int_{\Omega:\ w_i \ge u_{i*}} |[w_i(s) - u_{i*}(t)]^+|^{r-1}\, dx \\
&\le C^*|w_i(s)|_{L^r(\Omega)}|[w_i(s) - u_{i*}(t)]^+|^{r-1}_{L^r(\Omega)} \\
&\quad + C^*|[w_i(s) - u_{i*}(t)]^+|^{r-1}_{L^r(\Omega)} \\
&\le C^*(|w_i(s) - u_{i*}(t)|_{L^r(\Omega)} + |u_{i*}(t)|_{L^r(\Omega)})|[w_i(s) - u_{i*}(t)]^+|^{r-1}_{L^r(\Omega)} \\
&\quad + C^*|[w_i(s) - u_{i*}(t)]^+|^{r-1}_{L^r(\Omega)} \\
&\le C^*|[w_i(s) - u_{i*}(t)]^+|^{r-1}_{L^r(\Omega)}(|[w_i(s) - u_{i*}(t)]^+|_{L^r(\Omega)} + 1),
\end{aligned}
$$

where $C^* = C_{h,\mathbf{F},f,k_0,\Omega,\mathbf{c},\mathbf{p},d,T,u_0}$. Thus, we have from (15) in view of (17), (18) that

$$
a\frac{d}{ds}|[w_i(s) - u_{i*}(t)]^+|_{L^r(\Omega)} \le C^*(|[w_i(s) - u_{i*}(t)]^+|_{L^r(\Omega)} + 1). \tag{19}
$$

Integrating (19) with respect to s on $[0, t]$ and letting r tends to ∞, we deduce that

$$|[w_i(s) - u_{i*}(t)]^+|_{L^\infty(\Omega)} \le |[w_i(0) - u_{i*}(t)]^+|_{L^\infty(\Omega)} + aC^* \int_0^s (|[w_i(\tau) - u_{i*}(t)]^+|_{L^\infty(\Omega)} + 1)\, d\tau$$

and therefore $|[w_i(s) - u_{i*}(t)]^+|_{L^\infty(\Omega)} \le C_2$, $s \in [0,t]$ for each $t \in [0,T]$, where $C_2 = C_{h,\mathbf{F},f,k_0,\Omega,a,\mathbf{c},\mathbf{p},d,T,u_0}$. Consequently, $|[w_i(t)]^+|_{L^\infty(\Omega)} \le C_2$, $t \in [0,T]$. Analogously, it can be proved that $|[-w_i(t)]^+|_{L^\infty(\Omega)} \le C_2$, $t \in [0,T]$, and thus we obtain the required estimate

$$|w_i(t)|_{L^\infty(\Omega)} \le C_2, \quad t \in [0,T], \quad i = 1, \dots, m. \tag{20}$$

Now, in view of the estimates (13) and (20), taking $M \ge \max\{k_0 + 1, C_1, C_2\}$, we conclude that the solution of the problem (5)–(8) coincides with the solution of the following problem (without cut-off)

$$a\mathbf{w}' - \kappa\Delta\mathbf{w} + \partial\mathbf{I}^\mu_{K(u)}(\mathbf{w}) = \mathbf{F}(\mathbf{w}, u) \quad \text{in} \quad Q, \tag{21}$$

$$\mathbf{c} \cdot (\mathbf{J}_u\mathbf{w})' + du' - \Delta u = h(\mathbf{J}_u\mathbf{w}, u) \quad \text{in} \quad Q, \tag{22}$$

$$\frac{\partial \mathbf{w}}{\partial \nu} = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \tag{23}$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \quad u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \tag{24}$$

where

$$\begin{aligned}\partial\mathbf{I}^\mu_{K(u)}(\mathbf{w}) &= (\partial I^{\mu(1)}_u(w_1), \dots, \partial I^{\mu(m)}_u(w_m)) \\ &= \frac{1}{\mu}\big([w_1 - f_1^*(u)]^+ - [f_{1*}(u) - w_1]^+, \dots, \\ &\qquad [w_m - f_m^*(u)]^+ - [f_{m*}(u) - w_m]^+\big).\end{aligned}$$

Due to the bounds (13) and (20) all functions in the approximate problem (21)–(24) are globally Lipschitz continuous and the righthand sides are bounded.

Note that the function $(\mathbf{J}_u\mathbf{w})(t)$ belongs to the class $W^{1,2}(0,T;\mathbf{H})$ and:
(25)
$$|\mathbf{c}\cdot(\mathbf{J}_u\mathbf{w})'(t)|_H \le \sqrt{2}|\mathbf{c}|\left(|\mathbf{w}'(t)|_{\mathbf{H}} + C_{f'}|u'(t)|_H\right), \quad \text{a.e.} \quad t\in(0,T),$$

where $C_{f'} = \max\{|f_{i*}'|_\infty, |f_i^{*\prime}|_\infty,\ i=1,\dots,m\}$;

$$(26)\quad (\mathbf{c}\cdot(\mathbf{J}_u\mathbf{w})'(t), u'(t)) \ge -\sqrt{2}|\mathbf{c}||\mathbf{w}'(t)|_{\mathbf{H}}|u'(t)|_H, \quad \text{a.e.} \quad t\in(0,T).$$

The estimates (25), (26) can be proved using that

$$(J_u^{(i)}w_i)' = \begin{cases} f_{i*}'(u)u' & \text{if } w_i \le f_{i*}(u) \\ w_i' & \text{if } f_{i*}(u) < w_i < f_i^*(u) \\ f_i^{*\prime}(u)u' & \text{if } w_i \ge f_i^*(u), \end{cases}$$

$i=1,\dots,m$, and that $c_i f_{i*}'(u)\ge 0$, $c_i f_i^{*\prime}(u)\ge 0$.

In order to take the limit $\mu\to 0$ in problem (21)–(24), we calculate:

(i) multiply (22) by u',

(ii) multiply (22) by $-\Delta u$,

(iii) multiply (21) by $\mathbf{w}'$,

(iv) multiply (21) by $-\Delta\mathbf{w}$,

(v) multiply (21) by $\partial\mathbf{I}^\mu_{K(u)}$.

Now, we multiply equation (22) by u', using (26) and applying Young inequality, we obtain that

$$(27)\qquad d|u'|_H^2 + \frac{d}{dt}|\nabla u|_H^2 \le C_3(\kappa^2|\Delta\mathbf{w}|^2_{\mathbf{H}} + 1),$$

where $C_3 = C_{a,\mathbf{c},d,|\Omega|,|\mathbf{F}|_\infty,|h|_\infty}$; $|\Omega|$ is the Lebesgue measure of the set Ω.

Next we multiply (22) by $-\Delta u$; using (25) and the Young inequality, we conclude that

$$(28)\qquad d\frac{d}{dt}|\nabla u|_H^2 + |\Delta u|_H^2 \le C_4(|\mathbf{w}'|^2_{\mathbf{H}} + |u'|_H^2 + 1),$$

where $C_4 = C_{\mathbf{c},|h|_\infty,|\Omega|,C_{f'}}$.

Denote by $I^{\mu}_{K(u)}(\mathbf{w})$ the Yosida regularization of the indicator function $I_{K(u)}(\mathbf{w})$ of the set $K(u)$. Then

$$(I^{\mu}_{K(u)}(\mathbf{w}))(t) = \frac{1}{2\mu}\sum_{i=1}^{m}\left(|[w_i - f_i^*(u)]^+|_H^2 + |[f_{i*}(u) - w]^+|_H^2\right)$$

is absolutely continuous on $[0,T]$ and

(29)
$$\frac{d}{dt}I^{\mu}_{K(u)}(\mathbf{w}) \leq (\partial\mathbf{I}^{\mu}_{K(u)}(\mathbf{w}),\mathbf{w}')_{\mathbf{H}} + C_{f'}|\partial\mathbf{I}^{\mu}_{K(u)}(\mathbf{w})|_{\mathbf{H}}|u'|_H, \text{ a.e. in } (0,T).$$

See [**7, 9, 19**] for the proof of (29).

Next, we compute (iii). Multiplying (21) by $\mathbf{w}'$ and using (29), we obtain that

(30)
$$a|\mathbf{w}'|_{\mathbf{H}}^2 + \kappa\frac{d}{dt}|\nabla\mathbf{w}|_{\mathbf{H}}^2 + 2\frac{d}{dt}I^{\mu}_{K(u)}(\mathbf{w}) \leq C_5\left(|u'|_H^2 + \kappa^2|\Delta\mathbf{w}|_{\mathbf{H}}^2 + 1\right), \text{ a.e. on } (0,T),$$

where $C_5 = C_{a,C_{f'},|\mathbf{F}|_\infty,|\Omega|}$.

Now we proceed with (iv). To this end we note that since $f_{i*}(u), f_i^*(u) \in H^2(\Omega)$ almost everywhere in $(0,T)$, $i = 1,\dots,m$, the following estimate, cf. [**7, 19**], is valid

(31)
$$(\partial\mathbf{I}^{\mu}_{K(u)}(\mathbf{w}), -\Delta\mathbf{w})_{\mathbf{H}} \geq -\frac{1}{4}|\partial\mathbf{I}^{\mu}_{K(u)}(\mathbf{w})|_{\mathbf{H}}^2 - \frac{1}{2}\sum_{i=1}^{m}\left(|\Delta f_{i*}(u)|_H^2 + |\Delta f_i^*(u)|_H^2\right).$$

Also

(32)
$$(\mathbf{F}(\mathbf{w},u), -\Delta\mathbf{w})_{\mathbf{H}} \leq C_{\mathbf{F}}^*(|\nabla u|_H^2 + |\nabla\mathbf{w}|_{\mathbf{H}}^2),$$

where $C_{\mathbf{F}}^* = \max\{C_{F_i}^*,\ i = 1,\dots,m\}$,

$$C_{F_i}^* = \sum_{j=1}^{m}\left(\sup_{(\mathbf{w},u)\in R^m\times R}\left|\frac{\partial F_i}{\partial w_j}(\mathbf{w},u)\right|\right) + \sup_{(\mathbf{w},u)\in R^m\times R}\left|\frac{\partial F_i}{\partial u}(\mathbf{w},u)\right|, \quad i = 1,\dots,m.$$

Now, multiplying (21) by $-\Delta\mathbf{w}$, we get in view of (31) and (32) that

$$\begin{aligned}(33)\quad & \frac{a}{2}\frac{d}{dt}|\nabla\mathbf{w}|_{\mathbf{H}}^2+\kappa|\Delta\mathbf{w}|_{\mathbf{H}}^2 \\ &\qquad\le \frac{1}{4}|\partial\mathbf{I}_{K(u)}^{\mu}(\mathbf{w})|_{\mathbf{H}}^2+\frac{1}{2}\sum_{i=1}^{m}\left(|\Delta f_i^*(u)|_H^2+|\Delta f_{i*}(u)|_H^2\right) \\ &\qquad\quad + C_{\mathbf{F}}^*(|\nabla u|_H^2+|\nabla\mathbf{w}|_{\mathbf{H}}^2).\end{aligned}$$

Finally, we perform (v). Multiplying (21) by $\partial\mathbf{I}_{K(u)}^{\mu}(\mathbf{w})$ and using estimate (29), we have that

$$\begin{aligned}(34)\quad & a\frac{d}{dt}I_{K(u)}^{\mu}(\mathbf{w})+\frac{1}{2}|\partial\mathbf{I}_{K(u)}^{\mu}(\mathbf{w})|_{\mathbf{H}}^2\le\frac{1}{2}aC_{f'}(aC_{f'}+1)|u'|_H^2 \\ &+\frac{\kappa}{4}|\partial\mathbf{I}_{K(u)}^{\mu}(\mathbf{w})|_{\mathbf{H}}^2+\frac{\kappa}{2}\sum_{i=1}^{m}\left(|\Delta f_i^*(u)|_H^2+|\Delta f_{i*}(u)|_H^2\right)+C_1(\mathbf{F}),\end{aligned}$$

where $C_1(\mathbf{F})=(aC_{f'}+1)|\mathbf{F}|_\infty^2|\Omega|$. Note that

$$|\Delta f_{i*}(u)|_H^2\le C_6(|\nabla u|_{L^4}^4+|\Delta u|_H^2),\quad |\Delta f_i^*(u)|_H^2\le C_6(|\nabla u|_{L^4}^4+|\Delta u|_H^2),$$

$i=1,\dots,m$, where $C_6=2\max\{|f_{i*}''|_\infty^2,|f_{i*}'|_\infty^2,|f_i^{*\prime\prime}|_\infty^2,|f_i^{*\prime}|_\infty^2,\ i=1,\dots,m\}$. By the Gagliardo-Nirenberg inequality, cf. [**7**], we have that

$$|\nabla u|_{L^4}^4\le C_\Omega|u|_{H^2(\Omega)}^2|u|_\infty^2\le C_\Omega(|u|_H^2+|\Delta u|_H^2)|u|_\infty^2\le C_7(1+|\Delta u|_H^2),$$

where the constant C_7 depends on C_1. Thus, we obtain that

$$(35)\qquad \sum_{i=1}^{m}\left(|\Delta f_{i*}(u)|_H^2+|\Delta f_i^*(u)|_H^2\right)\le C_8(1+|\Delta u|_H^2),$$

with $C_8=mC_6(C_7+1)$.

Adding (33) and (34), we get in view of (35) that

$$\begin{aligned}(36)\quad & \frac{d}{dt}\left\{aI_{K(u)}^{\mu}(\mathbf{w})+\frac{a}{2}|\nabla\mathbf{w}|_{\mathbf{H}}^2\right\}+\kappa|\Delta\mathbf{w}|_{\mathbf{H}}^2+\frac{1-\kappa}{4}|\partial\mathbf{I}_{K(u)}^{\mu}(\mathbf{w})|_{\mathbf{H}}^2 \\ &\qquad\le C_9\left(1+|\nabla u|_H^2+|\nabla\mathbf{w}|_{\mathbf{H}}^2+|u'|_H^2+(2+\kappa)(1+|\Delta u|_H^2)\right),\end{aligned}$$

where $C_9=C_{C_1(\mathbf{F}),C_{\mathbf{F}}^*,a,C_{f'},C_8}$.

Let ε_1, ε_2, ε_3 be positive numbers to be specified later. We calculate

$$(27) + \varepsilon_1 \times (28) + \varepsilon_2 \times (30) + \varepsilon_3 \times (36).$$

We have that

$$\begin{aligned}
(37) \quad & (d - \varepsilon_1 C_4 - \varepsilon_2 C_5 - \varepsilon_3 C_9)\,|u'|_H^2 \\
& + (\varepsilon_2 a - \varepsilon_1 C_4)|\mathbf{w}'|_{\mathbf{H}}^2 + (\varepsilon_1 - \varepsilon_3 C_9(2+\kappa))|\Delta u|_H^2 \\
& + \kappa(\varepsilon_3 - C_3\kappa - \varepsilon_2 C_5 \kappa)|\Delta \mathbf{w}|_{\mathbf{H}}^2 + \varepsilon_3 \frac{1-\kappa}{4}|\partial \mathbf{I}^{\mu}_{K(u)}(\mathbf{w})|_{\mathbf{H}}^2 \\
& + \frac{d}{dt}\{(1+\varepsilon_1 d)|\nabla u|_H^2 + (\varepsilon_2 \kappa + \varepsilon_3 \frac{a}{2})|\nabla \mathbf{w}|_{\mathbf{H}}^2 + (2\varepsilon_2 + a\varepsilon_3) I^{\mu}_{K(u)}(\mathbf{w})\} \\
& \qquad \le C_{10} + \varepsilon_3 C_9 \left(3 + \kappa + |\nabla u|_H^2 + |\nabla \mathbf{w}|_{\mathbf{H}}^2\right),
\end{aligned}$$

where $C_{10} = C_3 + \varepsilon_1 C_4 + \varepsilon_2 C_5$. Now, we will fix ε_i, $i = 1, 2, 3$, as well the constant κ_0 in the statement of Theorem 4.1, so that the coefficients in the first three lines of (37), i.e.,

$$\begin{gathered}
d - \varepsilon_1 C_4 - \varepsilon_2 C_5 - \varepsilon_3 C_9, \\
\varepsilon_2 a - \varepsilon_1 C_4, \quad \varepsilon_1 - \varepsilon_3 C_9 (2+\kappa), \\
\kappa(\varepsilon_3 - C_3\kappa - \varepsilon_2 C_5\kappa), \quad \varepsilon_3 \frac{1-\kappa}{4}
\end{gathered}$$

are all positive whenever $\kappa \in (0, \kappa_0)$. For instance, we can take

$$\varepsilon_2 = \frac{d}{4C_5}, \qquad \varepsilon_1 = \frac{1}{4C_4} \min\{d, 2\varepsilon_2 a\},$$

and then

$$\varepsilon_3 = \frac{1}{4C_9} \min\left\{d, \frac{2\varepsilon_1}{2+\kappa_0}\right\}$$

with

$$\kappa_0 = \min\left\{\frac{1}{2}, \frac{1}{8C_9}\min\{d, \varepsilon_1\} \frac{1}{C_3 + \varepsilon_2 C_5}\right\}.$$

We note that in this case

$$(38) \qquad d - \varepsilon_1 C_4 - \varepsilon_2 C_5 - \varepsilon_3 C_9 \ge \frac{d}{4},$$

$$\varepsilon_2 a - \varepsilon_1 C_4 \geq \varepsilon_2 \frac{a}{2} \tag{39}$$

along with

$$\varepsilon_1 - \varepsilon_3 C_9(2+\kappa) \geq \frac{\varepsilon_1}{2} \quad \text{for all} \quad \kappa \in (0, \kappa_0), \tag{40}$$

so that the above coefficients are all bounded from below uniformly with respect to κ. Moreover, concerning the coefficients in the third line of (37), we have that

$$\kappa(\varepsilon_3 - C_3\kappa - \varepsilon_2 C_5 \kappa) \geq \kappa \frac{\varepsilon_3}{2}, \qquad \varepsilon_3 \frac{1-\kappa}{4} \geq \frac{\varepsilon_3}{4}$$

for all $\kappa \in (0, \kappa_0)$. Moreover, it is easily checked that

$$\begin{aligned} E_\mu &:= (1+\varepsilon_1 d)|\nabla u|_H^2 + (\varepsilon_2\kappa + \frac{\varepsilon_3 a}{2})|\nabla \mathbf{w}|_{\mathbf{H}}^2 + (2\varepsilon_2 + a\varepsilon_3) I_{K(u)}^\mu(\mathbf{w}) \\ &\geq |\nabla u|_H^2 + \kappa\varepsilon_2 |\nabla \mathbf{w}|_H^2 \geq \varepsilon_0(|\nabla u|_H^2 + |\nabla \mathbf{w}|_H^2), \end{aligned}$$

where $\varepsilon_0 = \min\{1, \kappa\varepsilon_2\}$. Therefore, from (37) it follows that

$$E_\mu(t) \leq e^{(C_{10}\varepsilon_3/\varepsilon_0)T} E_\mu(0) + \frac{C_{11}\varepsilon_0}{C_{10}\varepsilon_3}(e^{(C_{10}\varepsilon_3/\varepsilon_0)T} - 1)$$

for all $t \in [0, T]$, where $C_{11} = C_{10} + 4\varepsilon_3 C_9$. Since $E_\mu(0) = (1+\varepsilon_1 d)|\nabla u_0|_H^2 + (\varepsilon_2\kappa + \varepsilon_3(a/2))|\nabla \mathbf{w}_0|_{\mathbf{H}}^2$, we conclude uniform estimates for the approximate solutions $\{\mathbf{w}_\mu, u_\mu\}$, that is, there is a positive constant R_0, depending only on the initial data and the quantities in assumptions H1–H5, such that

$$\begin{aligned} &|u_\mu'|_{L^2(0,T;H)} + |\mathbf{w}_\mu'|_{L^2(0,T;\mathbf{H})} + |\Delta u_\mu|_{L^2(0,T;H)} + \kappa^{1/2}|\Delta \mathbf{w}_\mu|_{L^2(0,T;\mathbf{H})} \\ &\quad + |\partial \mathbf{I}_{K(u_\mu)}^\mu(\mathbf{w}_\mu)|_{L^2(0,T;\mathbf{H})} + |\nabla u_\mu|_{L^\infty(0,T;H)} \\ &\quad + |\nabla \mathbf{w}_\mu|_{L^\infty(0,T;\mathbf{H})} + |I_{K(u_\mu)}^\mu(\mathbf{w}_\mu)|_{L^\infty(0,T)} \\ &\quad \leq R_0 \end{aligned} \tag{41}$$

for all $\mu \in (0, 1]$ and all $\kappa \in (0, \kappa_0]$.

On account of the uniform estimates (41) for the approximate solutions $\{\mathbf{w}_\mu, u_\mu\}$, it follows that there is a subsequence $\{\mu_n\}$ with $\mu_n \searrow 0$ such that

$$\begin{aligned} \mathbf{w}_{\mu_n} \longrightarrow \mathbf{w} \quad &\text{weakly in} \quad W^{1,2}(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \\ &\text{and weakly star in} \quad L^\infty(0,T;\mathbf{V}), \end{aligned} \tag{42}$$

$$(43)\quad u_{\mu_n} \longrightarrow u \quad \text{weakly in} \quad W^{1,2}(0,T;H) \cap L^2(0,T;H^2(\Omega)) \quad \text{and weakly star in} \quad L^\infty(0,T;V)$$

and

$$(44)\qquad \partial \mathbf{I}^{\mu_n}_{K(u_{\mu_n})}(\mathbf{w}_{\mu_n}) \longrightarrow \mathbf{z} \quad \text{weakly in} \quad L^2(0,T;\mathbf{H})$$

for some functions $\mathbf{w}$, u, $\mathbf{z}$, and moreover

$$(45)\qquad \{I^{\mu_n}_{K(u_{\mu_n})}(\mathbf{w}_{\mu_n})\} \quad \text{is bounded in} \quad L^\infty(0,T);$$

in these cases, by Ascoli-Arzelà's theorem it follows that

$$(46)\quad \mathbf{w}_{\mu_n} \longrightarrow \mathbf{w} \quad \text{in} \quad C([0,T];\mathbf{H}), \quad u_{\mu_n} \longrightarrow u \quad \text{in} \quad C([0,T];H),$$

and

$$(47)\qquad \begin{aligned} &\mathbf{J}_{u_{\mu_n}}(\mathbf{w}_{\mu_n}) \longrightarrow \mathbf{w} \quad \text{in} \quad C([0,T];\mathbf{H}), \\ &(\mathbf{J}_{u_{\mu_n}}(\mathbf{w}_{\mu_n}))' \longrightarrow \mathbf{w}' \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}), \end{aligned}$$

which show that

$$\mathbf{z} \in \partial \mathbf{I}_{K(u)}(\mathbf{w}) \quad \text{in} \quad L^2(0,T;\mathbf{H}), \qquad \mathbf{w} \in K(u) \quad \text{a.e. in} \quad Q.$$

Now, taking $\mu = \mu_n$ in (21)–(24) and passing to the limit in n, we see easily from (42)–(47), that the pair $\{\mathbf{w}, u\}$ is a solution of the system (1)–(4). $\square$

4.2 System (1)–(4) with $\kappa = 0$. In this section consider the system (1)–(4) with $\kappa = 0$, namely, the following system

$$(48)\qquad a\mathbf{w}_t + \partial \mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u) \quad \text{in} \quad Q,$$

$$(49)\qquad \mathbf{c} \cdot \mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u) \quad \text{in} \quad Q.$$

$$(50)\qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Sigma = (0,T) \times \partial\Omega,$$

$$\mathbf{w}(0,x) = \mathbf{w}_0(x), \quad u(0,x) = u_0(x) \quad \text{in} \quad \Omega. \tag{51}$$

Definition 4.2. A pair of functions $\{\mathbf{w}, u\}$ is called a solution of the system (48)–(51) if:

(i) $w_i \in L^\infty(0,T;L^\infty(\Omega)) \cap W^{1,2}(0,T;H)$, $i = 1, \dots, m$.

(ii) $u \in L^\infty(0,T;V \cap L^\infty(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;H)$.

(iii) $a\mathbf{w}_t + \partial \mathbf{I}_{K(u)}(\mathbf{w}) \ni \mathbf{F}(\mathbf{w}, u)$ in $\mathbf{H}$, almost everywhere in $(0,T)$.

(iv) $\mathbf{c} \cdot \mathbf{w}_t + du_t - \Delta u = h(\mathbf{w}, u)$ in H, almost everywhere in $(0,T)$.

(v) $(\partial u/\partial \nu) = 0$ in $L^2(\partial\Omega)$, almost everywhere in $(0,T)$.

(vi) $\mathbf{w}(0) = \mathbf{w}_0$, $u(0) = u_0$.

Theorem 4.3. *Suppose that assumptions* H1–H5 *are satisfied. Then the system* (48)–(51) *possesses a unique solution. Moreover,* $\mathbf{w} \in L^\infty(0,T;\mathbf{V})$.

Proof. (i) *Uniqueness of solutions.* The uniqueness proof is similar to the proof of [**7**, Theorem 2.3]. We present the details below.

Suppose that $\{\mathbf{w}^{(i)}, u^{(i)}\}$, $i = 1,2$, are two different solutions of the system (48)–(51) in the sense of Definition 4.2, and denote $\mathbf{w} = \mathbf{w}^{(1)} - \mathbf{w}^{(2)}$, $u = u^{(1)} - u^{(2)}$. Let us test the difference between the respective equations by any measurable selection $s_u \in \operatorname{sign}(u^{(1)} - u^{(2)})$. Noting that $-(\Delta(u^{(1)} - u^{(2)}), s_u) \geq 0$ almost everywhere in $(0,T)$, we obtain that

$$\begin{aligned}
&\left(\sum_{i=1}^{m} c_i (w_i^{(1)} - w_i^{(2)})'(t), s_u(t)\right) + d((u^{(1)} - u^{(2)})'(t), s_u(t)) \\
&\leq L\left\{\sum_{i=1}^{m} |(w_i^{(1)} - w_i^{(2)})(t)|_{L^1(\Omega)} + |(u^{(1)} - u^{(2)})(t)|_{L^1(\Omega)}\right\}, \\
&\qquad\qquad \text{a.e.} \quad t \in (0,T),
\end{aligned} \tag{52}$$

where we denote by $L > 0$ the common Lipschitz constant of $\mathbf{F}$ and h.

Put $F_j^{(i)} = F_j(\mathbf{w}^{(i)}, u^{(i)})$, $i = 1, 2$; $j = 1, \dots, m$. We claim that there exist $s_{w_j} \in L^\infty(Q)$ such that $s_{w_j} \in \operatorname{sign}(w_j^{(1)} - w_j^{(2)})$, $j = 1, \dots, m$ and

$$
\begin{aligned}
&a(w_j^{(1)} - w_j^{(2)})' s_{w_j} - (F_j^{(1)} - F_j^{(2)}) s_{w_j} \\
&\leq \operatorname{sign}(c_j) \left(a(w_j^{(1)} - w_j^{(2)})' s_u - (F_j^{(1)} - F_j^{(2)}) s_u \right), \\
&\text{a.e. in} \quad Q.
\end{aligned} \tag{53}
$$

In view of assumptions H1, H2, there are two cases to be considered here:

Case 1. $c_j > 0$ and f_{j*}, f_j^* are nondecreasing functions on R.

Case 2. $c_j < 0$ and f_{j*}, f_j^* are nonincreasing functions on R.

For Case 1, we directly refer to Theorem 2.3 of [**7**]. Case 2 is similar to the method presented in Theorem 2.3 of [**7**]. We give details for this case below.

We can take $s_{w_j} = -s_u$ on the subsets of $(t, x) \in Q$ where $\{w_j^{(1)} < w_j^{(2)},\ u^{(1)} > u^{(2)}\}$ or $\{w_j^{(1)} > w_j^{(2)},\ u^{(1)} < u^{(2)}\}$. Let us check the case when $\{w_j^{(1)} < w_j^{(2)},\ u^{(1)} \leq u^{(2)}\}$. In this case we have that $w_j^{(1)} < f_j^*(u^{(1)})$ (otherwise $w_j^{(1)} = f_j^*(u^{(1)}) \geq f_j^*(u^{(2)}) \geq w_j^{(2)}$, which yields a contradiction) and $w_j^{(2)} > f_{j*}(u^{(2)})$ (otherwise $w_j^{(2)} = f_{j*}(u^{(2)}) \leq f_{j*}(u^{(1)}) \leq w_j^{(1)}$, which is a contradiction). Since

$$
\begin{gathered}
(a w_j^{(i)\prime} - F_j^{(i)})(w_j^{(i)} - z_j) \leq 0 \quad \text{for all} \quad z_j \in [f_{j*}(u^{(i)}), f_j^*(u^{(i)})], \\
i = 1, 2, \quad \text{a.e. in} \quad Q,
\end{gathered}
$$

by suitably chosen test numbers z_j, we conclude that

$$
a(w_j^{(1)} - w_j^{(2)})' - (F_j^{(1)} - F_j^{(2)}) \geq 0 \quad \text{a.e. in} \quad \{w_j^{(1)} < w_j^{(2)},\ u^{(1)} \leq u^{(2)}\}.
$$

Therefore (53) is fulfilled since $-s_{w_j} = 1 \geq s_u \in [-1, 1]$ in $\{w_j^{(1)} < w_j^{(2)},\ u^{(1)} \leq u^{(2)}\}$.

In the set $\{w_j^{(1)} > w_j^{(2)},\ u^{(1)} \geq u^{(2)}\}$ we obtain the reverse inequalities and again conclude that (53) holds true.

Multiplying inequality (53) by c_j and noting that $|s_u| \leq 1$, $|s_{w_j}| \leq 1$, we have that

$$c_j(w_j^{(1)} - w_j^{(2)})' s_u \geq -2\frac{|c_j|}{a}|F_j^{(1)} - F_j^{(2)}| + |c_j|(w_j^{(1)} - w_j^{(2)})' s_{w_j}$$
$$\text{a.e. in} \quad Q,\ j = 1, \dots, m.$$

Therefore,

(54)
$$\left(\sum_{j=1}^m c_j(w_j^{(1)} - w_j^{(2)})', s_u\right) \geq -2\frac{L}{a}\left(\sum_{j=1}^m |c_j||w_j|_{L^1(\Omega)} + |u|_{L^1(\Omega)}\right)$$
$$+ \left(\sum_{j=1}^m |c_j|(w_j^{(1)} - w_j^{(2)})', s_{w_j}\right)$$
$$\text{a.e. in} \quad (0,T).$$

In view of (52), we conclude from (54) that

$$\frac{d}{dt}\left\{\sum_{j=1}^m |c_j||w_j|_{L^1(\Omega)} + d|u|_{L^1(\Omega)}\right\} \tag{55}$$
$$\leq L\left(1 + \frac{2}{a}\right)\left\{\sum_{j=1}^m |c_j||w_j|_{L^1(\Omega)} + |u|_{L^1(\Omega)}\right\}, \quad \text{a.e. in} \quad (0,T).$$

Integrating (55) with respect to t we obtain the uniqueness of solutions.

(ii) *Existence of solutions.* Let us note that the validity of the estimates for the approximate solutions from subsection 4.1 extends to the constructed solution $\{\mathbf{w}^{(\kappa)}, u^{(\kappa)}\}$ of the system (1)–(4) whenever $0 < \kappa < \kappa_0$, cf. (37)–(40). Therefore, in view of the uniqueness part and arguing as in [**9**] it could be shown that $\{\mathbf{w}^{(\kappa)}, u^{(\kappa)}\}$ converges in a suitable sense to the unique solution $\{\mathbf{w}, u\}$ of the system (1)–(4) with $\kappa = 0$. Moreover, $\mathbf{w}$ satisfies $\mathbf{w} \in L^\infty(0,T;\mathbf{V})$. $\square$

4.3 Local solutions.

Remark 1. If we suppose in Theorems 4.1 and 4.3 that the functions $\mathbf{F}, h$ are locally Lipschitz continuous functions on $R^m \times R$ (without any growth conditions), existence of local solutions of the respective systems can be proved, namely reasoning analogously as above, the following theorems hold true:

Theorem 4.4. *Suppose that assumptions* H1–H4 *are satisfied. Then there exist a positive number* T_0 *(depending only on* $|\mathbf{w}_0|_\infty$ *and* $|u_0|_\infty$*) as well as a constant* $\kappa_0 > 0$ *such that for each* $0 < \kappa < \kappa_0$ *the system* (1)–(4) *possesses at least one solution on* $[0, T_0] \times \Omega$.

Theorem 4.5. *Suppose that assumptions* H1–H4 *are satisfied. Then there exists a positive number* T_0 *(depending only on* $|\mathbf{w}_0|_\infty$ *and* $|u_0|_\infty$*) such that the system* (48)–(51) *possesses a unique solution on* $[0, T_0] \times \Omega$.

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