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SMOOTHNESS PROPERTIES OF QUASI-MEASURES

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ABSTRACT. We construct several examples of simple quasi-measures that show that the strong and weak smoothness properties for quasi-measures proposed by Boardman are distinct. These examples also show that in general, only the obvious implications hold between these properties. We describe a general construction of product quasi-measures that yields further examples that are not simple.

We also provide characterizations of the strong smoothness properties in terms of the action of the induced Borel quasi-measure on the Stone-Čech compactification and show that the dimension of the Stone-Čech remainder influences the smoothness properties of the Baire quasi-measures on X. Finally, we explore the effects of different topological properties on the various classes of smooth quasi-measures.

1. Introduction. Quasi-measures were first studied by Johan Aarnes [**1**] on compact spaces as set functions that represent functionals which are linear on singly generated subalgebras of the collection of real-valued continuous functions. They are generalizations of the regular measures which appear in the Riesz representation theorem. Later, Boardman [**5, 6**], generalized Aarnes' results to the case where the underlying space is completely regular and asked several questions relating to the smoothness properties of quasi-measures in this context. The goal of this paper is to answer those questions. Further information about quasi-measures in this setting can be found in [**4**] where the representation theorem of Boardman is proved in a cleaner way and in [**9**] where some topological properties of the collection of quasi-measures are addressed.

All spaces X under consideration are assumed to be completely regular. A *Baire quasi-measure* on X is a real-valued, finite, non-negative set function μ defined on $\mathcal{A} = \{A \subseteq X : A$ is either a zero set or cozero set of $X\}$ that satisfies the following axioms:

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1. If $A \in \mathcal{A}$, then $\mu(A) + \mu(X \backslash A) = \mu(X)$.

2. If $A_1, A_2 \in \mathcal{A}$ and $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$.

3. If $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 = \varnothing$, and $A_1 \cup A_2 \in \mathcal{A}$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

4. If U is a cozero set of X, then $\mu(U) = \sup\{\mu(Z) : Z \subseteq U$ and Z is a zero set$\}$.

Similarly, a *Borel quasi-measure* is a set function defined on $\{A \subseteq X : A$ is a closed or open subset of $X\}$ that satisfies the same properties for open and closed sets. As described in [**13**], every Baire quasi-measure μ on X induces a Borel quasi-measure $\bar{\nu}$ on βX, the Stone-Čech compactification of X. A quasi-measure is said to be *simple* if it takes only the values 0 and 1.

It is crucial to note that a Baire quasi-measure need not be the restriction of a Baire measure to $\mathcal{A}$. Indeed, as shown in [**13**], a Baire quasi-measure μ extends to a Baire measure on $\mathcal{A}$ if and only if μ is subadditive on $\mathcal{A}$. Moreover, if μ is not subadditive, then it cannot be extended to a monotone additive set function on the collection of differences of elements of $\mathcal{A}$.

These classes of quasi-measures were introduced by Boardman in [**5, 6**], and studied further by Wheeler in [**13**]. Boardman also generalized the notions of σ-smoothness, τ-smoothness, and tightness for measures as in Varadarajan [**12**].

Definition 1.1. Let μ be a Baire quasi-measure on X.

1. μ is *σ-smooth* if whenever $\{Z_n\}$ is a sequence of zero sets with $Z_n \searrow \varnothing$, then $\mu(Z_n) \to 0$.

2. μ is *strongly σ-smooth* if whenever $\{Z_n\}$ is a sequence of zero sets, Z is a zero set, and $Z_n \searrow Z$, then $\mu(Z_n) \to \mu(Z)$.

3. μ is *τ-smooth* if whenever $\{Z_\alpha\}$ is a net of zero sets with $Z_\alpha \searrow \varnothing$, then $\mu(Z_\alpha) \to 0$.

4. μ is *strongly τ-smooth* if whenever $\{Z_\alpha\}$ is a net of zero sets, Z is a zero set, and $Z_\alpha \searrow Z$, then $\mu(Z_\alpha) \to \mu(Z)$.

5. μ is *tight* if for every $\varepsilon > 0$ there is a compact $K \subseteq X$ such that $\bar{\nu}(K) > \mu(X) - \varepsilon$.

6. μ is *strongly tight* if for every cozero $U \subseteq X$ and $\varepsilon > 0$, there is a compact $K \subseteq U$ such that $\bar{\nu}(K) > \mu(U) - \varepsilon$.

There is an equivalent internal definition of tight quasi-measure. We use the external version because its connection with 6 (which Boardman did not consider) is more apparent. Notice that $6 \Rightarrow 4 \Rightarrow 2$ and that $5 \Rightarrow 3 \Rightarrow 1$. Also, $2 \Rightarrow 1$, $4 \Rightarrow 3$ and $6 \Rightarrow 5$. For measures, we have $1 \Leftrightarrow 2$, $3 \Leftrightarrow 4$ and $5 \Leftrightarrow 6$.

In this paper we first construct several examples that show that the "weak" smoothness properties (1), (3) and (5) are distinct from the corresponding strong versions and that, moreover, only the above implications hold in general. We provide additional examples by establishing a general technique for constructing product quasi-measures. We then prove theorems that characterize the strong smoothness properties in terms of the action of the corresponding Borel measure on the Stone-Čech compactification and (for strong τ-smoothness) in terms of the behavior of the corresponding quasi-linear functional on $C(X)$ (this proof makes use of an interesting generalization of the Monotone Convergence Theorem). We also show that the dimension of the Stone-Čech remainder influences the smoothness properties of the quasi-measures on X. Finally, we explore the effects of different topological properties on the various classes of smooth quasi-measures.

We will need the following characterizations of σ- and τ-smoothness, which are due to Boardman and generalize the results for measures obtained by Knowles in [**10**].

Theorem 1.2. *Let μ be a Baire quasi-measure on X and $\bar{\nu}$ the induced Borel quasi-measure on βX.*

1. *μ is σ-smooth if and only if whenever Z is a zero set of βX and $Z \cap X = \varnothing$, then $\bar{\nu}(Z) = 0$.*

2. *μ is τ-smooth if and only if whenever F is a compact subset of βX and $F \cap X = \varnothing$, then $\bar{\nu}(F) = 0$.*

In Theorem 3.1 below, we will provide similar characterizations of the strong smoothness properties.

2. Examples. Most of our examples will make use of the long line L or its one-point compactification $L \oplus 1$. Recall that ω_1 is the least uncountable ordinal number and that L is the connected space obtained by inserting a copy of the unit interval $(0,1)$ between each $\alpha \in \omega_1$ and its successor. In the natural way, L is also linearly ordered, so we can use interval notation to describe subsets of L.

The first examples we construct below will be simple quasi-measures. To construct a simple Baire quasi-measure μ in a given space X, we generally use Aarnes' method of solid set functions, with which we assume that the reader is familiar (otherwise, see [**3**]). We will begin by defining a solid set function ν on a closed, connected and locally connected subset of βX. This extends to a quasi-measure on the closed set, which we construe as a Borel quasi-measure $\bar{\nu}$ on all of βX. We then take μ to be the Baire quasi-measure induced by $\bar{\nu}$.

We will also take advantage of the compactness of the space of all simple Borel quasi-measures on βX to obtain a Borel quasi-measure $\bar{\nu}$ on βX (and the corresponding Baire quasi-measure μ on X) as a w^*-limit point of a net of quasi-measures $\{\bar{\nu}_\alpha\}$ defined in the above manner. Explicitly, let βX^* denote the collection of all simple Borel quasi-measures on βX. The w^*-topology on βX^* can be described as follows. For each open $U \subseteq \beta X$, let $U^* = \{\bar{\nu} \in \beta X^* : \bar{\nu}(U) = 1\}$. Topologize βX^* by using the U^*'s as subbasic open sets. Thus, if $\{\bar{\nu}_\alpha\}$ is a net in βX^*, then $\bar{\nu}$ is a w^*-limit of $\{\bar{\nu}_\alpha\}$ if and only if whenever $U_1, \ldots, U_n$ are open in βX and $\bar{\nu}(U_i) = 1$ for $i = 1, \ldots, n$, then there is an α such that $\bar{\nu}_\beta(U_i) = 1$ for all $\beta \geq \alpha$ and $i = 1, \ldots, n$. See the paper of Aarnes [**2**] for more details.

These procedures are described in some detail in our first example.

Example 2.1. A Baire quasi-measure μ that is not σ-smooth. This is essentially Example 7.3 of [**6**]. Let $X = [0,1] \times (0,1]$. For each $n \geq 2$, define a Borel quasi-measure $\bar{\nu}_n$ on βX as follows. Set $F_n = [0,1] \times \{1/n\}$ and $p = (1/2, 1)$. Define a solid set function ν_n on the closed solid, i.e., connected and co-connected, subsets A of $X_n = [0,1] \times [1/n, 1]$ by

$$\nu_n(A) = \begin{cases} 0 & \text{if } A \cap F_n = \varnothing, \\ 1 & \text{if } F_n \subseteq A, \\ 1 & \text{if } p \in A \text{ and } A \cap F_n \neq \varnothing. \end{cases}$$

Then ν_n determines a Borel quasi-measure on X_n (see [**3**]) and hence a Borel quasi-measure $\bar{\nu}_n$ on βX (defined by $\bar{\nu}_n(F) = \nu_n(F \cap X_n)$). Let $\bar{\nu}$ be a w^*-limit of the family $\{\bar{\nu}_n : n \geq 2\}$, and let μ be the Baire quasi-measure on X corresponding to $\bar{\nu}$. For each $n \in N$, let $Z_n = [0,1] \times (0, 1/n]$. Then each Z_n is a zero set and $\mu(Z_n) = 1$, but $\cap Z_n = \varnothing$, so μ is not σ-smooth.

Alternatively, notice that since $\beta X \backslash X$ is a compact G_δ in βX, it is a zero set in βX. We claim that $\bar{\nu}(\beta X \backslash X) = 1$. Otherwise, there is a compact K contained in X with $\bar{\nu}(K) = 1$, hence an open $U \subseteq \beta X$ with $\mathrm{cl}_{\beta X} U \subseteq X$ and $\bar{\nu}(U) = 1$. But eventually $\bar{\nu}_n(U) = 0$, contradicting the fact that $\bar{\nu}$ is a w^*-limit of the $\{\bar{\nu}_n\}$. Thus, by Theorem 1.2, μ is not σ-smooth.

In the above example, we call $\bar{\nu}_n$ the Aarnes measure determined by p and F_n. In the sequel, we will define Aarnes measures by simply giving the appropriate point and closed subset of the space in question. The reader should note that it is not the case that in every space every combination of point and closed set can be used to define an Aarnes measure; there are definite topological restrictions that apply. All of the spaces on which we define Aarnes measures will have the property that the union of disjoint closed co-connected sets is co-connected; this condition suffices for the construction of Aarnes measures. For example, the square of the unit interval has this property, while the one-holed annulus does not. For more details and examples of the topological difficulties that can arise (and methods for avoiding them), see [**8**].

Example 2.2. A Baire quasi-measure μ that is σ-smooth, but neither τ- nor strongly σ-smooth. Let $X = L \times (0, 1]$. Notice that, because the restriction of every real-valued continuous function to a zero set of the form $L \times \{r\}$ is eventually constant, we can consider $\{\omega_1\} \times (0, 1]$ to be a subset of βX in the natural way.

For $n \geq 2$, set $X_n = (L \oplus 1) \times [1/n, 1]$, and define a Borel quasi-measure ν_n on X_n by using $p = (0, 1)$ and $F_n = ((L \oplus 1) \times \{1/n\}) \cup (\{\omega_1\} \times [1/n, 1])$. As before, this gives a net $\{\bar{\nu}_n\}$ of Borel quasi-measures on βX; let $\bar{\nu}$ be a w^*-limit of this net, and take μ to be the corresponding Baire quasi-measure on X.

Now, X is locally compact, so $\beta X \backslash X$ is closed in βX and $\bar{\nu}(\beta X \backslash X) = 1$, so μ is not τ-smooth. On the other hand, any zero set contained in $\beta X \backslash X$ is disjoint from $\{\omega_1\} \times (0,1]$, and hence gets $\bar{\nu}$-measure 0, so μ is σ-smooth. Set $Z = [1,\omega_1) \times (0,1]$ and (for $n \geq 2$) $Z_n = ([0,1] \times (0,1/n]) \cup Z$. Then $Z_n \searrow Z$, each $\mu(Z_n) = 1$, but $\mu(Z) = 0$, so μ is not strongly σ-smooth.

Example 2.3. A Baire quasi-measure μ that is tight but not strongly σ-smooth. Let $X = (L \times (0,1]) \cup \{(\omega_1, 1)\}$. Since X and the space from the previous example have the same Stone-Čech compactification, we can use the $\bar{\nu}$ from the previous example to induce a Baire quasi-measure μ on X. For the same reasons as before, μ is not strongly σ-smooth. Since $(L \oplus 1) \times \{1\}$ is a compact subset of X with $\bar{\nu}$-measure 1, μ is tight.

Example 2.4. A Baire quasi-measure μ that is strongly σ-smooth but not τ-smooth. Let $X = (L \oplus 1) \times L$. Then $\beta X = (L \oplus 1) \times (L \oplus 1)$ is connected and locally connected, so we can define an Aarnes measure $\bar{\nu}$ on βX by setting $F = (L \oplus 1) \times \{\omega_1\}$ and $p = (0,0)$. Let μ be the induced quasi-measure on X. Because X is pseudo-compact, every Baire quasi-measure on X is strongly σ-smooth (see [**6**]); however, $\bar{\nu}(\beta X \backslash X) = \bar{\nu}(F) = 1$, so μ is not τ-smooth.

This repairs an error in Example 7.4 of [**6**]. Indeed, as our Theorem 3.5 shows, every proper Baire quasi-measure on the space from that example is strongly tight.

Example 2.5. A Baire quasi-measure that is tight and strongly σ-smooth, but not strongly τ-smooth. Set $X = ((L \oplus 1) \times L) \cup \{(0, \omega_1)\}$. As above, $\beta X = (L \oplus 1) \times (L \oplus 1)$; we define $\bar{\nu}$ on βX as above and take μ to be the induced Baire quasi-measure on X. Then $\{0\} \times (L \oplus 1)$ is a compact subset of X with $\bar{\nu}$-measure 1, so μ is tight. The space X is psuedo-compact, so μ is strongly σ-smooth. If $Z = \{0\} \times [1, \omega_1]$ and $Z_\alpha = Z \cup [(L \oplus 1) \times [\alpha, \omega_1)]$, then $Z_\alpha \searrow Z$, each $\mu(Z_\alpha) = 1$, but $\mu(Z) = 0$, so μ is not strongly τ-smooth.

The next two examples make use of a special pathological subset of the plane.

Definition 2.6. A set $B \subseteq [0,1]^2$ is a *Bernstein set* if whenever $K \subseteq [0,1]^2$ is uncountable and compact, then $B \cap K \neq \varnothing$ and $K \backslash B \neq \varnothing$.

Under the assumption of the Axiom of Choice, Bernstein sets can be constructed by an easy transfinite induction. By definition, every compact subset of a Bernstein set is countable.

Example 2.7. A Baire quasi-measure μ that is strongly τ-smooth but not tight. Let B be a Bernstein set in $[0,1]^2$. Since $[0,1]^2 \backslash B$ is also Bernstein, we can assume that $(1/2, 1/2) \notin B$. Let $X = (L \times [0,1]^2) \cup (\{\omega_1\} \times B)$. Then $\beta X = (L \oplus 1) \times [0,1]^2$. Let $\bar{\nu}$ be an Aarnes measure on βX with $p = (\omega_1, (1/2, 1/2))$ and $F = \{\omega_1\} \times \partial$, where ∂ is the boundary of $[0,1]^2$. Let μ be the induced Baire quasi-measure on X.

Because $\beta X \backslash X$ is zero-dimensional, Theorem 3.5 shows that μ is strongly τ-smooth. To see that μ is not tight, suppose that $K \subseteq X$ is compact. Then because B is Bernstein, $K \cap (\{\omega_1\} \times [0,1]^2)$ is at most countable, so there is a compact connected $H \subseteq \{\omega_1\} \times [0,1]^2$ with $H \cap K = \varnothing$, $p \in H$, and $H \cap \partial \neq \varnothing$. Thus, $\bar{\nu}(H) = 1$, so $\bar{\nu}(K) = 0$, hence μ is not tight.

Example 2.8. A Baire quasi-measure μ that is strongly τ-smooth and tight but not strongly tight. Set $X = (L \times [0,1]^2) \cup (\{\omega_1\} \times B) \cup F$. Then as above, $\beta X = (L \oplus 1) \times [0,1]^2$, so we can use the $\bar{\nu}$ from the previous example to induce a Baire quasi-measure μ on X. As before, μ is strongly τ-smooth. Because $F \subseteq X$ and $\bar{\nu}(F) = 1$, μ is tight. To see that μ is not strongly tight, set $U = [(L \oplus 1) \times ([1/3, 2/3] \times [0, 2/3])] \cap X$. Then U is cozero and $\mu(U) = 1$. If $K \subseteq U$ is compact, then $K \cap (\{\omega_1\} \times [0,1]^2)$ is again countable, so there is a compact connected $H \subseteq \{\omega_1\} \times [0,1]^2$ with $H \cap K = \varnothing$, $p \in H$, and $H \cap F \neq \varnothing$. Again, this gives $\bar{\nu}(H) = 1$ and $\bar{\nu}(K) = 0$, so μ is not strongly tight.

The next example requires a generalized Bernstein set in the square of the long line.

Example 2.9. A Baire quasi-measure μ that is strongly σ-smooth, τ-smooth, not strongly τ-smooth, and not tight. Let $Y = (L \times L) \cup \{(0, \omega_1)\}$. We call a compact subset $K \subseteq Y$ *thin* if $K \cap [0, \alpha]^2$ is always countable. Notice that a thin subset of Y is zero-dimensional.

Lemma 2.10. *There is a $B \subseteq Y$ such that whenever $K \subseteq Y$ is compact and not thin, then $B \cap K \neq \varnothing$ and $K \backslash B \neq \varnothing$.*

Proof. If K is not thin, then some $K \cap [0, 1]^2$ is uncountable, and hence has cardinality $\mathbf{c}$ (the cardinality of the set of real numbers). Let $\mathcal{K} = \{K \subseteq Y : K$ is compact and not thin$\}$. Then $|\mathcal{K}| = \mathbf{c}$, so we can enumerate $\mathcal{K}$ as $\{K_\alpha : \alpha < \mathbf{c}\}$. We construct B by transfinite induction on $\alpha < \mathbf{c}$. Our induction hypothesis is that, for every $\beta < \alpha$, we have chosen a_β and b_β such that $a_\beta \neq b_\beta$ and $a_\beta, b_\beta \in K_\beta$. Given $\alpha < \mathbf{c}$, take $a_\alpha \neq b_\alpha$ in $K_\alpha \backslash (\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\})$. Then $\{b_\alpha : \alpha < \mathbf{c}\}$ is as desired. $\square$

Let B be as in the lemma; we can assume that $(0, 0) \notin B$ and that $(0, \omega_1) \in B$. Notice that any compact subset of B is thin. Let L' be the long line on ω_2, i.e., insert a copy of $(0, 1)$ between each ordinal $\alpha \in \omega_2$ and its successor, and set $X = [L' \times (L \oplus 1)^2] \cup (\{\omega_2\} \times B)$. Then $\beta X = (L' \oplus 1) \times (L \oplus 1)^2$. Set $p = (\omega_2, (0, 0))$ and $F = \{\omega_2\} \times [(L \oplus 1) \times \{\omega_1\}]$. Let $\bar{\nu}$ be an Aarnes measure on βX with p and F, and let μ be the induced Baire quasi-measure on X. Then μ is τ-smooth because any compact subset of $\beta X \backslash X$ gets $\bar{\nu}$-measure 0. Let $K \subseteq X$ be compact, then $K \cap (\{\omega_2\} \times Y)$ is thin (as a subset of Y) and does not contain p, so we can find a compact connected H with $p \in H$, $H \cap F \neq \varnothing$, and $H \cap K = \varnothing$. As before, this implies that μ is not tight. To see that μ is not strongly τ-smooth, set $Z = ((L' \oplus 1) \times (\{0\} \times [1, \omega_1]) \cap X$ and $Z_{\alpha\beta} = Z \cup [([\beta, \omega_2] \times (L \oplus 1) \times [\alpha, \omega_1]) \cap X$ for each $\alpha < \omega_1$ and $\beta < \omega_2$. Order the pairs (α, β) by $(\alpha', \beta') \leq (\alpha, \beta)$ if and only if $\alpha' \leq \alpha$ and $\beta' \leq \beta$. Then $Z_{\alpha\beta} \searrow Z$, each $\mu(Z_{\alpha\beta}) = 1$, but $\mu(Z) = 0$, so μ is not strongly τ-smooth.

Because all of the examples constructed above are proper simple quasi-measures, the following question is natural. (Recall that a Baire (Borel) quasi-measure μ is *proper* if whenever λ is a Baire (Borel) measure and $\lambda \leq \mu$, then $\lambda = 0$.)

Question 1. Suppose every proper simple Baire quasi-measure on X satisfies a particular smoothness property. Does every proper Baire quasi-measure satisfy the same property?

In many of the above situations, the space βX is not locally connected. As there is no construction theorem presently available for spaces that are not locally connected, in these cases we have had to resort to limit procedures to obtain the quasi-measures we are interested in. However, it seems likely that these examples can be construed as Aarnes measures.

Question 2. Under what conditions can Aarnes measures be defined directly in compact spaces that are not locally compact?

Our next examples utilize a procedure for constructing product quasi-measures that may be of independent interest. The procedure generalizes the method of [**7**] and is applicable in the situations first considered in [**5**]. The examples we obtain are alternatives for Examples 2.7 and 2.9 that are not simple.

We assume that the reader is familiar with [**7**]. Let μ and ν be Baire quasi-measures on spaces X and Y, respectively, with either μ a simple quasi-measure or ν a measure. Let ρ and η be the corresponding quasi-linear functionals on $C(X)$ and $C(Y)$. To define the desired product quasi-measure $\mu \times_l \nu$ on $X \times Y$, we show how to define the corresponding quasi-linear functional ζ on $C(X \times Y)$.

Fix $y \in Y$ and apply ρ to the function $x \mapsto f(x, y)$ to obtain a value $T_\rho(f)(y)$. If the function $T_\rho(f) : Y \to R$ is continuous on Y, we can then apply η to obtain the value of the product functional ζ at f (quasi-linearity follows from the assumption that either μ is simple or ν is a measure). Because integration with respect to a quasi-measure is continuous, the continuity of $T_\rho(f)$ depends essentially on the continuity of the map $y \mapsto f(\cdot, y)$ from Y to $C(X)$. The following well-known lemma gives a sufficient condition for this to occur.

Lemma 2.11. *With notation as above, if the projection $(x, y) \mapsto y$ is a closed map from $X \times Y$ to Y, then the map $y \mapsto f(\cdot, y)$ is continuous.*

Thus, if either X is compact or X is countably compact and Y is first countable, then the conclusion of the lemma will hold. This is the situation in the next two examples.

Example 2.12. A Baire quasi-measure $\mu \times_l \nu$ that is strongly τ-smooth but not tight. This is essentially Example 7.5 of [**6**]. Let $X = [0,1]^2$, and let μ be an Aarnes measure on X with $p = (1/2, 1/2)$ and closed set ∂, the boundary of X. Let $Y = [0,1]$ with the Sorgenfrey topology (basic open neighborhoods of $r \in Y$ are half-open intervals $[r, r+\epsilon)$). Let ν be Lebesgue measure on Y. We are in the situation of Lemma 2.11, so we can construct $\mu \times_l \nu$.

By Theorem 3.7 below, to show that $\mu \times_l \nu$ is strongly τ-smooth, it suffices to show that $\mu \times_l \nu$ is strongly σ-smooth. This follows easily from the facts that X is compact and that ν satisfies the Lebesgue dominated convergence theorem. To show that $\mu \times_l \nu$ is not tight, suppose that $K \subseteq X \times Y$ is compact. Then the projection $\pi_2(K)$ of K onto Y is countable, so $(\mu \times_l \nu)(K) \leq (\mu \times_l \nu)(X \times \pi_2(K)) = \mu(X)\nu(\pi_2(K)) = 1 \cdot 0 = 0$.

Example 2.13. A Baire quasi-measure $\mu \times_l \nu$ that is strongly σ-smooth and τ-smooth but neither tight nor strongly τ-smooth. Let X and μ be as in Example 2.5, and let Y and ν be as in the previous example. Because X is countably compact and Y is first countable, Lemma 2.11 applies and we can construct $\mu \times_l \nu$. By considering the functionals associated with μ and ν and the definition of the product quasi-measure, one sees that $\mu \times_l \nu$ is strongly σ-smooth and τ-smooth. To see that $\mu \times_l \nu$ is not strongly τ-smooth, let Z and Z_α be as in Example 2.5, set $W = Z \times [0,1]$ and $W_\alpha = Z_\alpha \times [0,1]$, then apply $\mu \times_l \nu$. The fact that $\mu \times_l \nu$ is not tight follows as in the previous example.

As Boardman remarks, it is easy to see that a strongly σ-smooth set function that satisfies the first three quasi-measure axioms also satisfies the fourth, and hence is a Baire quasi-measure. Our final example shows that strong σ-smoothness is essential for this result, even if the underlying space is compact.

Example 2.14. A simple σ-smooth set function ν on a compact X that satisfies (1)–(3) of the definition of quasi-measure but not (4). Let $X = [0,1]^2$. Define ν first on cozero sets by setting $\nu(U) = 1$ if U contains a punctured neighborhood of $p = (1/2, 1/2)$ and $\nu(U) = 0$ otherwise. That is, $\nu(U) = 1$ if and only if $U \cup \{p\}$ is a neighborhood of p. Extend ν to zero sets by setting $\nu(Z) = 1 - \nu(X \backslash Z)$. Thus, $\nu(Z) = 1$ if and only if $X \backslash Z$ doesn't contain a punctured neighborhood of p.

It is easy to see that ν is σ-smooth and satisfies (1)–(3). However, $\nu(X \backslash \{p\}) = 1$, and every zero set $Z \subseteq X \backslash \{p\}$ gets ν-measure 0, so ν does not satisfy (4).

3. Theorems. Our first result gives characterizations of the strong σ- and τ-smoothness properties in terms of the action of the induced Borel quasi-measure $\bar{\nu}$ on βX.

Theorem 3.1. *Let μ be a Baire quasi-measure on X, with $\bar{\nu}$ the corresponding Borel quasi-measure on βX.*

1. *μ is strongly σ-smooth if and only if whenever $Z \subseteq \beta X$ is a zero set, we have $\mu(Z \cap X) = \bar{\nu}(Z)$.*

2. *μ is strongly τ-smooth if and only if whenever $F \subseteq \beta X$ is closed and $F \cap X$ is a zero set in X, we have $\mu(F \cap X) = \bar{\nu}(F)$.*

Proof. To show (1), we first assume that μ is strongly σ-smooth. Suppose $f : \beta X \to [0,1]$ is continuous and $Z = f^{-1}(0)$. Fix $n \in N$, and let H be any zero set in X such that $H \subseteq X \backslash f^{-1}[0, 1/n] = f^{-1}(1/n, 1] \cap X$. Then $(\mathrm{cl}_{\beta X} H) \cap f^{-1}[0, 1/n] = \varnothing$, so we have $\mu(H) = \bar{\nu}(\mathrm{cl}_{\beta X} H) \le 1 - \bar{\nu}(f^{-1}[0, 1/n])$. Taking the supremum over all such zero sets of X contained in $f^{-1}(1/n, 1] \cap X$ gives $\mu(f^{-1}(1/n, 1] \cap X) \le 1 - \bar{\nu}(f^{-1}[0, 1/n])$, so $\bar{\nu}(Z) \le \bar{\nu}(f^{-1}[0, 1/n]) \le 1 - \mu(f^{-1}(1/n, 1] \cap X) = \mu(f^{-1}[0, 1/n] \cap X)$. Now, because μ is strongly σ-smooth, we have $\mu(f^{-1}[0, 1/n] \cap X) \to \mu(Z \cap X)$, so $\bar{\nu}(Z) \le \mu(Z \cap X) \le \bar{\nu}(Z)$ (the final inequality is true in general), so $\bar{\nu}(Z) = \mu(Z \cap X)$.

Now suppose that $\bar{\nu}(Z) = \mu(Z \cap X)$ whenever Z is a zero set in βX. Suppose we have a family $\{W_n\}$ of zero sets in X such that $W_n \searrow W$. Set $Z_n = \mathrm{cl}_{\beta X} W_n$ and $Z = \mathrm{cl}_{\beta X} W$. Then $\mu(W_n) = \bar{\nu}(Z_n)$ and $\bar{\nu}(Z_n) \to \bar{\nu}(\cap Z_n)$, see [**2**]. But $(\cap Z_n) \cap X = Z \cap X = W$, so $\bar{\nu}(\cap Z_n) = \mu(W)$, and so μ is σ-smooth.

To prove (2), we first assume that μ is strongly τ-smooth. Suppose F is closed in βX and that $Z = F \cap X$ is a zero set in X. Because βX is completely regular, there is a directed family $\{Z_\alpha\}$ of zero sets in βX such that $F = \cap Z_\alpha$. Because μ is strongly σ-smooth, the first part of this theorem implies that $\mu(Z_\alpha \cap X) = \bar{\nu}(Z_\alpha)$. We also have $\bar{\nu}(Z_\alpha) \to \bar{\nu}(F)$ and $(\cap Z_\alpha) \cap X = F \cap X = Z$. Thus, by strong τ-smoothness of μ, we have $\mu(Z_\alpha \cap X) \to \mu(Z)$. Putting all of this together yields $\bar{\nu}(F) = \mu(Z)$.

The proof of the converse is very similar to the proof of the converse of part (1) and is left to the reader. $\square$

We will next establish a characterization of strong τ-smoothness in terms of the action of the corresponding quasi-linear functional on $C(X)$. Similar characterizations for σ-, strong σ-, and τ-smoothness were obtained by Boardman. We will require the following lemma, which may be of independent interest. We say that a net $\{f_\alpha\}$ is *monotone* if $\alpha < \beta$ implies $f_\beta(x) \leq f_\alpha(x)$ for all x in the domain of f.

Lemma 3.2. *Let $\{f_\alpha\}$ be a monotone net of decreasing functions, with each $f_\alpha : [a,b] \to [0,1]$. Suppose that $f_\alpha \searrow f$ pointwise, with $f : [a,b] \to [0,1]$. Then $\int_a^b f_\alpha \, dx \to \int_a^b f \, dx$.*

Proof. By monotonicity, f has only countably many points of discontinuity. By extending the interval $[a,b]$ and the definitions of the functions, we can assume that a and b are points of continuity for f. Let $\varepsilon > 0$ be given, then there only finitely many points, say $x_1, \ldots, x_m$, for which $f(x_k^-) - f(x_k^+) \geq \varepsilon/2$. Pick $y_1, z_1, y_2, z_2, \ldots, y_m, z_m$ points of continuity for f such that $y_1 < x_1 < z_1 < y_2 < x_2 < z_2 < \cdots < y_m < x_m < z_m$ and $z_k - y_k < (\varepsilon/2m)$, for $k = 1, \ldots, m$.

For each $k = 1, \ldots, m$, pick points $x_{k0}, \ldots, x_{kl}$, where l depends on k and $z_k = x_{k0} < x_{k1} < \cdots < x_{kl} = y_{k+1}$ and for each j,

$$f(x_{kj}) - f(x_{k(j+1)}) < \frac{\varepsilon}{4(b-a)}.$$

Finally, fix α_0 such that for all k and j, $f_{\alpha_0}(x_{kj}) - f(x_{kj}) < (\varepsilon/4(b-a))$. From now on, we assume $\alpha \geq \alpha_0$.

Now, for $k = 1, \ldots, m$, we have $\int_{y_k}^{z_k} (f_\alpha(x) - f(x))\, dx \le 1 \cdot (\varepsilon/2m)$, so if $A = \cup_{k=1}^{m} [y_k, z_k]$, then

$$\int_A (f_\alpha - f)\, dx < \frac{\varepsilon}{2m} \cdot m = \frac{\varepsilon}{2}.$$

If x is such that $x_{kj} < x < x_{k(j+1)}$, then

$$\begin{aligned} f_\alpha(x) - f(x) &\le f_\alpha(x_{kj}) - f(x_{k(j+1)}) \\ &< \frac{\varepsilon}{4(b-a)} + f(x_{kj}) - f(x_{k(j+1)}) \\ &< \frac{\varepsilon}{4(b-a)} + \frac{\varepsilon}{4(b-a)} \\ &= \frac{\varepsilon}{2(b-a)}, \end{aligned}$$

so

$$\int_{x_{kj}}^{x_{k(j+1)}} (f_\alpha - f)\, dx \le \frac{\varepsilon}{2(b-a)} (x_{k(j+1)} - x_{kj}).$$

Thus, if $B = \cup_{k=1}^{m-1} [z_k, y_{k+1}]$, then

$$\int_B (f_\alpha - f)\, dx \le \frac{\varepsilon}{2(b-a)} (b-a) = \frac{\varepsilon}{2}.$$

Therefore,

$$\int_a^b (f_\alpha - f) = \int_A (f_\alpha - f)\, dx + \int_B (f_\alpha - f)\, dx \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\int_a^b f_\alpha\, dx \to \int_a^b f\, dx$. $\square$

Theorem 3.3. *Let μ be a Baire quasi-measure on X with corresponding quasi-linear functional ρ. Then μ is strongly τ-smooth if and only if whenever $f_\alpha \searrow f$ pointwise in $C(X)$, then $\rho(f_\alpha) \to \rho(f)$.*

Proof. Suppose first that ρ is strongly τ-smooth in the above sense, and that $Z_\alpha \searrow Z$ are zero sets. Let $Z \preceq g$ with $g \in C(X)$. Define $\mathcal{D} = \{f \in C(X) : g \le f$ and there is an α_0 with $f_{\alpha_0} < f\}$. Order $\mathcal{D}$ by pointwise inequality, then $\mathcal{D}$ is a net with $f \searrow g$, so $\rho(f) \to \rho(g)$.

Let $\varepsilon > 0$ be given, and pick $f \in \mathcal{D}$ so that $\rho(f) < \rho(g) + \varepsilon$. Then there is an α_0 such that $Z_{\alpha_0} \prec f$, so $\mu(Z_{\alpha_0}) \le \rho(f) < \rho(g) + \varepsilon$. Hence, $\lim \mu(Z_\alpha) \le \rho(g)$. Since g is arbitrary, $\lim \mu(Z_\alpha) = \mu(Z)$.

Conversely, suppose μ is strongly τ-smooth and that $g_\alpha \searrow g$ in $C(X)$, where we may assume that $g \ge 0$. For each $x \in R$, $\hat{g}_\alpha(x) = \mu(g_\alpha^{-1}[x, +\infty)) \to \mu(g^{-1}[x, +\infty)) = \hat{g}(x)$. Since each $\hat{g}_\alpha$ is monotone decreasing and $\hat{g}_\alpha \searrow \hat{g}$, by the lemma $\rho(g_\alpha) = \int \hat{g}_\alpha(x)\, dx \to \int \hat{g}(x)\, dx = \rho(g)$. $\square$

The next result gives easy to check conditions that guarantee that every proper quasi-measure on a space satisfies the strong smoothness properties. We will use the following lemma from [**11**].

Lemma 3.4. *Let ν be a proper Borel quasi-measure on a compact X. Suppose F and W are closed subsets of X with W zero-dimensional. Then $\nu(F \cup W) = \nu(F)$.*

Theorem 3.5. 1. *If every zero set in βX that is disjoint from X is zero-dimensional, then every proper Baire quasi-measure on X is strongly σ-smooth.*

2. *If every closed subset of βX that is disjoint from X is zero-dimensional, then every proper Baire quasi-measure on X is strongly τ-smooth.*

3. *If $\mathrm{cl}_{\beta X}(\beta X \backslash X)$ is zero-dimensional, then every proper Baire quasi-measure on X is strongly tight.*

Proof. We prove (2) first; the proof of (1) is similar and left to the reader. Assume that compact subsets of $\beta X \backslash X$ are zero-dimensional, and let μ be a proper Baire quasi-measure on X with $\bar{\nu}$ the associated proper Borel quasi-measure on βX. Suppose F is a closed subset of βX and that $Z = F \cap X$ is a zero set in X. Then $K = \mathrm{cl}_{\beta X} Z$ is a zero set in βX and $\mu(Z) = \bar{\nu}(K)$.

Fix $\varepsilon > 0$ and find a closed $H \subseteq \beta X$ disjoint from K with $\bar{\nu}(K) + \bar{\nu}(H) > \bar{\nu}(\beta X) - \varepsilon$. By normality in βX, there is an open $U \supseteq H$ with $\mathrm{cl}_{\beta X} U \cap K = \varnothing$. Then $F = (F \backslash U) \cup (\mathrm{cl}_{\beta X} U \cap F)$. Both sets in this union are closed and the second is zero-dimensional, so by the lemma, we have $\bar{\nu}(F) = \bar{\nu}(F \backslash U) \leq \bar{\nu}(\beta X) - \bar{\nu}(H) \leq \bar{\nu}(K) + \varepsilon$. So $\bar{\nu}(F) = \bar{\nu}(K) = \mu(Z)$, so by part (2) of Theorem 3.1, μ is strongly τ-smooth.

To prove (3), suppose that $W = \mathrm{cl}_{\beta X}(\beta X \backslash X)$ is zero-dimensional and that U is a cozero subset of X. Set $Z = X \backslash U$ and $F = \mathrm{cl}_{\beta X} Z$. Then $\bar{\nu}(F) = \mu(Z)$ and by the lemma, $\bar{\nu}(F \cup W) = \bar{\nu}(F)$. If $V = \beta X \backslash (F \cup W)$, then $V \subseteq U$ and $\bar{\nu}(U) = \bar{\nu}(\beta X) - \bar{\nu}(F \cup W) = \mu(X) - \mu(Z) = \mu(U)$. So for every $\varepsilon > 0$ there is a compact $K \subseteq U$ with $\bar{\nu}(K) > \mu(U)$, hence μ is strongly τ-smooth. □

We remark that the proof of the previous result actually gives a generalization of the lemma: if ν is a proper Borel quasi-measure on a compact X and $F = K \cup W$ with F and K closed and every compact subset of W zero-dimensional, then $\nu(F) = \nu(K)$.

The next results are the "strong" analogs of Boardman's Theorem 5.16 and Corollary 5.17.

Theorem 3.6. *Suppose μ is a strongly σ-smooth quasi-measure on X. Then μ is strongly τ-smooth if and only if whenever U is a cozero set in X and $\{U_\alpha\}$ is a cozero cover of U, there is a countable subfamily $\{U_{\alpha_n}\}$ of $\{U_\alpha\}$ such that $\mu(\cup_{n \in N} U_{\alpha_n}) = \mu(U)$.*

Proof. $\Rightarrow$. Suppose μ is strongly τ-smooth. Let $\{U_\alpha\}$ be a cozero cover of $U = X \backslash Z$, which we assume is closed under finite unions. Let $Z_\alpha = X \backslash U_\alpha$; by strong τ-smoothness, $\mu(Z_\alpha) \to \mu(Z)$. For each $n \in N$, find an α_n such that $\mu(Z_{\alpha_n}) > \mu(Z) - 1/n$. Since the U_α's are closed under finite unions, we can assume that the U_{α_n}'s are increasing, so that $\mu(\cup_{n \in N} U_{\alpha_n}) = 1 - \mu(Z)$.

$\Leftarrow$. Suppose that whenever U is a cozero set in X and $\{U_\alpha\}$ is a cozero cover of U, there is a countable subfamily $\{U_{\alpha_n}\}$ of $\{U_\alpha\}$ such that $\mu(\cup_{n \in N} U_{\alpha_n}) = \mu(U)$. Let $Z_\alpha \searrow Z$ and set $U_\alpha = X \backslash Z_\alpha$ and $U = X \backslash Z$. By hypothesis, there is a countable subcollection $\{U_{\alpha_n}\}$ of the U_α's such that $\mu(\cup_{n \in N} U_{\alpha_n}) = \mu(U)$. We can assume the U_{α_n}'s are increasing in n; by strong σ-smoothness we have $\mu(U_{\alpha_n}) \to \mu(\cup_{n \in N} U_{\alpha_n})$. Thus, $\mu(Z) = 1 - \mu(\cup_{n \in N} U_{\alpha_n}) = \lim_n (1 - \mu(U_{\alpha_n}) \leq \lim_\alpha (1 - \mu(U_\alpha)) \leq \mu(Z)$. So μ is strongly τ-smooth. □

The assumption of strong σ-smoothness in this result is necessary, as can be seen from Example 2.3. A comparison with Theorem 5.16 of [**6**] suggests that for strongly σ-smooth quasi-measures, the difference between τ-smoothness and strong τ-smoothness is similar to the difference between Lindelöf and hereditary Lindelöf. However, the following result shows that this is not quite true.

Theorem 3.7. *Suppose X is Lindelöf. Then every strongly σ-smooth Baire quasi-measure on X is strongly τ-smooth.*

Proof. Let μ be a strongly σ-smooth Baire quasi-measure on a Lindelöf X, and suppose that we have a family of zero sets $\{Z_\alpha : \alpha \in A\}$ with $Z_\alpha \searrow Z$. By way of contradiction, suppose that there is an $\varepsilon > 0$ such that $\mu(Z_\alpha) > \mu(Z) + \varepsilon$ for all α. Let U be a cozero set such that $Z \subseteq U$ and $\mu(U) < \mu(Z) + \varepsilon$. Then $\{U\} \cup \{X \backslash Z_\alpha : \alpha \in A\}$ is an open cover of X, let $\{U\} \cup \{X \backslash Z_{\alpha_n}\}$ be a countable subcover. Then $\cap Z_{\alpha_n} \subseteq U$; by strong σ-smoothness $\mu(\cap Z_{\alpha_n}) \geq \mu(Z) + \varepsilon$, which contradicts $\mu(U) < \mu(Z) + \varepsilon$. Thus, μ is strongly τ-smooth. $\square$

Our final result shows that in locally compact spaces, the notions of (strong) τ-smoothness and (strong) tightness merge.

Theorem 3.8. *Suppose X is locally compact and that μ is a Baire quasi-measure on X. Then μ is (strongly) τ-smooth if and only if μ is (strongly) tight.*

Proof. We prove the forward implication in the case when μ is strongly σ-smooth. Let U be cozero in X, then by local compactness of X, U is open in βX and $F = (\beta X \backslash X) \cup (X \backslash U)$ is a closed set in βX whose intersection with X is a zero set $Z = X \backslash U$. By strong τ-smoothness, $\bar{\nu}(F) = \mu(F \cap X) = 1 - \mu(U)$, so $\bar{\nu}(U) = 1 - \bar{\nu}(F) = \mu(U)$. That is, $\bar{\nu}(U) = 1 - \bar{\nu}(F) = \mu(U)$. Thus, for every $\varepsilon > 0$, there is a compact $K \subseteq U$ such that $\bar{\nu}(K) > \mu(U) - \varepsilon$, so that μ is strongly tight. $\square$

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