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FOURTH ORDER OPERATORS WITH GENERAL WENTZELL BOUNDARY CONDITIONS

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ABSTRACT. Let Ω be a bounded subset of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ in C^4, $a \in C^4(\overline{\Omega})$ with $a > 0$ in $\overline{\Omega}$, and let A be the fourth order operator defined by $Au := \Delta(a\Delta u)$, respectively $Au := B^2 u$, where $Bu := \nabla\cdot(a\nabla u))$, with general Wentzell boundary condition of the type

$$Au + \beta\frac{\partial(a\Delta u)}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega,$$

$$\left(\text{respectively } Au + \beta\frac{\partial(Bu)}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega\right).$$

We prove that, under additional boundary conditions, if $\beta, \gamma \in C^{3+\varepsilon}(\partial\Omega)$, $\beta > 0$, then the realization of the operator A on a suitable Hilbert space of L^2 type, with a suitable weight on $\partial\Omega$, is essentially self-adjoint and bounded below.

0. Introduction. Consider problems involving the Laplacian Δ on a smooth bounded domain Ω in $\mathbf{R}^N$. The usual boundary conditions are of Robin type, i.e.,

$$\beta\frac{\partial u}{\partial n} + \gamma u = 0,$$

where $(\beta(x), \gamma(x))$ is a nonzero vector for each $x \in \partial\Omega$, the boundary of Ω, and n is the unit outer normal to $\partial\Omega$. But by working in $C(\overline{\Omega})$ rather than in $L^p(\Omega)$ one can use Wentzell boundary conditions of the form

$$\alpha\Delta u + \beta\frac{\partial u}{\partial n} + \gamma u = 0,$$

where $(\alpha(x), \beta(x), \gamma(x))$ is a nonzero vector in $\mathbf{R}^3$ for each x in $\partial\Omega$. The resolvent equation $\Delta u - \lambda u = h$ on the boundary cannot distinguish between $u = 0$ on $\partial\Omega$ and $\Delta u = 0$ on $\partial\Omega$ when $h = 0$ on $\partial\Omega$; such functions h are dense in $L^2(\Omega)$ but not in $C(\overline{\Omega})$. In the previous work

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[**6**] we showed how to remedy this, replacing the ambient base space $L^2(\Omega)$ by $\mathcal{H} = L^2(\Omega) \oplus L^2(\partial\Omega, w\,dS)$ with a suitable weight function w depending naturally on the boundary conditions.

In [**6**] we showed how to solve linear parabolic equations of the form $du/dt = Au$ (with A a second order elliptic operator) with boundary conditions of the form $\alpha Au + \beta(\partial u/\partial n) + \gamma u = 0$ on $\partial\Omega$. Here β is positive and $w = 1/\beta$. In this paper we find the corresponding results for a fourth order operator A of the type $Au := \Delta(a\Delta u)$, respectively $Au := B^2 u$, where $Bu := \nabla \cdot (a\nabla u))$ with general Wentzell boundary condition

$$Au + \beta \frac{\partial(a\Delta u)}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega,$$
$$\left(\text{respectively } Au + \beta \frac{\partial(Bu)}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega\right).$$

Indeed, we obtain essential self-adjointness and semi-boundedness of A on $\mathcal{H}$, when a suitable additional boundary condition is included in the domain of A.

A classification of general boundary conditions for symmetry, semi-boundedness and quasiaccretivity of the operator $Au = u''''$ will be studied in the paper [**7**].

1. The operator $\Delta(a\Delta)$. Here we deal with the operator $Au := \Delta(a\Delta u)$, where we assume that Ω is an open bounded subset of $\mathbf{R}^N$ with C^4 boundary and such that the following assumptions hold:

(A1) $a \in C^4(\overline{\Omega})$, $a(x) > 0$ in $\overline{\Omega}$,

(A2) $\beta \in C^{3+\varepsilon}(\partial\Omega)$, $\beta(x) > 0$ for $x \in \partial\Omega$, $\gamma \in C^{3+\varepsilon}(\partial\Omega)$ (here $C^{k+\varepsilon}(\Lambda)$ denotes, as usual, the space of functions in $C^k(\Lambda)$ whose kth derivatives are Hölder continuous on Λ with exponent $\varepsilon \in (0,1)$),

(A3) $\mathcal{H} := L^2(\Omega, dx) \oplus L^2(\partial\Omega, (dS/\beta)) = X_2$, the completion of $C(\overline{\Omega})$ with respect to the norm $||\cdot||_{X_2}$ associated to the inner product

$$\langle u, v\rangle_{X_2} := \int_\Omega u(x)\overline{v(x)}\,dx + \int_{\partial\Omega} u(x)\overline{v(x)}\frac{dS}{\beta(x)}.$$

Note that if $u \in H^1(\Omega)$, then u has a trace $v \in H^{1/2}(\partial\Omega)$, and u can be identified with $(u, v) \in \mathcal{H}$.

In addition, we consider the following boundary conditions

$(\mathcal{BC})_1$ $\Delta(a\Delta u)(x) + \beta(x)(\partial(a\Delta u))/\partial n(x) + \gamma(x)u(x) = 0$ on $\partial\Omega$,

$(\mathcal{BC})_2$ Γ_1, Γ_2 are open subsets of $\partial\Omega$, $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, each $\overline{\Gamma_i} \setminus \Gamma_i$, $i = 1, 2$, is an S-null subset of $\partial\Omega$, and

$$\begin{cases} \Delta u = 0 & \text{on } \Gamma_1 \\ (\partial u/\partial n) = 0 & \text{on } \Gamma_2. \end{cases}$$

Then the following result holds.

Theorem 1.1. *Under the assumptions* (A1)–(A3), *the operator A with domain*

$$D(A) := \{u \in H^4(\Omega) \cap C^4(\Omega \cup \Gamma_1 \cup \Gamma_2) : \quad (\mathcal{BC})_1 \quad \textit{and} \quad (\mathcal{BC})_2 \quad \textit{hold}\}$$

is essentially self-adjoint on $\mathcal{H}$.

Proof. Let $u \in D(A)$, $v \in H^2(\Omega) \cap C^2(\Omega \cup \Gamma_1 \cup \Gamma_2)$ and evaluate, using the divergence theorem,

$$\begin{aligned} \langle Au, v\rangle_{X_2} &= \int_\Omega \Delta(a\Delta u)\overline{v}\,dx + \int_{\partial\Omega} \Delta(a\Delta u)\overline{v}\frac{dS}{\beta} \\ &= -\int_\Omega \nabla(a\Delta u)\cdot\nabla\overline{v}\,dx + \int_{\partial\Omega} \frac{\partial(a\Delta u)}{\partial n}\overline{v}\,dS \\ &\quad + \int_{\partial\Omega} \Delta(a\Delta u)\overline{v}\frac{dS}{\beta} \\ (\text{by } (\mathcal{BC})_1) &= -\int_\Omega \nabla(a\Delta u)\cdot\nabla\overline{v}\,dx - \int_{\partial\Omega} \gamma u\overline{v}\frac{dS}{\beta} \\ &= \int_\Omega a\Delta u\Delta\overline{v}\,dx - \int_{\partial\Omega} (a\Delta u)\frac{\partial\overline{v}}{\partial n}\,dS - \int_{\partial\Omega} \gamma u\overline{v}\frac{dS}{\beta} \\ (\text{by } (\mathcal{BC})_2) &= \int_\Omega a\Delta u\Delta\overline{v}\,dx - \int_{\Gamma_2} (a\Delta u)\frac{\partial\overline{v}}{\partial n}\,dS - \int_{\partial\Omega} \gamma u\overline{v}\frac{dS}{\beta}. \end{aligned} \tag{1.1}$$

If also $v \in D(A)$, this becomes

$$\begin{aligned} \langle Au, v\rangle_{X_2} &= \int_\Omega \Delta u(a\Delta\overline{v})\,dx - \int_{\partial\Omega} \gamma u\overline{v}\frac{dS}{\beta} \\ &= \langle u, Av\rangle_{X_2} \end{aligned}$$

by $(\mathcal{BC})_2$. Hence, $(A, D(A))$ is symmetric. To prove that $(A, D(A))$ is essentially self-adjoint, it suffices to show that the range of $\lambda I + A$ is dense for sufficiently large (real) λ. To that end, let h be in the dense set $C^{4+\varepsilon}(\overline{\Omega})$, $\lambda > 0$, and consider

$$\lambda u + Au = h \quad \text{in} \quad \overline{\Omega}. \tag{1.2}$$

We seek a solution $u \in D(A)$ which satisfies (1.2). From $(\mathcal{BC})_1$ and (1.2), it follows that

$$-\beta \frac{\partial(a\Delta u)}{\partial n} + (\lambda - \gamma)u = h \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

We begin by finding a weak solution u of (1.2). Let $v \in H^2(\Omega) \cap C^2(\Omega \cup \Gamma_1 \cup \Gamma_2)$; multiply (1.2) by $\overline{v}$ and integrate to get

$$\lambda \int_\Omega u\overline{v}\, dx + \int_\Omega \Delta(a\Delta u)\overline{v}\, dx = \int_\Omega h\overline{v}\, dx.$$

Using the divergence theorem gives

$$\begin{aligned} \lambda \int_\Omega u\overline{v}\, dx - \int_\Omega \nabla(a\Delta u)\cdot\nabla\overline{v}\, dx + \int_{\partial\Omega} \beta\frac{\partial(a\Delta u)}{\partial n}\overline{v}\frac{dS}{\beta} \\ = \int_\Omega h\overline{v}\, dx. \end{aligned} \tag{1.4}$$

By using (1.3), (1.4) becomes

$$\begin{aligned} \lambda \int_\Omega u\overline{v}\, dx - \int_\Omega \nabla(a\Delta u)\cdot\nabla\overline{v}\, dx + \int_{\partial\Omega} (\lambda-\gamma)u\overline{v}\frac{dS}{\beta} \\ = \int_\Omega h\overline{v}\, dx + \int_{\partial\Omega} h\overline{v}\frac{dS}{\beta}. \end{aligned}$$

Again by the divergence theorem together with $(\mathcal{BC})_2$, we obtain

$$\begin{aligned} \lambda \int_\Omega u\overline{v}\, dx + \int_\Omega a\Delta u\Delta\overline{v}\, dx - \int_{\Gamma_2} a\Delta u\frac{\partial\overline{v}}{\partial n}\, dS + \int_{\partial\Omega} (\lambda-\gamma)u\overline{v}\frac{dS}{\beta} \\ = \langle h, v\rangle_{X_2}. \end{aligned}$$

This reduces to

$$\lambda \int_\Omega u\overline{v}\,dx + \int_\Omega \Delta u \Delta \overline{v}\, a\,dx + \int_{\partial\Omega} (\lambda - \gamma) u\overline{v} \frac{dS}{\beta} = \langle h, v\rangle_{X_2} \tag{1.5}$$

if $\partial v/\partial n = 0$ on Γ_2. Now suppose $\lambda > 0$, $\lambda > \max_{x\in\partial\Omega} \gamma(x)$ and $v \in D_{\Gamma_2}$, where

$$D_{\Gamma_2} := \Big\{ w \in C^2(\Omega \cup \Gamma_2) \cap H^2(\Omega) : \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_2 \Big\}. \tag{1.6}$$

To get the desired weak solution, we need to apply the Lax-Milgram lemma. This is slightly complicated, so we recall how it is done in an easier case.

As before, let Ω be a smooth bounded domain in $\mathbf{R}^N$ and let Γ_1, Γ_2 be as in $(\mathcal{BC})_2$. Let

$$\mathcal{D}_*(\Delta) := \Big\{ u \in H^2(\Omega) \cap C^2(\Omega \cup \Gamma_1 \cup \Gamma_2) : u = 0 \text{ on } \Gamma_1,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \Big\}$$

and

$$\widetilde{D} := \{ v \in H^2(\Omega) \cap C^1(\Omega \cup \Gamma_1 \cup \Gamma_2) : v = 0 \text{ on } \Gamma_1 \}.$$

For $u \in \mathcal{D}_*(\Delta), v \in \widetilde{D}$, and $\lambda u - \Delta u = h \in C(\overline{\Omega})$, we have

$$\langle \lambda u - \Delta u, v\rangle_{L^2(\Omega)} = \langle h, v\rangle_{L^2(\Omega)},$$

whence

$$\lambda \int_\Omega u\overline{v}\,dx + \int_\Omega \nabla u \cdot \nabla \overline{v}\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \overline{v}\,dS = \int_\Omega h\overline{v}\,dx.$$

But

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\overline{v}\,dS = \int_{\Gamma_1} \frac{\partial u}{\partial n}\overline{v}\,dS + \int_{\Gamma_2} \frac{\partial u}{\partial n}\overline{v}\,dS = 0$$

since $\partial u/\partial n = 0$ on Γ_2 and $v = 0$ on Γ_1. Then

$$\lambda \int_\Omega u\overline{v}\,dx + \int_\Omega \nabla u \nabla \overline{v}\,dx = \int_\Omega h\overline{v}\,dx \tag{1.7}$$

holds for all $u \in \mathcal{D}_*(\Delta)$ and for all v in the closure V of $\widetilde{D}$ in $H^1(\Omega)$; in the above set, the trace of v on $\partial\Omega$ is a well-defined function. Under the $H^1(\Omega)$ norm, V is a Hilbert space, indeed, a closed subspace of $H^1(\Omega)$ satisfying

$$H_0^1(\Omega) \subset V \subset H^1(\Omega).$$

$V = H^1(\Omega)$ when Δ has the Neumann boundary condition ($\Gamma_1 = \varnothing$, $\Gamma_2 = \partial\Omega$), and $V = H_0^1(\Omega)$ when Δ has the Dirichlet boundary condition ($\Gamma_1 = \partial\Omega$, $\Gamma_2 = \varnothing$). Rewrite (1.7) as

$$L(u,v) = F(v), \quad u, v \in V.$$

For $\lambda > 0$, L satisfies the hypotheses of the Lax-Milgram lemma; thus, there is a unique $u \in V$ satisfying (1.7) for all $v \in V$. If $h \in C^{\varepsilon}(\overline{\Omega})$, then by elliptic regularity, see [**1, 9, 11, 12**], $u \in H^2(\Omega) \cap C^2(\Omega \cup \Gamma_1 \cup \Gamma_2)$. Thus, the weak solution u belongs to $\mathcal{D}_*(\Delta)$, so $(\Delta, \mathcal{D}_*(\Delta))$ is essentially self-adjoint on $L^2(\Omega)$.

We now return to (1.5). Let $L(u,v)$, respectively $F(v)$, be the left, respectively right, hand side of (1.5). Suppose that $h \in X_2$. Let

$$V_0 := \left\{ u \in H^4(\Omega) \cap C^4(\Omega \cup \Gamma_1 \cup \Gamma_2) : \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \right\}, \tag{1.8}$$

Let $\mathcal{K}$ be the completion of V_0 in the norm

$$||u||_{\mathcal{K}} := [||u||_{X_2}^2 + ||\Delta u||_{L^2(\Omega, a\,dx)}^2]^{1/2}.$$

Note that L is a bounded sesquilinear form on $\mathcal{K}$ and F is a bounded conjugate linear functional on $\mathcal{K}$. Indeed, if $u, v \in \mathcal{K}$, then

$$\begin{aligned} |L(u,v)| &\le \lambda ||u||_{L^2(\Omega)} ||v||_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega, adx)} \|\Delta v\|_{L^2(\Omega, adx)} \\ &\quad + \Big(\lambda - \min_{x \in \partial\Omega} \gamma(x)\Big) \|u\|_{L^2(\partial\Omega, (dS/\beta))} \|v\|_{L^2(\partial\Omega, (dS/\beta))} \\ &\le C(\lambda, \gamma, a, \beta) \|u\|_{\mathcal{K}} \|v\|_{\mathcal{K}}, \\ |F(v)| &\le ||h||_{X_2} ||v||_{X_2} \le ||h||_{X_2} ||v||_{\mathcal{K}}. \end{aligned}$$

Moreover,

$$\operatorname{Re} L(u,u) \ge c_0 ||u||_{\mathcal{K}}^2,$$

where $c_0 := \min\{\lambda, \lambda - \max_{x \in \partial\Omega} \gamma(x), 1\} > 0$. Thus, the Lax-Milgram lemma gives a unique weak solution u in $\mathcal{K}$ satisfying $L(u,v) = F(v)$ for all $v \in \mathcal{K}$, provided that λ is real and large enough.

For h in a dense set, we want to show that our weak solution $u \in \mathcal{K}$ is in $D(A)$. We know

$$\lambda u + \Delta(a\Delta u) = h \in C^{4+\varepsilon}(\overline{\Omega}), \tag{1.9}$$

$$-\beta \frac{\partial}{\partial n}(a\Delta u) + (\lambda - \gamma)u = h \quad \text{on} \quad \partial\Omega. \tag{1.10}$$

$$\Delta u = 0 \quad \text{on} \quad \Gamma_1, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_2, \tag{1.11}$$

with (1.10) holding in a weak sense and $u \in H^1(\Omega)$. Moreover u satisfies the uniformly elliptic problem

$$\lambda v + \Delta(a\Delta v) = h \quad \text{in} \quad \Omega, \tag{1.12}$$

$$v = k_1 \quad \text{on} \quad \partial\Omega, \quad \frac{\partial v}{\partial n} = k_2 \quad \text{on} \quad \partial\Omega, \tag{1.13}$$

where $k_1 = u|_{\partial\Omega}$, $k_2 = (\partial u/\partial n)|_{\partial\Omega}$. This implies that $v = u \in H^2(\Omega)$. Next $z := a\Delta u$ satisfies

$$\Delta z = h - \lambda v \in C^{4+\varepsilon}(\overline{\Omega}) + H^2(\Omega) = H^2(\Omega) \tag{1.14}$$

$$\frac{\partial z}{\partial n} = e_1 \quad \text{on} \quad \partial\Omega, \tag{1.15}$$

where $e_1 = (1/\beta)(\lambda v - \gamma v - h) \in H^{1/2}(\partial\Omega)$. Therefore $z = a\Delta u \in H^{3/2}(\Omega)$.

It follows that $v \in H^4(\Omega)$, and so $v \in C^{3+\delta}(\overline{\Omega})$ if $N = 1$, $v \in C^{2+\delta}(\overline{\Omega})$ if $N = 2$, and $v \in W^{3,2N/(N-2)}(\Omega)$ if $N \geq 3$; here and below, δ is a positive constant that may change from line to line. The Sobolev embeddings that we need, cf., e.g., [**1**, Theorem 5.4, page 17], are

$$W^{k,p}(\Omega) \subset C^{(k-1)+\delta}(\overline{\Omega}), \qquad \text{if} \quad p > N$$

in which case $\delta = 1 - (N/p) > 0$,

$$W^{k,N}(\Omega) \subset W^{k-1,q}(\Omega) \quad \text{for any} \quad N \le q < \infty,$$
$$W^{k,p}(\Omega) \subset W^{k-1,Np/(N-p)}(\Omega) \quad \text{if} \quad p < N,$$

and $u \mapsto u|_{\partial\Omega}$ continuously maps $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$.

For $N \ge 3$, $v \in C^{2+\delta}(\overline{\Omega})$ if $N = 3$, $v \in W^{2,p}(\Omega)$ if $N \ge 4$ where $p < \infty$ is arbitrary if $N = 4$ and $p = 2N/(N-4)$ if $N \ge 5$. Thus, see (1.14), (1.15), $z \in W^{2,2N/(N-4)}(\Omega)$ if $N \ge 5$. For $N \le 5$, $z \in C^{\delta}(\overline{\Omega})$ whence $v \in C^{4+\delta}(\overline{\Omega})$ by (1.12). For $N \ge 7$, $z \in W^{1,2N/(N-6)}(\Omega)$ (or $z \in W^{1,p}(\Omega)$ for any finite p if $N = 6$), whence $e_1 \in L^{2N/(N-6)}(\partial\Omega)$, see (1.15). Thus, $z \in W^{2,2N/(N-6)}(\Omega)$, whence $z \in C^{\delta}(\overline{\Omega})$ if $N \le 7$, and thus $v \in C^{4+\delta}(\overline{\Omega})$ in this case. For $N = 8$, $z \in W^{1,p}(\Omega)$ for all $p < \infty$ and $z \in W^{1,2N/(N-8)}(\Omega)$ for $N \ge 9$. Then $e_1 \in L^{2N/(N-8)}(\partial\Omega)$ for $N \ge 9$. Continuing in this way, we conclude $v \in C^{4+\delta}(\overline{\Omega})$ in all dimensions. It follows that $u \in C^4(\Omega \cup \Gamma_1 \cup \Gamma_2)$ and the assertion holds. □

Remark 1.2. Notice that in the one-dimensional case for $\Omega := (0,1)$, the condition $(\mathcal{BC})_2$ reduces to either $\partial^2 u/\partial x^2 = 0$ or $\partial u/\partial x = 0$ at each endpoint in $\partial\Omega = \{0,1\}$. Let us consider the operator $A_1 u := (au'')''$ on $C^4[0,1]$, where

$$a \in C^4[0,1], \quad a(x) > 0 \quad \text{for all} \quad x \in [0,1].$$

We equip A_1 with general Wentzell boundary conditions $(\mathcal{BC})_{1,j}$, given by

$$(\mathcal{BC})_{1,j} \quad A_1 u(j) + \beta_j (au'')'(j) + \gamma_j u(j) = 0, \quad j = 0,1$$

where $\gamma_j \in \mathbf{R}$, $\beta_0 < 0 < \beta_1$, and with boundary conditions $(\mathcal{BC})_2$, i.e.,

$$u''(0) = 0 = u'(1) \quad (\Gamma_1 = \{0\}, \quad \Gamma_2 = \{1\}), \tag{1.16}$$

or

$$u'(0) = 0 = u''(1) \quad (\Gamma_1 = \{1\}, \quad \Gamma_2 = \{0\}), \tag{1.17}$$

or

$$u''(0) = 0 = u''(1) \quad (\Gamma_1 = \{0,1\}, \quad \Gamma_2 = \varnothing), \tag{1.18}$$

or

$$u'(0) = 0 = u'(1) \quad (\Gamma_1 = \varnothing, \quad \Gamma_2 = \{0,1\}). \tag{1.19}$$

Then A_1 is essentially self-adjoint and $A_1 \geq \varepsilon I$ on $\mathcal{H}$, for a suitable $\varepsilon \in \mathbf{R}$. In addition, $\varepsilon \geq 0$ if $\gamma_0, \gamma_1 \leq 0$ and $\varepsilon > 0$ if $\gamma_0, \gamma_1 < 0$. Here $\mathcal{H} = L^2(0,1) \oplus \mathbf{C}^2$, with inner product $\langle u, v\rangle = \int_0^1 u(x)\overline{v(x)}\, dx + \sum_{j=0}^1 w_j u(j)\overline{v(j)}$, for u, v in the dense subset $C[0,1]$ of $\mathcal{H}$, and $w_j = (-1)^{j+1}/\beta_j$, $j = 0, 1$.

Also, for $a \equiv 1$, let us consider the operator $B := d^2/dx^2$ on $C^2[0,1]$. It is essentially self-adjoint in $\mathcal{H}$ if the boundary conditions are

$$Bu(j) + \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1.$$

Then $A_2 := B^2$ on $\mathcal{H}$ has its boundary conditions

$$u''''(j) + \beta_j u'''(j) + \gamma_j u''(j) = 0, \quad j = 0, 1 \tag{1.20}$$

$$u''(j) + \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1. \tag{1.21}$$

All of these operators, i.e., A_1 for $a \equiv 1$ with $(\mathcal{BC})_{1,j}$, $j = 0, 1$ and $(\mathcal{BC})_2$, and $A_2 = B^2$ with (1.20) and (1.21), agree on the same domain

$$\widetilde{C_0^4}(0,1) := \{u \in C^4[0,1] \ : \ u^{(k)}(j) = 0, \ 0 \leq k \leq 4, \ j = 0, 1\}.$$

Moreover, for any of these A's we have

$$\dim\, D(A)/\widetilde{C_0^4}(0,1) < \infty,$$

see the Appendix. Thus, if $\lambda \in \rho(A_1) \cap \rho(A_2)$, then

$$(\lambda - A_1)^{-1} - (\lambda - A_2)^{-1}$$

is a finite rank operator. Since $A_2 = B^2$ has a compact resolvent (since B does by [**2**]), so do all of our A_k.

Remark 1.3. When $\Gamma_1 = \overline{\Gamma_1}$ and $\Gamma_2 = \overline{\Gamma_2}$ in $(\mathcal{BC})_2$ are disjoint and $\partial\Omega = \Gamma$, then we may use as domain

$$D(A) := \{u \in C^4(\overline{\Omega}) : (\mathcal{BC})_1 \text{ and } (\mathcal{BC})_2 \text{ hold}\}.$$

This is the case when Ω is an interval as in Remark 1.2 and when Ω is an annulus, with, say, Γ_1 being the inner boundary and Γ_2 the outer boundary. That is

$$\Omega := \{x \in \mathbf{R}^N : 0 < a < |x| < b < \infty\},$$

with

$$\Gamma_1 := \{x \in \mathbf{R}^N : |x| = a\}, \quad \Gamma_2 := \{x \in \mathbf{R}^N : |x| = b\}.$$

When the boundary of Ω is connected, we normally take one of Γ_1, Γ_2 to be empty. To see why, take

$$\Omega := \{x \in \mathbf{R}^2 : |x| < 1\}$$

and identify $x = (x_1, x_2)$ in $\mathbf{R}^2$ with $x_1 + i\, x_2$ in $\mathbf{C}$. Using obvious notation and identifications, let

$$\Gamma_1 := \{e^{i\theta} : 0 < \theta < \pi\}, \quad \Gamma_2 := \{e^{i\theta} : \pi < \theta < 2\pi\}$$

be the top and bottom half unit circle, respectively. Then $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, and $\overline{\Gamma_i} \setminus \Gamma_i$, for $i = 1, 2$, consists of two points, the right point $R = 1 + i\,0 = (1, 0)$ and the left point $L = -1 + i\,0 = (-1, 0) = e^{i\pi}$. If we wish to impose a boundary condition (as in $(\mathcal{BC})_2$) at L and/or R, it is not clear which one to use, or neither, or both. Since $\{L, R\}$ forms a null set in $\partial\Omega$, it doesn't matter for the X_2 theory. In this case, it would be hard to prove that $u \in D(A)$ is C^4 at L or R; that is why we used the complicated (but effective and usable) definition of $D(A)$.

Remark 1.4. Concerning semi-boundedness, let us observe that

$$\langle Au, u \rangle_{X_2} = \|\Delta u\|^2_{L^2(\Omega, a\, dx)} - \int_{\partial\Omega} \gamma |u|^2 \frac{dS}{\beta}$$

and hence

$$\langle Au, u \rangle_{X_2} \geq \min\Big\{ -\max_{x \in \partial\Omega} \gamma(x), 0 \Big\} ||u||^2_{X_2}.$$

The question of the N-dimensional Sobolev inequality without boundary conditions is relevant. We have $A \geq \varepsilon I$ for some $\varepsilon > 0$ if $\gamma < 0$ on

$\partial\Omega$. Hence, in all cases the closure of the operator A is self-adjoint and bounded below. By spectral theorem $(-A, D(A))$ generates a cosine function, and the cosine function $\{C(t) : t \in \mathbf{R}\}$ is uniformly bounded on $\mathbf{R}$ if $\gamma \leq 0$ on $\partial\Omega$. This implies that the initial value problem for the beam equation $u_{tt} = (au_{xx})_{xx}$ with boundary conditions $(\mathcal{BC})_1$ and $(\mathcal{BC})_2$ is well posed, even if γ is not nonnegative on $\partial\Omega$.

For the meaning of cosine functions and their relations with the well-posedness of second order Cauchy problems, we refer to [**10**, Chapter 2 Section 8].

2. The operator $\nabla \cdot (a\nabla Bu)$ with $Bu := \nabla \cdot (a\nabla u)$. Let us set

$$Au := \nabla \cdot (a\nabla Bu), \quad \text{where} \quad Bu := \nabla \cdot (a\nabla u)$$

and assume that

(α_1) $a \in C^4(\overline{\Omega})$, $a(x) > 0$ in $\overline{\Omega}$;

(α_2) $\beta \in C^{3+\varepsilon}(\partial\Omega)$, $\beta(x) > 0$ for $x \in \partial\Omega$, $\gamma \in C^{3+\varepsilon}(\partial\Omega)$.

(α_3) $\mathcal{H} = L^2(\Omega, dx) \oplus L^2(\partial\Omega, (adS/\beta)) := X_2$, the completion of $C(\overline{\Omega})$ with respect to the norm associated to the inner product

$$\langle u, v\rangle_{X_2} := \int_\Omega u(x)\overline{v(x)}\,dx + \int_{\partial\Omega} u(x)\overline{v(x)}\,\frac{a(x)\,dS}{\beta(x)}.$$

In addition, we consider the following boundary conditions

$(bc)_1$ $Au(x) + \beta(x)(\partial(Bu)/\partial n)(x) + \gamma(x)u(x) = 0$ on $\partial\Omega$

$(bc)_2$ $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, Γ_1, Γ_2 are open subsets of $\partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, $\overline{\Gamma_i} \setminus \Gamma_i$ is an S-null set for each $i = 1, 2$, and

$$\begin{cases} Bu = 0 & \text{on } \Gamma_1 \\ \partial u/\partial n = 0 & \text{on } \Gamma_2. \end{cases}$$

Then we have

Theorem 2.1 *Under the assumptions* (α_1)–(α_3), *the operator* A *with domain*

$$D(A) := \{u \in C^4(\Omega \cup \Gamma_1 \cup \Gamma_2) \cap H^4(\Omega) : (bc)_1 \quad and \quad (bc)_2 \quad hold\}$$

is essentially self-adjoint on $\mathcal{H}$.

Proof. The proof follows similar lines as in the proof of Theorem 1.1. Let us consider $u \in D(A)$, $v \in H^2(\Omega) \cap C^2(\Omega \cup \Gamma_1 \cup \Gamma_2)$, and evaluate

$$\begin{aligned}\langle Au, v\rangle_{X_2} &= \int_\Omega \nabla \cdot (a\nabla Bu)\overline{v}\, dx + \int_{\partial\Omega} Au\overline{v}\,\frac{adS}{\beta} \\ &= -\int_\Omega (a\nabla Bu) \cdot \nabla\overline{v}\, dx + \int_{\partial\Omega} \beta\frac{\partial}{\partial n}(Bu)\overline{v}\,\frac{adS}{\beta} \\ &\quad + \int_{\partial\Omega} Au\overline{v}\frac{adS}{\beta} \\ &= -\int_\Omega (\nabla Bu) \cdot (a\nabla\overline{v})\, dx - \int_{\partial\Omega} \gamma u\overline{v}\,\frac{adS}{\beta}\end{aligned}$$

by $(bc)_1$ and the divergence theorem. Again applying the divergence theorem and $(bc)_2$ in the above equality, we have that

$$\langle Au, v\rangle_{X_2} = \int_\Omega (Bu)(\nabla \cdot (a\nabla\overline{v}))\, dx - \int_{\Gamma_2} Bu\frac{\partial v}{\partial n}\, adS - \int_{\partial\Omega} \gamma u\overline{v}\,\frac{adS}{\beta}.$$

If also $v \in D(A)$, then we obtain

$$\begin{aligned}\langle Au, v\rangle_{X_2} &= \int_\Omega (Bu)(B\overline{v})\, dx - \int_{\partial\Omega} \gamma u\overline{v}\,\frac{adS}{\beta} \\ &= \langle u, Av\rangle_{X_2}.\end{aligned}$$

Hence $(A, D(A))$ is symmetric. Concerning the range condition, let us assume $h \in C^{4+\varepsilon}(\overline{\Omega})$ for some positive ε and for $\lambda > 0$ consider

$$\lambda u + Au = h \quad \text{in} \quad \Omega. \tag{2.1}$$

If $u \in D(A)$, then, on $\partial\Omega$ we have

$$-\beta\frac{\partial}{\partial n}(Bu) + (\lambda - \gamma)u = h \quad \text{on} \quad \partial\Omega \tag{2.2}$$

by $(bc)_1$. Multiply (2.1) by $\overline{v}$, for $v \in H^2(\Omega)$, and integrate to get

$$\lambda \int_\Omega u\overline{v}\,dx + \int_\Omega \nabla \cdot (a\nabla Bu)\overline{v}\,dx = \int_\Omega h\overline{v}\,dx.$$

Using the divergence theorem gives

$$\lambda \int_\Omega u\overline{v}\,dx - \int_\Omega a\nabla(Bu) \cdot \nabla\overline{v}\,dx + \int_{\partial\Omega} \beta \frac{\partial Bu}{\partial n}\overline{v}\frac{adS}{\beta} = \int_\Omega h\overline{v}\,dx. \tag{2.3}$$

By (2.2), (2.3) becomes

$$\lambda \int_\Omega u\overline{v}\,dx - \int_\Omega a\nabla(Bu) \cdot \nabla\overline{v}\,dx - \int_{\partial\Omega} (\gamma - \lambda)u\overline{v}\,\frac{adS}{\beta} = \int_\Omega h\overline{v}\,dx + \int_{\partial\Omega} h\overline{v}\,\frac{adS}{\beta}. \tag{2.4}$$

Again using the divergence theorem and $(bc)_2$, we obtain

$$\lambda \int_\Omega u\overline{v}\,dx + \int_\Omega (Bu)B\overline{v}\,dx - \int_{\Gamma_2} Bu\frac{\partial\overline{v}}{\partial n}a\,dS - \int_{\partial\Omega} (\gamma - \lambda)u\overline{v}\,\frac{adS}{\beta} = \langle h, v\rangle_{X_2}. \tag{2.5}$$

Now suppose $\lambda > 0$, $\lambda > \max_{x\in\partial\Omega}\gamma(x)$ and $v \in D_{\Gamma_2}$, see (1.6). Let $L(u,v)$ be the lefthand side of (2.5), and let $F(v) = \langle h, v\rangle_{X_2}$. Let V_0 be as in (1.8) and define $\mathcal{K}$ to be the completion of V_0, in the norm

$$|||u|||_{\mathcal{K}} := [||u||^2_{X_2} + ||Bu||^2_{L^2(\Omega,dx)}]^{1/2}.$$

We show that L is a bounded sesquilinear form on $\mathcal{K}$ and F is a bounded conjugate linear functional on $\mathcal{K}$ for $\lambda > 0$, $\lambda > \max_{x\in\partial\Omega}\gamma(x)$. Indeed, if $u, v \in \mathcal{K}$, then

$$\begin{aligned} |L(u,v)| &\le \max\{|\lambda|, 1\}|||u|||_{\mathcal{K}}|||v|||_{\mathcal{K}} + \Big(\max_{x\in\partial\Omega}|\gamma(x)| + |\lambda|\Big)|||u|||_{\mathcal{K}}|||v|||_{\mathcal{K}} \\ &\le C(\lambda,\gamma,a,\beta)|||u|||_{\mathcal{K}}|||v|||_{\mathcal{K}}. \end{aligned}$$

Also

$$|F(v)| \leq ||h||_{X_2}||v||_{X_2} \leq ||h||_{X_2}|||v|||_{\mathcal{K}}.$$

Finally, for $\lambda > 0$, $\lambda > \max_{x\in\partial\Omega}\gamma(x)$, we have

$$\operatorname{Re} L(u,u) \geq \min\Big\{\lambda, \min_{x\in\partial\Omega} \lambda - \gamma(x), 1\Big\}|||u|||^2_{\mathcal{K}}.$$

Thus, by the Lax-Milgram lemma, there is a unique $u \in \mathcal{K}$ such that

$$L(u,v) = F(v) \quad \text{for all} \quad v \in \mathcal{K}.$$

This u is our weak solution of (2.1) satisfying $(bc)_1$ and $(bc)_2$. By using similar arguments as in the proof of Theorem 1.1, provided that we replace (1.10) by (2.2), (1.12) by

$$\lambda v + \nabla \cdot (a\nabla Bv) = h,$$

and $z := Bv$, we conclude that $u \in D(A)$, so that $u \in D(A)$ and $(A, D(A))$ is essentially self-adjoint on X_2.

Remark 2.2. Concerning semi-boundedness, let us observe that

$$\langle Au, u\rangle_{X_2} = \|Bu\|^2_{L^2(\Omega,dx)} - \int_{\partial\Omega} \gamma|u|^2\,\frac{adS}{\beta},$$

and hence

$$\langle Au, u\rangle_{X_2} \geq \min\Big\{-\max_{x\in\partial\Omega}\gamma(x), 0\Big\}||u||^2_{X_2}.$$

Thus, by the spectral theorem, the closure of the operator $(-A, D(A))$ generates a cosine function, and the cosine function $\{C(t) : t \in \mathbf{R}\}$ is uniformly bounded on $\mathbf{R}$ if $\gamma \leq 0$ on $\partial\Omega$.

APPENDIX

Let $n \in \mathbf{N}$ and define

$$X := \Big\{u \in C^{n+1}[0,1] : u^{(j)}(0) = u^{(j)}(1) = 0 \text{ for } j = 0, 1, \ldots, n\Big\}.$$

We claim that $C^{n+1}[0,1]/X$ is finite dimensional.

To show this, let

$$(\mathcal{A}) \qquad u(x) := \varphi(x) \sum_{j=0}^{n} a_j \frac{x^j}{j!} + (1-\varphi(x)) \sum_{j=0}^{n} b_j \frac{(x-1)^j}{j!},$$

where φ is a C^∞-function on $[0,1]$ such that $\varphi \equiv 1$ on $[0,1/3]$ and $\varphi \equiv 0$ on $[2/3,1]$. Then $u^{(k)}(0) = a_k$ and $u^{(k)}(1) = b_k$. Given $f \in C^{n+1}[0,1]$, let

$$a_j := f^j(0) \quad \text{and} \quad b_j := f^j(1), \quad j = 0, \dots, n.$$

Then for u as in $(\mathcal{A})$, $f - u$ is in X. Thus $C^{n+1}[0,1]/X$ is $2n+2$-dimensional.

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REFERENCES

1. R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. P.A. Binding, P.J. Browne and B.A. Watson, *Spectral problems for non-linear Sturm-Liouville equations with eigenparameter dependent boundary conditions*, Canad. J. Math. **52** (2000), 248–264.

3. K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Grad. Texts Math. **94**, Springer-Verlag, New York, 2000.

4. A. Favini, G. Ruiz Goldstein, J.A. Goldstein and S. Romanelli, C_0 *semigroups generated by second order differential operators with generalized Wentzell boundary conditions*, Proc. Amer. Math. Soc. **128** (2000), 1981–1989.

5. ———, *Degenerate second order differential operators generating analytic semigroups in L^p and $W^{1,p}$*, Math. Nachr. **238** (2002), 78–102.

6. ———, *The heat equation with generalized Wentzell boundary condition*, J. Evol. Equations **2** (2002), 1–19.

7. ———, *Classification of general Wentzell boundary conditions for fourth order operators in one space dimension*, J. Math. Anal. Appl. **333** (2007), 219–235.

8. A. Favini, G. Ruiz Goldstein, J.A. Goldstein and S. Romanelli, *The heat equation with nonlinear general Wentzell boundary condition*, Adv. Differential Equations **11** (2006), 481–510.

9. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001.

10. J.A. Goldstein, *Semigroups of linear operators and applications*, Oxford University Press, Oxford, 1985.

11. O.A. Ladyzhenskaja, V.A. Solonnikov, and N.N. Uralseva, *Linear and quasi-linear equations of parabolic type*, Trans. Math. Mono. **33**, American Mathematical Society, Providence, R.I., 1967.

12. O.A. Ladyzhenskaja and N.N. Uralseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

13. H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Publishing, Amsterdam, 1978.

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