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TRIGONOMETRIC SPLINES WITH VARIABLE SHAPE PARAMETER

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ABSTRACT. A new class of trigonometric splines has recently been introduced by Xuli Han [**3, 4**], where basis functions consist of a shape parameter that can be effectively used to control the shape of resulting trigonometric spline curves. A change in shape parameter, however, affects the curve globally, which may not be so suitable for CAGD applications. Keeping this in view, we have introduced a variable shape parameter in the trigonometric quadratic spline curves, in the present paper, which in its turn allows to manipulate the shape of the curve locally in each segment. We also study the approximation properties of these curves by determining the distance of the curve from the control points. For this, we employ a simpler approach, compared to the one used in [**3**]. We further study interpolation by these spline curves over a given knot sequence and corresponding data. A similar construction can also be presented for cubic trigonometric spline curves.

1. Introduction. Trigonometric B splines were first introduced in [**10**] and were subsequently studied from various perspectives, see e.g., [**5, 6, 7, 9, 11, 12**] and references therein. In recent years special attention has been paid to applications of trigonometric splines in geometric modeling, as it was observed that many problems of surface modeling could be better handled by trigonometric splines (especially those relating to data fitting on spherical objects). This has led to the introduction of various types of trigonometric splines having different features suitable for CAGD applications, see e.g., [**8, 13, 14, 15**]. Keeping in view the application potentialities of trigonometric splines, Han [**3**] has recently introduced a class of C^1-quadratic trigonometric spline curves with basis functions having a shape parameter. This parameter helps in better control over the shape of the resulting curve in

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the sense that, by varying the parameter, the curve can be made closer to the control polygon than the usual C^1-quadratic B-spline curve. This idea has also been extended to define a cubic trigonometric spline with a shape parameter in a very recent paper by Han [**4**].

The construction of trigonometric spline curves, however, allows a single shape parameter to be used for shape control. Due to this, a change in parameter value affects the entire curve leading to a global change, which is not so suitable for CAGD applications. Keeping this in view, we have introduced a variable shape parameter in the construction of C^1-quadratic trigonometric spline curves in the present paper. This allows to manipulate the shape of the curve locally in each segment. We also show that the curve can be made sufficiently close to its control polygon by appropriately choosing the shape parameter. It may be mentioned that we have employed a much simpler method for determining the distance of the curve from its control polyline, compared to the one used in Han [**3**].

Our splines, therefore, serve as an alternative to the well known rational B-spline curves which also allow local shape control by way of change of weight factors. However, we observe that for no choice of the shape parameter, the trigonometric spline curve retraces the control polygon, even if we employ a variable shape parameter in the curve. Note that this is possible with the use of rational B-spline curves which allow use of weight factors for shape control. In view of this we also study a rational analogue of the trigonometric spline curves and study the approximability of these splines. We have shown that the weight factors present in the rational curves could be used effectively to make the curve approach the control polygon. It is worthwhile to mention that rational curves have found interesting applications in problems of constrained interpolation where the curve is required to remain in a certain region. Such problems arise, e.g., in robotics, to determine the motion profiles of an arm of robot.

Finally we study the problem of interpolation by these C^1-quadratic trigonometric splines over a given knot sequence and corresponding data.

The paper is organized as follows. In Section 2, we present the construction of basis functions for C^1-trigonometric splines with variable shape parameters. We study the properties of the resulting spline curves which are suitable for CAGD applications. Section 3 deals with approximation properties of the curves while Section 4 is devoted to problem of determining a C^1-trigonometric spline curve which interpolates given data at a specified sequence of parameter values. Finally in Section 5, we also present a rational analogue of these trigonometric curves and study the approximation property of the curves. As expected, the weight factors here play a crucial role in making the rational spline curve closer to the control polygon than polynomial trigonometric spline studied in [**3**]. Our results are supported by numerical examples in each section.

2. C^1-trigonometric splines with variable shape parameter: Construction. We shall mostly use the notations introduced in [**3**]. Let $u_0 < u_1 < u_2 < \cdots < u_{n+3}$ be a given knot sequence with $\triangle_i = u_{i+1} - u_i$. Define the local parameter of the ith interval by

$$t_i(u) = \frac{\pi(u - u_i)}{2\triangle_i}, u_i \le u \le u_{i+1}.$$

Further,

$$\alpha_i = \frac{\triangle_i}{(\triangle_{i-1} + \triangle_i)}, \quad \text{and} \quad \beta_i = \frac{\triangle_i}{(\triangle_i + \triangle_{i+1})}.$$

For each relevant value of i, let λ_i be a local C^1-shape function satisfying $-1 \le \lambda_i(t_i(u)) \le 1$. We further assume that

$$\lambda_i(0) = \lambda_{i-1}(\pi/2), \quad \text{for all } i. \tag{1}$$

We next define

$$c_i(u) = (1 - \sin(t_i(u))(1 - \lambda_i(t_i(u)) \sin t_i(u)),$$
$$d_i(u) = (1 - \cos(t_i(u))(1 - \lambda_i(t_i(u)) \cos t_i(u)).$$

As defined in [**3**], the associated basis functions are given by

$$b_i(u) = \begin{cases} \beta_i d_i(u) & u \in [u_i, u_{i+1}); \\ 1 - \alpha_{i+1} c_{i+1}(u) - \beta_{i+1} d_{i+1}(u) & u \in [u_{i+1}, u_{i+2}); \\ \alpha_{i+2} c_{i+2}(u) & u \in [u_{i+2}, u_{i+3}); \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to see that the following properties of b_i continue to hold with variable shape parameter $\lambda_i(t_i)$, cf. [**3**].

$$b_i(u) > 0, \quad u_i < u < u_{i+3}. \tag{3}$$

$$\sum_{i=0}^{n} b_i(u) = 1, \quad u \in [u_2, u_{n+1}]. \tag{4}$$

The fact that the b_is are continuous follows at once, if we appeal to [**3**, Theorem 2], combined with the assumption (1). We can further show that the b_is are actually C^1. To see this, we observe that

$$\begin{aligned}
\frac{db_i}{du}(u_i+) &= 0,\\
\frac{db_i}{du}(u_{i+1}-) &= \frac{\pi\beta_i}{2\triangle_i}[1+\lambda_i(\pi/2)],\\
\frac{db_i}{du}(u_{i+1}+) &= \frac{\pi\alpha_{i+1}}{2\triangle_{i+1}}[1+\lambda_{i+1}(0)],\\
\frac{db_i}{du}(u_{i+2}-) &= \frac{-\pi\beta_{i+1}}{2\triangle_{i+1}}[1+\lambda_{i+1}(\pi/2)],\\
\frac{db_i}{du}(u_{i+2}+) &= \frac{-\pi\alpha_{i+2}}{2\triangle_{i+1}}[1+\lambda_{i+2}(0)],\\
\frac{db_i}{du}(u_{i+3}-) &= 0.
\end{aligned}$$

Using (1) and the fact that $\triangle_{i+1}\beta_i = \triangle_i\alpha_{i+1}$, we establish our main claim that the b_is are C^1. These basis functions for two different choice shape functions are shown in Figure 1.

Let $\{P_0, P_1, \dots, P_n\}$ be a given set of points in R^2 or R^3, $n \geq 2$. Define the C^1 quadratic trigonometric spline with variable shape parameter by

$$T(u) = \sum_{j=0}^{n} P_j b_j(u), \quad u \in [u_2, u_{n+1}]. \tag{5}$$

The points P_i are called control points of the curve and the polyline formed by joining successively P_i with P_{i+1}, $i = 0, 1, \dots, n-1$, is called control polyline of the curve. It is easy to see that on a subinterval $[u_i, u_{i+1}]$, $T(u)$ can be written as

$$T(u) = T_i(u) = P_{i-2}b_{i-2}(u) + P_{i-1}b_{i-1}(u) + P_i b_i(u). \tag{6}$$

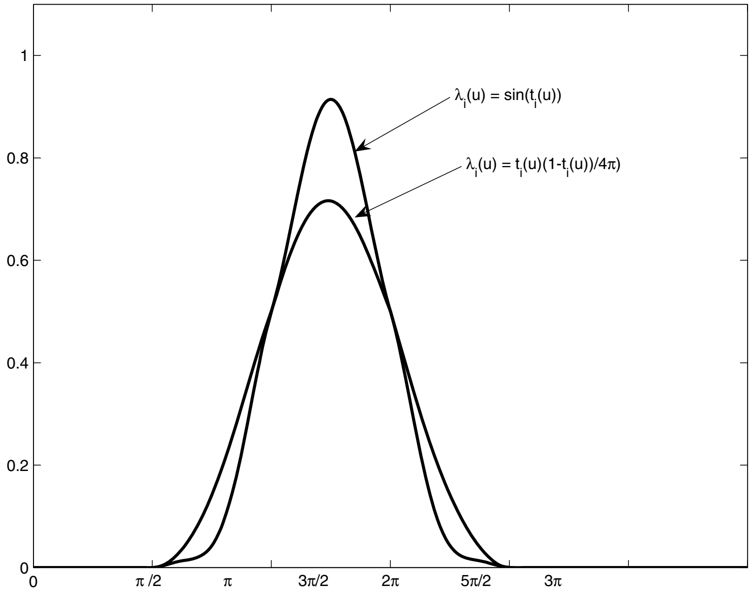

FIGURE 1. Basis functions for C^1 trigonometric splines.

Further, by virtue of (2), we have

$$T_i(u_i) = \alpha_i P_{i-2} + (1-\alpha_i)P_{i-1}, \tag{7a}$$

$$T_i(u_{i+1}) = (1-\beta_i)P_{i-1} + \beta_i P_i, \tag{7b}$$

$$\frac{dT_i}{du}(u_i) = \frac{\pi\alpha_i}{2\triangle_i}(1+\lambda_i(0))(P_{i-1}-P_{i-2}),$$

$$\frac{dT_i}{du}(u_{i+1}) = \frac{\pi\beta_i}{2\triangle_i}(1+\lambda_i(\pi/2))(P_i-P_{i-1}).$$

The open and closed trigonometric splines with variable shape parameters can also be defined as described in [**3**] with appropriate modifications. We next turn to discuss the approximability of these splines, which is important for geometric modeling.

3. Approximation properties. On the interval $[u_i, u_{i+1}]$, T(u) is given by (6). By virtue of (3) and (4), $T(u)$ always remains in the convex hull of its control points, in each segment. It is clear that $T_i(u)$

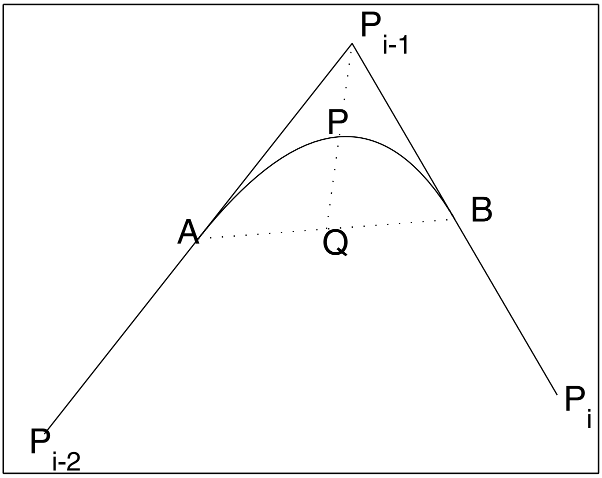

FIGURE 2. A curve segment on the interval $[u_i, u_{i+1}]$.

remains in the triangle formed by the vertices A, P_{i-1} and B, where $A = T_i(u_i)$ and $B = T_i(u_{i+1})$ given by (7a) and (7b) respectively. Let Q be a point on the line segment joining A and B given by

$$Q = (1-\alpha)A + \alpha B, \quad 0 < \alpha < 1.$$

Draw a line from Q to join P_{i-1} which meets the curve segment in the point $T_i(\widehat{u})$ for some parameter value $\widehat{u}$, see Figure 2.

Let $T_i(\widehat{u})$ divide the line segment $\overline{QP_{i-1}}$ in the ratio $w : 1-w$. We may write

$$\begin{aligned} T_i(\widehat{u}) &= (1-w)Q + wP_{i-1} \\ &= (1-w)[(1-\alpha)A + \alpha B] + wP_{i-1} \\ &= (1-w)(1-\alpha)[\alpha_i P_{i-2} + (1-\alpha_i)P_{i-1}] \\ &\quad + (1-w)\alpha[(1-\beta_i)P_{i-1} + \beta_i P_i] + wP_{i-1}. \end{aligned}$$

Hence,

(8)
$$\begin{aligned} T_i(\widehat{u}) = (1-w)(1-\alpha)\alpha_i P_{i-2} & \\ + [(1-w)(1-\alpha)(1-\alpha_i) + (1-w)\alpha(1-\beta_i) + w]P_{i-1} & \\ + (1-w)\alpha\beta_i P_i. & \end{aligned}$$

Comparing this form with (6), we get

$$(1-w)(1-\alpha)\alpha_i = \alpha_i c_i(\widehat{u}),$$
$$(1-w)\alpha\beta_i = \beta_i d_i(\widehat{u}).$$

This in its turn gives

$$w = (1+\lambda_i(t_i(\widehat{u}))(\sin\ t_i(\hat{u}) + \cos t_i(\widehat{u}) - 1)). \tag{9}$$

This gives the distance of the curve from the control polygon. For, if we let $K = \|Q - P_{i-1}\|$, then $\|T_i(\widehat{u}) - P_{i-1}\| = (1-w)K$. Therefore, the curve can be made sufficiently close to its control polygon by choosing a λ_i such that it achieves its maximum at the midpoint of the interval $[u_i, u_{i+1}]$. This is because the curve segment is closest to P_{i-1} for a mid point parameter value and if λ_i assumes maximum value at this point, the curve would be closest to the point P_{i-1}. Note that the case $\lambda_i = \lambda$ (constant) for all i reduces to the C^1-quadratic trigonometric spline studied in [**3**].

For this case, an exact value of the parameter $\widehat{u}$ can be obtained after a little computation as follows.

$$\widehat{u} = \cos^{-1}\left(\frac{m+\sqrt{2-m^2}}{2}\right),$$

where $m = 1 + w/(1+\lambda)$. For details of the analysis, reference may be made to [**1**].

We now turn to compare it with the usual C^1-quadratic polynomial B-spline curve for the same control polygon. One may note that the spline segment on $[u_i, u_{i+1}]$ may be written as

$$B_i(u) = P_{i-2}\tilde{b}_{i-2}(u) + P_{i-1}\tilde{b}_{i-1}(u) + P_i\tilde{b}_i(u), \quad u \in [u_i, u_{i+1}], \tag{10}$$

where

$$\tilde{b}_{i-2}(u) = \alpha_i(\pi - 2t_i)^2/\pi^2,$$
$$\tilde{b}_{i-1}(u) = 1 - \tilde{b}_{i-2}(u) - \tilde{b}_i(u),\ \tilde{b}_i(u) = \beta_i(2t_i)^2/\pi^2.$$

Since B_i also lies in the triangle formed by A, P_{i-1}, B, we may use the same argument (as we did in the previous case) to write the point on the curve $B_i(\tilde{u})$ as

(11)

$$\begin{aligned} B_i(\tilde{u}) &= (1-\widetilde{w})(1-\alpha)\alpha_i P_{i-2} \\ &\quad + [(1-\widetilde{w})(1-\alpha)(1-\alpha_i) + (1-\widetilde{w})\alpha(1-\beta_i) + \widetilde{w}]P_{i-1} \\ &\quad + (1-\widetilde{w})\alpha\beta_i P_i. \end{aligned}$$

Comparing (10) and (11), we obtain

$$\widetilde{w} = 4t_i(\pi - 2t_i)/\pi^2.$$

Therefore, $\|B_i(u) - P_{i-1}\| = (1-\widetilde{w})K$ and hence

$$\|T_i(u) - P_{i-1}\| = \frac{1-w}{1-\widetilde{w}}\|B_i(u) - P_{i-1}\|. \tag{12}$$

This gives that T_i will be closer to the control polygon than B_i provided $w \geq \widetilde{w}$ for all $0 \leq t_i \leq \pi/2$. By appropriately choosing λ_i we may obtain values of w greater than $\widetilde{w}$. Note that $\max w = 1/2$. We thus obtain the following

Theorem 3.1. *Let $u_0 < u_1 < \cdots < u_{n+3}$ be a given sequence of parameter values and $\{P_0, P_1, \ldots, P_n\}$ the corresponding set of control points. Then the C^1- trigonometric spline curve $T(u)$ defined by* (5) *over $[u_2, u_{n+1}]$ will be closer to the control polygon than the usual C^1 quadratic B-spline curve defined over the same knot sequence and the same control polygon provided $(1+\lambda_i(t_i(u)))[\cos t_i(u) + \sin t_i(u) - 1] \geq 4t_i(u)(\pi - 2t_i(u))/\pi^2$ for all u and for relevant values of i.*

We now turn to compare these trigonometric splines with the usual rational quadratic B-spline curves. The main reason for this comparison is that by adjusting the weights in a rational B-spline curve, we can bring the rational segments pretty close to the control polygon. But choice of weights for rational B-spline curves is not an easy job, cf. [**2**, page 240].

We begin with the ith segment of a rational B-spline curve, which is inside the triangle $\widetilde{A}P_{i-1}\widetilde{B}$ with vertices $\widetilde{A}$ and $\widetilde{B}$ given by:

$$\widetilde{A} = R_i(u_i) = \frac{P_{i-2}w_{i-2}\alpha_i + P_{i-1}w_{i-1}(1-\alpha_i)}{w_{i-2}\alpha_i + w_{i-1}(1-\alpha_i)}$$

$$\widetilde{B} = R_i(u_{i+1}) = \frac{P_{i-1}w_{i-1}(1-\beta_i) + P_i w_i\beta_i}{w_{i-1}(1-\beta_i) + w_i\beta_i}.$$

If we want to make a rational segment sufficiently close to the edge $P_{i-1}P_i$ of the polyline, we need to choose weight w_{i-2} sufficiently small compared to w_{i-1}. In this case the enclosing triangle $\widetilde{A}P_{i-1}\widetilde{B}$ will be decreasing in area and would eventually approach to the straight line segment $P_{i-1}P_i$ as the ratio w_{i-2}/w_{i-1} approaches 0. However, the triangle $AP_{i-1}B$ of trigonometric spline segment remains unchanged and hence the segment also. Therefore locally it is not possible to compare the trigonometric curve segment by segment with rational quadratic spline curve, since the two pieces are defined inside different triangles. None the less, a global comparison can be made as follows. By choosing an increasing (decreasing) sequence of weight factors with sufficiently low (high) ratio w_{i-1}/w_i, one can see that the rational quadratic B-spline curve can be made pretty close to the control polyline. The larger the ratio, the closer the curve. This however is not possible with trigonometric curves. Whatever may be the choice of shape parameter, the curve would never trace the control polyline.

Remark on choice of λ. One may note that the curve no longer remains quadratic in nature if a variable shape parameter is chosen. However, the main motivation for the choice of a variable shape parameter has been the need for obtaining local control in the trigonometric splines introduced in [**3**].

A useful choice for λ_i could be $\lambda_i(t_i(u)) = \sin(2t_i(u))$, which is nonnegative on $[u_i, u_{i+1}]$ with a maximum at the midpoint of the interval. Another choice could be using polynomials, say $\lambda_i(t_i(u)) = t_i(u)(2\pi - t_i(u))/2\pi^2$. In both the cases, the λ_i satisfies the condition that $\lambda_i(0) = \lambda_i(\pi/2)$. However, this is not a necessary condition. For example, one might like to choose

$$\lambda_{i-1}(t_{i-1}(u)) = (2t_{i-1}(u)/\pi)^2, \quad u \in [u_{i-1}, u_i]$$

and

$$\lambda_i(t_i(u)) = 1 - (2t_i(u)/\pi)^2, \quad u \in [u_i, u_{i+1}].$$

4. Interpolation. The present section is devoted to the problem of interpolation of given data at prescribed parameter values, by C^1 trigonometric spline curves studied in the previous section. Let

$$u_2 < x_2 < u_3 < \cdots < u_n < x_n < u_{n+1}$$

and

$$\{f_2,\ f_3, \dots , f_n\}$$

be the corresponding data in R^2 or R^3. To determine a C^1-quadratic trigonometric spline T of the form (5), which satisfies $T(x_i) = f_i$, $i = 2, \dots , n$, we first need to show the existence of such a spline curve. Since T has $n + 1$ unknown control points and there are only $n - 1$ interpolatory conditions, two additional boundary conditions are needed to completely determine the spline. Suppose two boundary conditions have somehow been obtained; we observe that the interpolatory condition imposes the following constraint on each segment

$$P_{i-2}b_{i-2}(x_i) + P_{i-1}b_{i-1}(x_i) + P_i b_i(x_i) = f_i, \quad i = 2, \dots , n. \tag{13}$$

This system of equations would provide a unique solution for P_i if it is strictly diagonally dominant. This imposes the following conditions.

$$b_{i-1}(x_i) > b_{i-2}(x_i) + b_i(x_i), \quad i = 2, \dots , n.$$

Substituting the values of b_{i-2}, b_{i-1} and b_i we obtain

$$1 - 2[\alpha_i c(\theta_i) + \beta_i d(\theta_i)] > 0, \tag{14}$$

where $\theta_i = t_i(x_i)$, $c(\theta_i) = c_i(x_i)$; and $d(\theta_i)$ has a similar meaning. Let $\gamma_i = \max\{\alpha_i, \beta_i\}$. Then (14) is satisfied if

$$2\gamma_i[c(\theta_i) + d(\theta_i)] < 1.$$

This is equivalent to saying that

$$2\gamma_i[1 + (1 + \lambda_i(\theta_i))\{1 - \cos(\theta_i) - \sin(\theta_i)\}] < 1. \tag{15}$$

We thus have the following.

Theorem 4.1. *Let $u_2 < x_2 < \cdots < u_n < x_n < u_{n+1}$ and $\{f_2, \dots , f_n\}$ be a set of data given in R^2 or R^3. There exists a unique C^1 trigonometric spline curve T of the form* (5), *which satisfies the interpolatory conditions*:

$$T(x_i) = f_i, \quad i = 2, \dots , n,$$

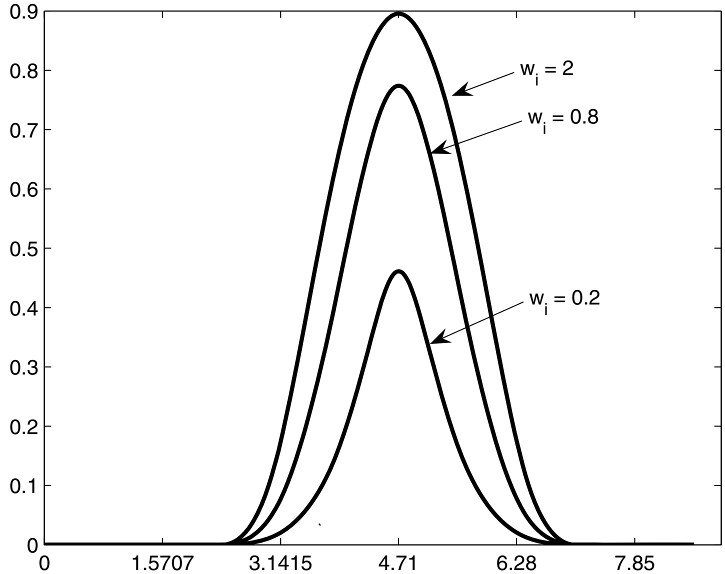

FIGURE 3. Basis functions for rational quadratic trigonometric splines.

provided condition (15) is satisfied and the end control points P_0 and P_n can be specified by suitable end conditions.

In the foregoing theorem, by 'suitable end conditions' we mean to say that the diagonal dominance of the resulting system of equations is not affected.

It can be seen that if $\triangle_i$ is a constant and each x_i is the midpoint of the subinterval $[u_i, u_{i+1}]$, i.e., $x_i = (u_i + u_{i+1})/2$, then condition (15) is clearly satisfied since in this case the condition reduces to

$$-(1 + \lambda_i(\theta_i))(\sqrt{2} - 1) < 0,$$

which is trivially true.

5. Rational trigonometric spline curves. Let $\{w_0, w_1, \ldots, w_n\}$ be a sequence of positive weight factors chosen in such a way that

$$\sum_{i=0}^{n} w_i b_i(u) \neq 0,$$

at any point in the domain interval $[u_2, u_{n+1}]$. Define the C^1-rational quadratic trigonometric spline curve for the control points $\{P_0, P_1, \dots, P_n\}$ as follows:

$$r(u) = \frac{\sum_{i=0}^{n} w_i b_i(u) P_i}{\sum_{i=0}^{n} w_i b_i(u)}, \tag{16}$$

where basis functions are as defined earlier. Although the shape parameter in the definition of b_i may be chosen to be variable, for the sake of convenience we shall assume $\lambda_i(u) = \lambda(\text{constant})$ for all i as in [**3**]. In Figure 3, we have demonstrated the basis functions for the rational trigonometric spline. The analysis for variable λ_i follows the same lines.

On the interval $[u_i, u_{i+1}]$, $r(u)$ is given by

$$r_i(u) = \frac{p_i(u)}{q_i(u)},$$

where

$$r_i(u) = \frac{P_{i-2}w_{i-2}b_{i-2}(u) + P_{i-1}w_{i-1}b_{i-1}(u) + P_i w_i b_i(u)}{w_{i-2}b_{i-2}(u) + w_{i-1}b_{i-1}(u) + w_i b_i(u)}. \tag{17}$$

It is clear that for the interval $[u_i, u_{i+1}]$ the curve remains in the convex hull of points A_{i-1}, P_{i-1}, A_i where A_i is now given by

$$\begin{aligned} A_i &= r(u_{i+1}) \\ &= \frac{w_{i-1}(1-\beta_i)P_{i-1} + w_i\beta_i P_i}{w_{i-1}(1-\beta_i) + w_i\beta_i}, \\ &= \frac{w_{i-1}\triangle_{i+1}P_{i-1} + w_i\triangle_i P_i}{w_{i-1}\triangle_{i+1} + w_i\triangle_i}. \end{aligned}$$

Let Q be a point on the line segment $\overline{A_{i-1}A_i}$ which divides it in the ratio $(1-\alpha) : \alpha$ so that

$$Q = (1-\alpha)A_{i-1} + \alpha A_i.$$

We further assume that the line segment QP_{i-1} meets the curve in the point $r_i(\tilde{u})$ for some parameter value $\tilde{u}$. Let $r_i(\tilde{u})$ divide the line segment $\overline{P_{i-1}Q}$ in ratio $1-\widetilde{w}:\widetilde{w}$. We may write

$$\begin{aligned} r_i(\tilde{u}) &= (1-\widetilde{w})Q + \widetilde{w}P_{i-1} \\ &= (1-\widetilde{w})(1-\alpha)\left[\frac{w_{i-2}\triangle_i P_{i-2} + w_{i-1}\triangle_{i-1}P_{i-1}}{w_{i-2}\triangle_i + w_{i-1}\triangle_{i-1}}\right] \\ &\quad + (1-\widetilde{w})\alpha\left[\frac{w_{i-1}\triangle_{i+1}P_{i-1} + w_i\triangle_i P_i}{w_{i-1}\triangle_{i+1} + w_i\triangle_i}\right] + \widetilde{w}P_{i-1}. \end{aligned} \tag{18}$$

Proceeding as in Section 3 and comparing the coefficients of P_{i-2}, P_{i-1} and P_i in (17) and (18) we get

$$(1-\widetilde{w})(1-\alpha) = \frac{b_{i-2}(\tilde{u})}{\triangle_i q_i(\tilde{u})}[w_{i-2}\triangle_i + w_{i-1}\triangle_{i-1}] \tag{19a}$$

$$(1-\widetilde{w})\alpha = \frac{b_i(\tilde{u})}{\triangle_i q_i(\tilde{u})}[w_{i-1}\triangle_{i+1} + w_i\triangle_i]. \tag{19b}$$

Combining (19a) and (19b) we get

$$(1-\widetilde{w}) = \frac{(w_{i-2}\triangle_i + w_{i-1}\triangle_{i-1})b_{i-2}(\tilde{u}) + (w_{i-1}\triangle_{i+1} + w_i\triangle_i)b_i(\tilde{u})}{q_i(\tilde{u})\triangle_i}.$$

After a little simplification we obtain

$$\widetilde{w} = \frac{w_{i-1}}{q_i(\tilde{u})}[(1+\lambda)(\sin\ t_i(\tilde{u}) + \cos\ t_i(\tilde{u})) - 1]. \tag{20}$$

This gives a measure of the distance of the curve segment from the control point P_{i-1}. We therefore have $\|r_i(\tilde{u}) - P_{i-1}\| = (1-\tilde{w})K$ where $K = \|Q - P_{i-1}\|$. Comparing with (9), we observe that in this case weight factors can be suitably chosen to obtain a curve sufficiently close to the control polygon. Figure 4 shows an example of rational quadratic spline curve. Taking $w_i = 1$ for all i we obtain the case of the polynomial trigonometric spline.

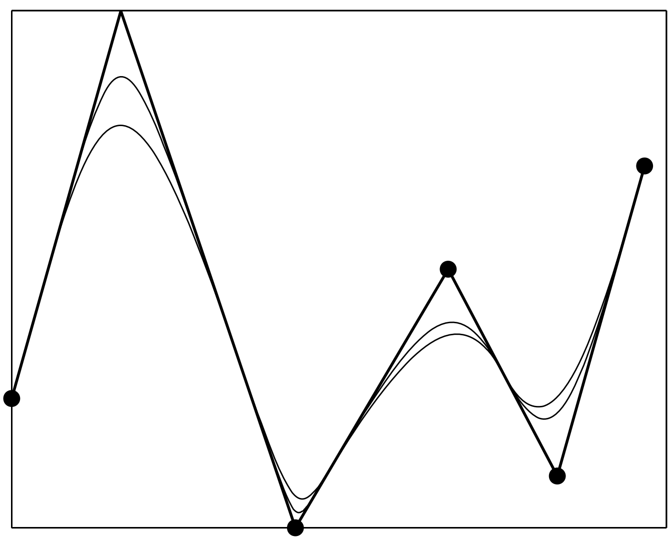

FIGURE 4. C^1 rational quadratic spline curves with two different sets of weight factors.

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