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CONDITIONAL ANALYTIC FEYNMAN INTEGRAL OVER PRODUCT SPACE OF WIENER PATHS IN ABSTRACT WIENER SPACE

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ABSTRACT. In this paper, we derive a simple formula for the conditional Wiener integrals over the product space of Wiener paths and evaluate analytic Feynman integrals and conditional analytic Feynman integrals of functionals in Banach algebras which are equivalent to the space of complex Borel measures on a real separable Hilbert space.

1. Introduction and preliminaries. Let $C_0[0,T]$ denote the classical Wiener space, that is, the space of real-valued continuous functions $x(t)$ defined on $[0,T]$ with $x(0) = 0$. The concept of conditional Wiener integral in this space was introduced by Yeh in [**14, 15**], and he used an inversion formula for evaluating some conditional Wiener integrals. On the other hand, Park and Skoug [**11**] derived a simple formula for evaluating conditional Wiener integrals, and Chung and Skoug introduced the concept of a conditional analytic Feynman integral on the classical Wiener space [**6**]. And then, using the simple formula, they evaluated the conditional analytic Feynman integrals of functions in the Banach algebra $\mathcal{S}$ which was introduced by Cameron and Storvick in [**2**]. Also, they proved that the conditional analytic Feynman integral of functions in $\mathcal{S}$ is a solution of the Schrödinger equation.

The space $C_0(\mathbf{B})$, which is the space of abstract Wiener space-valued continuous functions defined on $[0,T]$, was introduced by Kuelb and LePage in [**9**], and Ryu [**13**] introduced various properties on the space, which appear in the classical and abstract Wiener spaces. In [**3**], Chang, Cho and Yoo derived a simple formula for evaluating some conditional

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Wiener integrals on the space $C_0(\mathbf{B})$ and defined the conditional analytic Feynman integral on the space with the conditioning function $X_\tau(x) = (x(t_1), \dots, x(t_k))$. And then, they evaluated the conditional analytic Feynman integrals of various types of functions, in particular, functions which appear in quantum mechanics and Feynman integration theories.

In this paper, we derive a simple formula for the conditional Wiener integrals of functions defined on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ with the conditioning function $X_{\vec{\tau}}(x,y) = ((x(t_1), \dots, x(t_k)), (y(s_1), \dots, y(s_l)))$ and define the conditional analytic Feynman integral on the space $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ using the formula. Also, we introduce the Banach algebras $\mathcal{F}(C_0(\mathbf{B}); u)$, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ of functions defined on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$, which correspond to the Banach algebras $\mathcal{F}$, $\mathcal{F}_{A_1,A_2}$, respectively, in [**8**]. In fact, we show that the spaces $\mathcal{F}(C_0(\mathbf{B}); u)$, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$, $\mathcal{F}(\mathcal{H})$ ([**1**]), $\mathcal{F}$, $\mathcal{F}_{A_1,A_2}$ ([**8**]) and $\mathcal{M}(\mathcal{H})$ are all equivalent, where $\mathcal{M}(\mathcal{H})$ is the space of all complex Borel measures on a real separable Hilbert space $\mathcal{H}$. In particular, if $\mathcal{H} = L_2[0,T]$, then they all are equivalent to $\mathcal{S}$. And then, we evaluate the analytic Feynman integral and the conditional analytic Feynman integral of functions in $\mathcal{F}(C_0(\mathbf{B}); u)$ and $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let B be a real normed linear space and let $\mathcal{B}(B)$ be the Borel σ-field on B. Let $X : (\Omega, A, P) \to (B, \mathcal{B}(B))$ be a random variable, and let $F : \Omega \to \mathbf{C}$ be an integrable function. Let P_X be the probability distribution of X on $(B, \mathcal{B}(B))$, and let $\mathcal{D}$ be the σ-field $\{X^{-1}(A) : A \in \mathcal{B}(B)\}$. Let $P_{\mathcal{D}}$ be the probability measure induced by P, that is, $P_{\mathcal{D}}(E) = P(E)$ for $E \in \mathcal{D}$. By the definition of conditional expectation there exists a $\mathcal{D}$-measurable function $E[F|X]$ (the conditional expectation of F given X) defined on Ω such that the relation

$$\int_E E[F|X](\omega)\, dP_{\mathcal{D}}(\omega) = \int_E F(\omega)\, dP(\omega)$$

holds for every $E \in \mathcal{D}$. But there exists a P_X-integrable function ψ defined on B which is unique P_X almost everywhere such that $E[F|X](\omega) = (\psi \circ X)(\omega)$ for $P_{\mathcal{D}}$ almost everywhere ω in Ω. ψ is also called the conditional expectation of F given X and without loss of generality, it is denoted by $E[F|X](\xi)$ for $\xi \in B$. Throughout this paper, we will consider the function ψ as the conditional expectation of F given X.

2. Wiener paths in abstract Wiener space. Let $(\mathcal{H}, \mathbf{B}, m)$ be an abstract Wiener space [**10**]. Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that the e_js are in $\mathbf{B}^*$, the dual space of the real separable Banach space $\mathbf{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbf{B}$, define the stochastic inner product $(h, y)^\sim$ by

$$(h, y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j \rangle (y, e_j) & \text{if the limit exists;} \\ 0 & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbf{B}$ and $\mathbf{B}^*$ [**8**]. Note that for each h $(\neq 0)$ in $\mathcal{H}$, $(h, \cdot)^\sim$ is a Gaussian random variable on $\mathbf{B}$ with mean zero, variance $|h|^2$; also $(h, y)^\sim$ is essentially independent of the choice of the complete orthonormal set used in its definition and, further, $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda (h, y)^\sim$ for all $\lambda \in \mathbf{R}$. It is well known that, if $\{h_1, h_2, \ldots, h_n\}$ is an orthogonal set in $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$s are independent. Moreover, if both h and y are in $\mathcal{H}$, then $(h, y)^\sim = (h, y)$ where $(\cdot, \cdot)$ denotes the inner product h and y.

Let $C_0(\mathbf{B})$ denote the set of all continuous functions on $[0, T]$ into $\mathbf{B}$ which vanish at 0. Then $C_0(\mathbf{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbf{B})} \equiv \sup_{0 \leq t \leq T} \|x(t)\|_{\mathbf{B}}$. The minimal σ-field making the mapping $x \to x(t)$ measurable is $\mathcal{B}(C_0(\mathbf{B}))$, the Borel σ-field on $C_0(\mathbf{B})$. Further, Brownian motion in $\mathbf{B}$ induces a probability measure $m_{\mathbf{B}}$ on $(C_0(\mathbf{B}), \mathcal{B}(C_0(\mathbf{B})))$ which is mean-zero Gaussian [**13**]. We will introduce a concrete form of $m_{\mathbf{B}}$. Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \cdots < t_k \leq T$. Let $T_{\vec{t}} : \mathbf{B}^k \to \mathbf{B}^k$ be given by

$$\begin{aligned} &T_{\vec{t}}(x_1, x_2, \ldots, x_k) \\ &= \left(\sqrt{t_1 - t_0}\, x_1, \sqrt{t_1 - t_0}\, x_1 + \sqrt{t_2 - t_1}\, x_2, \ldots, \sum_{j=1}^{k} \sqrt{t_j - t_{j-1}}\, x_j \right). \end{aligned}$$

We define a set function $\nu_{\vec{t}}$ on $\mathcal{B}(\mathbf{B}^k)$ by

$$\nu_{\vec{t}}(B) = \left(\prod_1^k m \right) \left(T_{\vec{t}}^{-1}(B) \right)$$

for $B \in \mathcal{B}(\mathbf{B}^k)$. Then $\nu_{\vec{t}}$ is a Borel measure. Let $f_{\vec{t}} : C_0(\mathbf{B}) \to \mathbf{B}^k$ be the function defined by

$$f_{\vec{t}}(x) = (x(t_1), x(t_2), \ldots, x(t_k)).$$

For Borel subsets $B_1, B_2, \ldots, B_k$ of $\mathbf{B}$, $f_{\vec{t}}^{-1}(\prod_{j=1}^{k} B_j)$ is called the I-set with respect to $B_1, B_2, \ldots, B_k$. Then the collection $\mathcal{I}$ of all I-sets is a semi-algebra. We define a set function $m_{\mathbf{B}}$ on $\mathcal{I}$ by

$$m_{\mathbf{B}}\left(f_{\vec{t}}^{-1}\left(\prod_{j=1}^{k} B_j\right)\right) = \nu_{\vec{t}}\left(\prod_{j=1}^{k} B_j\right).$$

Then $m_{\mathbf{B}}$ is well defined and countably additive on $\mathcal{I}$. Using Carathéodory extension process, we have a Borel measure $m_{\mathbf{B}}$ on $\mathcal{B}(C_0(\mathbf{B}))$.

A complex-valued measurable function defined on $C_0(\mathbf{B})$ is said to be *Wiener measurable* and a Wiener measurable function is said to be *Wiener integrable* if it is integrable.

Definition 2.1. Let $F : C_0(\mathbf{B}) \to \mathbf{C}$ be Wiener integrable and let $X : (C_0(\mathbf{B}), \mathcal{B}(C_0(\mathbf{B})), m_{\mathbf{B}}) \to (B, \mathcal{B}(B))$ be a random variable, where B is a real normed linear space with the Borel σ-field $\mathcal{B}(B)$. The conditional expectation $E[F|X]$ of F given X defined on B is called the conditional Wiener integral of F given X.

Now, we introduce Wiener integration theorem without proof. For the proof, see [**13**].

Theorem 2.2 (Wiener integration theorem). *Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k \le T$, and let $f : \mathbf{B}^k \to \mathbf{C}$ be a Borel measurable function. Then*

$$\int_{C_0(\mathbf{B})} f(x(t_1), x(t_2), \ldots, x(t_k))\, dm_{\mathbf{B}}(x)$$
$$\stackrel{*}{=} \int_{\mathbf{B}^k} (f \circ T_{\vec{t}})(x_1, x_2, \ldots, x_k)\, d\left(\prod_{j=1}^{k} m\right)(x_1, x_2, \ldots, x_k),$$

where by $\stackrel{*}{=}$ we mean that, if either side exists, then both sides exist and they are equal.

For convenience, we adopt the following notations:

$$E_{C_0(\mathbf{B})}[F] = \int_{C_0(\mathbf{B})} F(x)\, dm_{\mathbf{B}}(x)$$

if F is integrable on $C_0(\mathbf{B})$ and

$$E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F] = \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} F(x,y)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y)$$

if F is integrable on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$.

A subset E of $C_0(\mathbf{B})$ is called a scale-invariant null set if λE is (Wiener) measurable and $m_{\mathbf{B}}(\lambda E) = 0$ for any $\lambda > 0$. A property is said to hold s almost everywhere if it holds except for a scale-invariant null set. Let F be defined on $C_0(\mathbf{B})$, and let $F^{\lambda}(x) = F(\lambda^{-1/2}x)$ for $\lambda > 0$. If F^{λ} is measurable for any $\lambda > 0$, then F is said to be scale-invariant measurable.

Suppose, for a scale-invariant measurable function F, $E_{C_0(\mathbf{B})}[F^{\lambda}]$ exists for every $\lambda > 0$ and it has an analytic extension $J^*_{\lambda}(F)$ on $\mathbf{C}_+ \equiv \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$. Then we call $J^*_{\lambda}(F)$ the analytic Wiener integral of F with parameter λ, and it is denoted by $E^{anw_{\lambda}}[F]$. Moreover, if for nonzero real q, $E^{anw_{\lambda}}[F]$ has a limit as λ approaches to $-iq$ through $\mathbf{C}_+$, then it is called the analytic Feynman integral of F with parameter q and is denoted by $E^{anf_q}[F]$.

Let $\tau_1 : 0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0,T]$, and let x be in $C_0(\mathbf{B})$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$[x](t) = \sum_{j=1}^{k} \chi_{(t_{j-1},t_j]}(t) \left[x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})) \right] \tag{2.1}$$

where $t \in [0,T]$. For each $\vec{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbf{B}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given by (2.1), replacing $x(t_j)$ by ξ_j for $j = 0, 1, \ldots, k$ ($\xi_0 = 0$). Note that both $[x] : [0,T] \to \mathbf{B}$ and $[\vec{\xi}] : [0,T] \to \mathbf{B}$ are in $C_0(\mathbf{B})$.

The following lemma is useful in defining the conditional analytic Wiener integral and Feynman integral. For the detailed proof, see [**3**].

Lemma 2.3. *Let F be defined and integrable on $C_0(\mathbf{B})$. Let $X_{\tau_1} : C_0(\mathbf{B}) \to \mathbf{B}^k$ be a random variable given by $X_{\tau_1}(x) = (x(t_1), \ldots, x(t_k))$. Then, for every Borel measurable subset B of $\mathbf{B}^k$,*

$$\int_{X_{\tau_1}^{-1}(B)} F(x)\, dm_{\mathbf{B}}(x) = \int_B E_{C_0(\mathbf{B})}[F(x - [x] + [\vec{\xi}])]\, dP_{X_{\tau_1}}(\vec{\xi}),$$

where $P_{X_{\tau_1}}$ is the probability distribution of X_{τ_1} on $(\mathbf{B}^k, \mathcal{B}(\mathbf{B}^k))$.

By the definition of the conditional Wiener integral (Definition 2.1) and Lemma 2.3, we have

$$E[F|X_{\tau_1}](\vec{\xi}) = E_{C_0(\mathbf{B})}[F(x - [x] + [\vec{\xi}])] \quad \text{for } P_{X_{\tau_1}} \quad \text{a.e.} \quad \vec{\xi}. \tag{2.2}$$

The equation (2.2) is called a simple formula for the conditional Wiener integral of F given X_{τ_1} on the space $C_0(\mathbf{B})$.

For $\lambda > 0$ let $X_{\tau_1}^{\lambda}(x) = X_{\tau_1}(\lambda^{-1/2}x)$ and for $\vec{\xi} \in \mathbf{B}^k$ suppose that $E[F^{\lambda}|X_{\tau_1}^{\lambda}](\vec{\xi})$ exists. From (2.2) we have

$$E[F^{\lambda}|X_{\tau_1}^{\lambda}](\vec{\xi}) = E_{C_0(\mathbf{B})}[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])]$$

for almost every $\vec{\xi} \in \mathbf{B}^k$. If, for $\vec{\xi} \in \mathbf{B}^k$, $E_{C_0(\mathbf{B})}[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])]$ has the analytic extension $J_{\lambda}^{*}(F)(\vec{\xi})$ on $\mathbf{C}_+$, then we write

$$E^{anw_{\lambda}}[F|X_{\tau_1}](\vec{\xi}) = J_{\lambda}^{*}(F)(\vec{\xi})$$

for $\lambda \in \mathbf{C}_+$. $E^{anw_{\lambda}}[F|X_{\tau_1}]$ is a version of the conditional analytic Wiener integral.

For nonzero real q and $\vec{\xi} \in \mathbf{B}^k$, if the limit

$$\lim_{\lambda \to -iq} E^{anw_{\lambda}}[F|X_{\tau_1}](\vec{\xi})$$

exists, where λ approaches to $-iq$ through $\mathbf{C}_+$, then we write

$$E^{anf_q}[F|X_{\tau_1}](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_{\lambda}}[F|X_{\tau_1}](\vec{\xi}).$$

$E^{anf_q}[F|X_{\tau_1}]$ is a version of the conditional analytic Feynman integral.

Let $\tau_2 : 0 = s_0 < s_1 < \cdots < s_l = T$ be another partition of $[0, T]$. Let $X_{\tau_2} : C_0(\mathbf{B}) \to \mathbf{B}^l$ be defined by $X_{\tau_2}(x) = (x(s_1), \ldots, x(s_l))$, and let $X_{\vec{\tau}} : C_0(\mathbf{B}) \times C_0(\mathbf{B}) \to \mathbf{B}^k \times \mathbf{B}^l$ be a random variable defined by

$$\begin{aligned} X_{\vec{\tau}}(x, y) &\equiv (X_{\tau_1}(x), X_{\tau_2}(y)) \\ &= ((x(t_1), \ldots, x(t_k)), (y(s_1), \ldots, y(s_l))). \end{aligned} \tag{2.3}$$

Note that, since $\mathbf{B}$ is separable, we have $\mathcal{B}(\mathbf{B}^k \times \mathbf{B}^l) = \mathcal{B}(\mathbf{B}^k) \times \mathcal{B}(\mathbf{B}^l)$.

Now, for a given random variable $X_{\vec{\tau}}$, we will define the conditional Wiener integral and the conditional Feynman integral of functions defined on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$. For this purpose, we need the following lemma.

Lemma 2.4. *Let F be defined and integrable on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$. Let $X_{\vec{\tau}}$ be given by* (2.3)*, and let $P_{X_{\vec{\tau}}}$ be the probability distribution of $X_{\vec{\tau}}$ on $(\mathbf{B}^k \times \mathbf{B}^l, \mathcal{B}(\mathbf{B}^k \times \mathbf{B}^l))$. Then, for any B in $\mathcal{B}(\mathbf{B}^k \times \mathbf{B}^l)$, we have*

$$\begin{aligned}&\int_{X_{\vec{\tau}}^{-1}(B)} F(x,y) d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y)\\ &= \int_B E_{C_0(\mathbf{B}) \times C_0(\mathbf{B})}[F(x - [x] + [\vec{\xi}_1], y - [y] + [\vec{\xi}_2])]\, dP_{X_{\vec{\tau}}}(\vec{\xi}_1, \vec{\xi}_2).\end{aligned}$$

Proof. Let $P_{X_{\tau_2}}$ be the probability distribution of X_{τ_2} on $(\mathbf{B}^l, \mathcal{B}(\mathbf{B}^l))$. Suppose that $F \geq 0$. For $B_1 \times B_2$ in $\mathcal{B}(\mathbf{B}^k) \times \mathcal{B}(\mathbf{B}^l)$, we have

$$\begin{aligned}&\int_{B_1 \times B_2} E[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2)\, dP_{X_{\vec{\tau}}}(\vec{\xi}_1, \vec{\xi}_2)\\ &\quad = \int_{X_{\tau_1}^{-1}(B_1)} \int_{X_{\tau_2}^{-1}(B_2)} F(x,y)\, dm_{\mathbf{B}}(y)\, dm_{\mathbf{B}}(x)\\ &\quad = \int_{X_{\tau_1}^{-1}(B_1)} \int_{B_2} E[F(x,\cdot)|X_{\tau_2}](\vec{\xi}_2)\, dP_{X_{\tau_2}}(\vec{\xi}_2)\, dm_{\mathbf{B}}(x)\\ &\quad = \int_{X_{\tau_1}^{-1}(B_1)} \int_{B_2} E_{C_0(\mathbf{B})}[F(x, y - [y] + [\vec{\xi}_2])]\, dP_{X_{\tau_2}}(\vec{\xi}_2)\, dm_{\mathbf{B}}(x),\end{aligned}$$

where the second equality follows from the definition of conditional expectation and the last equality from Lemma 2.3. Since F is nonnegative, we have

$$\begin{aligned}&\int_{B_1 \times B_2} E[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2)\, dP_{X_{\vec{\tau}}}(\vec{\xi}_1, \vec{\xi}_2)\\ &= \int_{B_2} \int_{X_{\tau_1}^{-1}(B_1)} E_{C_0(\mathbf{B})}[F(x, y - [y] + [\vec{\xi}_2])]\, dm_{\mathbf{B}}(x)\, dP_{X_{\tau_2}}(\vec{\xi}_2)\end{aligned}$$

$$= \int_{B_2} \int_{B_1} E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(x-[x]+[\vec{\xi}_1], y-[y]+[\vec{\xi}_2])] \, dP_{X_{\tau_1}}(\vec{\xi}_1)\, dP_{X_{\tau_2}}(\vec{\xi}_2)$$

$$= \int_{B_1\times B_2} E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(x-[x]+[\vec{\xi}_1], y-[y]+[\vec{\xi}_2])]\, dP_{X_{\vec{\tau}}}(\vec{\xi}_1,\vec{\xi}_2).$$

Since $\{B_1 \times B_2 : B_1 \in \mathcal{B}(\mathbf{B}^k), B_2 \in \mathcal{B}(\mathbf{B}^l)\}$ forms a semi-algebra, we have the result for the case where F is nonnegative.

The general case follows easily. □

For a function F defined on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$, let $F^{\lambda_1,\lambda_2}(x,y) = F(\lambda_1^{-1/2}x,\ \lambda_2^{-1/2}y)$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. F is said to be scale-invariant measurable if F^{λ_1,λ_2} is measurable for all $\lambda_1, \lambda_2 > 0$. Given two $\mathbf{C}$-valued scale-invariant measurable functions F and G on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$, F is equal to G s almost everywhere if, for each $\lambda_1, \lambda_2 > 0$, $(m_{\mathbf{B}} \times m_{\mathbf{B}})(\{(x,y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B}) : F^{\lambda_1,\lambda_2}(x,y) \neq G^{\lambda_1,\lambda_2}(x,y)\}) = 0$.

Let F be a $\mathbf{C}$-valued measurable function on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ such that the integral

$$J_{\lambda_1,\lambda_2}(F) = \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} F^{\lambda_1,\lambda_2}(x,y)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y)$$

exists as a finite number for $\lambda_1 > 0$, $\lambda_2 > 0$. If there exists a function $J^*_{\lambda_1,\lambda_2}(F)$, analytic in (λ_1,λ_2) on $\mathbf{C}_+ \times \mathbf{C}_+$ such that $J^*_{\lambda_1,\lambda_2}(F) = J_{\lambda_1,\lambda_2}(F)$ for all $\lambda_1 > 0$, $\lambda_2 > 0$, then $J^*_{\lambda_1,\lambda_2}(F)$ is defined to be the analytic Wiener integral of F over $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ with parameter (λ_1, λ_2) and we write

$$E^{anw_{\lambda_1,\lambda_2}}[F] = J^*_{\lambda_1,\lambda_2}(F)$$

for $(\lambda_1,\lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$. Let $(q_1,q_2) \in \mathbf{R}^2$, $q_1 \neq 0, q_2 \neq 0$, and let F be a $\mathbf{C}$-valued measurable function such that $E^{anw_{\lambda_1,\lambda_2}}[F]$ exists for all $(\lambda_1,\lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F over $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ with parameter (q_1,q_2), and we write

$$E^{anf_{q_1,q_2}}[F] = \lim_{\substack{\lambda_1 \to -iq_1 \\ \lambda_2 \to -iq_2}} E^{anw_{\lambda_1,\lambda_2}}[F]$$

where λ_j approaches to $-iq_j$ through $\mathbf{C}_+$ for each $j = 1, 2$.

By the definition of the conditional expectation and Lemma 2.4, we have

$$(2.4)\quad E[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) = E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(x-[x]+[\vec{\xi}_1], y-[y]+[\vec{\xi}_2])]$$

for $P_{X_{\vec{\tau}}}$ almost everywhere $(\vec{\xi}_1,\vec{\xi}_2)$ in $\mathbf{B}^k\times\mathbf{B}^l$. For $\lambda_1,\lambda_2>0$, let $X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x,y)=X_{\vec{\tau}}(\lambda_1^{-1/2}x,\,\lambda_2^{-1/2}y)$ and for $(\vec{\xi}_1,\vec{\xi}_2)\in\mathbf{B}^k\times\mathbf{B}^l$, suppose $E[F^{\lambda_1,\lambda_2}|X_{\vec{\tau}}^{\lambda_1,\lambda_2}](\vec{\xi}_1,\,\vec{\xi}_2)$ exists. From (2.4), we have

$$\begin{aligned}&E[F^{\lambda_1,\lambda_2}|X_{\vec{\tau}}^{\lambda_1,\lambda_2}](\vec{\xi}_1,\vec{\xi}_2)\\&\quad=E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(\lambda_1^{-1/2}(x-[x])+[\vec{\xi}_1],\lambda_2^{-1/2}(y-[y])+[\vec{\xi}_2])].\end{aligned}$$

If, for $(\vec{\xi}_1,\vec{\xi}_2)\in\mathbf{B}^k\times\mathbf{B}^l$, $E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(\lambda_1^{-1/2}(x-[x])+[\vec{\xi}_1],\lambda_2^{-1/2}(y-[y])+[\vec{\xi}_2])]$ has the analytic extension $J^*_{\lambda_1,\lambda_2}(F)(\vec{\xi}_1,\vec{\xi}_2)$ on $\mathbf{C}_+\times\mathbf{C}_+$, then we call it the conditional analytic Wiener integral of F given $X_{\vec{\tau}}$ over $C_0(\mathbf{B})\times C_0(\mathbf{B})$, and we write

$$E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)=J^*_{\lambda_1,\lambda_2}(F)(\vec{\xi}_1,\vec{\xi}_2)$$

for $(\lambda_1,\lambda_2)\in\mathbf{C}_+\times\mathbf{C}_+$. If, for $(q_1,q_2)(q_j\neq 0$ for $j=1,2)$ in $\mathbf{R}^2$ and for $(\vec{\xi}_1,\vec{\xi}_2)\in\mathbf{B}^k\times\mathbf{B}^l$, the limit

$$\lim_{\substack{\lambda_1\to -iq_1\\ \lambda_2\to -iq_2}} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)$$

exists, where λ_j approaches to $-iq_j$ through $\mathbf{C}_+$ for each $j=1,2$, then it is called the conditional analytic Feynman integral of F given $X_{\vec{\tau}}$ over $C_0(\mathbf{B})\times C_0(\mathbf{B})$, and we write

$$E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)=\lim_{\substack{\lambda_1\to -iq_1\\ \lambda_2\to -iq_2}} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2).$$

Next we state a well-known integration formula which we use several times below.

Lemma 2.5. *Let $(\mathcal{H},\mathbf{B},m)$ be an abstract Wiener space, and let $h\in\mathcal{H}$. Then we have*

$$\int_{\mathbf{B}}\exp\{i(h,x_1)^{\sim}\}\,dm(x_1)=\exp\Big\{-\frac{|h|^2}{2}\Big\}.$$

3. Evaluation formulas for conditional Feynman integrals of functions in the Banach algebra $\mathcal{F}(C_0(\mathbf{B});u)$. Let $\mathcal{H}$ be an infinite dimensional real separable Hilbert space, and let $\mathcal{M}(\mathcal{H})$ be the class of all $\mathbf{C}$-valued Borel measures on $\mathcal{H}$ with bounded variation. Then $\mathcal{M}(\mathcal{H})$ is a Banach algebra under the total variation norm and with convolution as multiplication.

Throughout the remainder of this paper, let $0 < u \le T$ be fixed, but arbitrarily.

Definition 3.1. Let $\mathcal{F}(C_0(\mathbf{B});u)$ be the space of all s-equivalence classes of functions F which for $\sigma \in \mathcal{M}(\mathcal{H})$ have the form

$$F(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^{\sim}\}\, d\sigma(h) \tag{3.1}$$

for $x \in C_0(\mathbf{B})$.

Theorem 3.2. *The space $\mathcal{F}(C_0(\mathbf{B});u)$ in Definition 3.1 is an algebra over $\mathbf{C}$ under point-wise addition, point-wise multiplication and complex scalar multiplication. Moreover, $\mathcal{F}(C_0(\mathbf{B});u)$ is a Banach algebra with a norm $\|\cdot\|$ defined by*

$$\|F\| = \|\sigma\|, \tag{3.2}$$

where F and σ are related by (3.1) and $\|\sigma\|$ is the total variation of the complex measure σ.

Proof. It is easily verified that the operations of point-wise addition, point-wise multiplication and scalar multiplication can be regarded as operations on the equivalence classes of $\mathcal{F}(C_0(\mathbf{B});u)$. Denoting by F_σ the function defined by (3.1) for given $\sigma \in \mathcal{M}(\mathcal{H})$, clearly $F_{\sigma_1} + F_{\sigma_2} = F_{\sigma_1+\sigma_2}$, $F_{\sigma_1} \cdot F_{\sigma_2} = F_{\sigma_1 * \sigma_2}$ and $\lambda F_{\sigma_1} = F_{\lambda\sigma_1}$, for all σ_1, σ_2 in $\mathcal{M}(\mathcal{H})$ and $\lambda \in \mathbf{C}$. Hence,

$$\sigma \longmapsto [F_\sigma], \tag{3.3}$$

where $[F_\sigma]$ denotes the s-equivalence class of F_σ, defines a map of $\mathcal{M}(\mathcal{H})$ onto $\mathcal{F}(C_0(\mathbf{B});u)$ which is an algebra homomorphism. It remains only to show that (3.3) is one to one. To prove this fact, it suffices to show that $F_\sigma = 0$ s almost everywhere with respect to $m_{\mathbf{B}}$ implies that $\sigma = 0$.

Note that if $F_1(x_1) = 0$ for almost every $x_1 \in \mathbf{B}$ with respect to the measure m, where for $\mu \in \mathcal{M}(\mathcal{H})$ the function F_1 is defined by

$$F_1(x_1) = \int_{\mathcal{H}} \exp\{i(h, x_1)^\sim\}\, d\mu(h), \tag{3.4}$$

then we have $\mu = 0$ [**8**, Proposition 2.1].

Let F_1 be the function given by (3.4) with replacing μ by σ. Then $0 = F_\sigma(x) = F_1(x(u))$ for s almost everywhere x in $C_0(\mathbf{B})$ with respect to the measure $m_{\mathbf{B}}$. To complete the proof we must show that $F_1(x_1) = 0$ almost everywhere x_1 in $\mathbf{B}$ with respect to the measure m. Let $B_1 = \{x_1 \in \mathbf{B} : F_1(x_1) \neq 0\}$, and let $B = \{x \in C_0(\mathbf{B}) : F_1(x(u)) \neq 0\}$. Since B is a scale-invariant null set, we have

$$\begin{aligned}
0 = m_{\mathbf{B}}(\sqrt{u}B) &= \int_{C_0(\mathbf{B})} \chi_{\sqrt{u}B}(x)\, dm_{\mathbf{B}}(x) \\
&= \int_{C_0(\mathbf{B})} \chi_B\left(\frac{1}{\sqrt{u}}x\right) dm_{\mathbf{B}}(x) \\
&= \int_{C_0(\mathbf{B})} \chi_{B_1}\left(\frac{1}{\sqrt{u}}x(u)\right) dm_{\mathbf{B}}(x) \\
&= \int_{\mathbf{B}} \chi_{B_1}\left(\frac{1}{\sqrt{u}}\sqrt{u}x_1\right) dm(x_1) = m(B_1)
\end{aligned}$$

by Theorem 2.2 which completes the proof as desired. □

Corollary 3.3. *Define the space $\mathcal{F}$ as the class of all functions on $\mathbf{B}$ of the form, for $\sigma \in \mathcal{M}(\mathcal{H})$,*

$$F(x_1) = \int_{\mathcal{H}} \exp\{i(h, x_1)^\sim\}\, d\sigma(h) \tag{3.5}$$

for s almost everywhere x_1 in $\mathbf{B}$ with respect to the measure m, and define the space $\mathcal{F}(\mathcal{H})$ as the class of all functions on $\mathcal{H}$ of the form, for $\sigma \in \mathcal{M}(\mathcal{H})$,

$$F(h) = \int_{\mathcal{H}} \exp\{i(h', h)\}\, d\sigma(h') \tag{3.6}$$

for any $h \in \mathcal{H}$. Then both $\mathcal{F}$ and $\mathcal{F}(\mathcal{H})$ are Banach algebras. Moreover, $\mathcal{M}(\mathcal{H})$, $\mathcal{F}$, $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(C_0(\mathbf{B});u)$ are all isomorphic as Banach algebras. In particular, if $\mathcal{H}=L_2[0,T]$, then they all are isomorphic to $\mathcal{S}$ which is the Banach algebra given in [**2**].

Proof. The results follow by Propositions 2.1 and 2.3 in [**8**], Theorems 2.1 and 2.3 in [**2**] and Theorem 3.2. □

Now we evaluate the analytic Wiener and Feynman integrals of functions in $\mathcal{F}(C_0(\mathbf{B});u)$.

Theorem 3.4. *Let $F \in \mathcal{F}(C_0(\mathbf{B});u)$ be given by* (3.1). *Then $E^{anw_\lambda}[F]$ exists for $\lambda \in \mathbf{C}_+$ and, for any nonzero real q, $E^{anf_q}[F]$ exists. Moreover, we have*

$$E^{anw_\lambda}[F] = \int_{\mathcal{H}} \exp\left\{-\frac{u}{2\lambda}|h|^2\right\} d\sigma(h)$$

and

$$E^{anf_q}[F] = \int_{\mathcal{H}} \exp\left\{-\frac{ui}{2q}|h|^2\right\} d\sigma(h).$$

Proof. Let $\lambda > 0$. Then we have

$$\begin{aligned}\int_{C_0(\mathbf{B})} F^\lambda(x)\,dm_{\mathbf{B}}(x) &= \int_{C_0(\mathbf{B})}\int_{\mathcal{H}} \exp\{i(h,\lambda^{-1/2}x(u))^\sim\}\,d\sigma(h)dm_{\mathbf{B}}(x)\\ &= \int_{\mathcal{H}}\int_{C_0(\mathbf{B})} \exp\{i(h,\lambda^{-1/2}x(u))^\sim\}\,dm_{\mathbf{B}}(x)d\sigma(h)\end{aligned}$$

by Fubini's theorem. By Theorem 2.2 and Lemma 2.5, we have

$$\begin{aligned}\int_{C_0(\mathbf{B})} F^\lambda(x)\,dm_{\mathbf{B}}(x) &= \int_{\mathcal{H}}\int_{\mathbf{B}} \exp\{i\lambda^{-1/2}(h,\sqrt{u}x_1))^\sim\}\,dm(x_1)\sigma(h)\\ &= \int_{\mathcal{H}} \exp\left\{-\frac{u}{2\lambda}|h|^2\right\}d\sigma(h).\end{aligned}$$

The results now follow by Morera's theorem and the dominated convergence theorem. □

Theorem 3.5. *Let $F \in \mathcal{F}(C_0(\mathbf{B}); u)$ be given by* (3.1)*, and let X_{τ_1} be as in Lemma* 2.3*. Then $E^{anw_\lambda}[F|X_{\tau_1}]$ exists for all $\lambda \in \mathbf{C}_+$ and, for any nonzero real q, $E^{anf_q}[F|X_{\tau_1}]$ exists. Moreover, when $t_{p-1} < u < t_p$ for some $p \in \{1, \dots, k\}$, we have*

$$E^{anw_\lambda}[F|X_{\tau_1}](\vec{\xi}) = \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\} d\sigma(h)$$

for s almost everywhere $\vec{\xi} \in \mathbf{B}^k$ and

$$E^{anf_q}[F|X_{\tau_1}](\vec{\xi}) = \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} d\sigma(h),$$

where $\Gamma = [(t_p - u)(u - t_{p-1})]/(t_p - t_{p-1})$. When $u = t_p$ for some $p \in \{1, \dots, k\}$, we have

$$E^{anw_\lambda}[F|X_{\tau_1}](\vec{\xi}) = E^{anf_q}[F|X_{\tau_1}](\vec{\xi}) = F([\vec{\xi}]).$$

Proof. Let $t_{p-1} < u < t_p$ for some $p \in \{1, \dots, k\}$. For $\lambda > 0$ and for s almost everywhere $\vec{\xi} \in \mathbf{B}^k$, we have

$$\begin{aligned}
&E_{C_0(\mathbf{B})}[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])] \\
&= \int_{C_0(\mathbf{B})} \int_{\mathcal{H}} \exp\{i(h, \lambda^{-1/2}(x(u) - [x](u)) + [\vec{\xi}](u))^\sim\} \, d\sigma(h) \, dm_{\mathbf{B}}(x) \\
&= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^\sim\} \int_{C_0(\mathbf{B})} \exp\bigg\{ i\lambda^{-1/2}\bigg(h, x(u) - x(t_{p-1}) \\
&\qquad\qquad - \frac{u - t_{p-1}}{t_p - t_{p-1}}(x(t_p) - x(t_{p-1}))\bigg)^\sim \bigg\} \, dm_{\mathbf{B}}(x) \, d\sigma(h)
\end{aligned}$$

by Fubini's theorem. Let $\alpha = [(t_p - u)(u - t_{p-1})^{1/2}]/(t_p - t_{p-1})$ and $\beta = -[(u - t_{p-1})(t_p - u)^{1/2}]/(t_p - t_{p-1})$. By Theorem 2.2, we have

$$\begin{aligned}
&E_{C_0(\mathbf{B})}[F(\lambda^{-1/2}(x - [x]) + [\vec{\xi}])] \\
&\quad = \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^{\sim}\} \\
&\quad \times \int_{\mathbf{B}^2} \exp\{i\lambda^{-1/2}(\alpha(h, x_1)^{\sim} + \beta(h, x_2)^{\sim})\}\, dm^2(x_1, x_2)\, d\sigma(h) \\
&\quad = \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^{\sim}\} \exp\left\{-\frac{\alpha^2 + \beta^2}{2\lambda}|h|^2\right\} d\sigma(h) \\
&\quad = \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u))^{\sim}\} \exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\} d\sigma(h),
\end{aligned}$$

where $\Gamma = [(t_p - u)(u - t_{p-1})]/(t_p - t_{p-1})$ and the second equality follows from Lemma 2.5. By Morera's theorem and the dominated convergence theorem, the results follow.

When $s = t_p$ for some $p \in \{1, \ldots, k\}$, the results follow trivially. □

Remark 3.6. (1) For each H in $\mathcal{B}(\mathcal{H})$, let

$$\mu_\lambda(H) = \int_H \exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\} d\sigma(h) \quad \text{and}$$
$$\mu_q(H) = \int_H \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} d\sigma(h),$$

and for s almost everywhere x in $C_0(\mathbf{B})$, let

$$F_\lambda(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^{\sim}\}\, d\mu_\lambda(h) \quad \text{and}$$
$$F_q(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^{\sim}\}\, d\mu_q(h).$$

Then both F_λ and F_q are in $\mathcal{F}(C_0(\mathbf{B}); u)$ and, by Theorem 3.5, we have

$$E^{anw_\lambda}[F|X_{\tau_1}](\vec{\xi}) = F_\lambda([\vec{\xi}]) \quad \text{and} \quad E^{anf_q}[F|X_{\tau_1}](\vec{\xi}) = F_q([\vec{\xi}]).$$

(2) Let A be a nonnegative bounded self-adjoint operator on $\mathcal{H}$. Then the function F which for $\sigma \in \mathcal{M}(\mathcal{H})$ has the form

$$F(x) = \int_{\mathcal{H}} \exp\{i(A^{1/2}h, x(u))^{\sim}\}\, d\sigma(h) \tag{3.7}$$

for s almost everywhere $x \in C_0(\mathbf{B})$ belongs to $\mathcal{F}(C_0(\mathbf{B});u)$ since $\sigma \circ (A^{1/2})^{-1} \in \mathcal{M}(\mathcal{H})$ and F can be rewritten as

$$F(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^\sim\}\, d(\sigma \circ (A^{1/2})^{-1})(h)$$

by the change of variable theorem. By Theorems 3.4 and 3.5, we have

$$E^{anf_q}[F] = \int_{\mathcal{H}} \exp\left\{-\frac{ui}{2q}(Ah, h)\right\} d\sigma(h) \tag{3.8}$$

and

(3.9)

$$E^{anf_q}[F|X_{\tau_1}](\vec{\xi}) = \int_{\mathcal{H}} \exp\{(A^{1/2}h, [\vec{\xi}](u))^\sim\} \exp\left\{-\frac{i\Gamma}{2q}(Ah, h)\right\} d\sigma(h).$$

4. Evaluation formulas for conditional Feynman integrals of functions in the Banach algebra $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$. In this section, we define a Banach algebra $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$, which corresponds to the class $\mathcal{F}_{A_1,A_2}$ of functions defined on $\mathbf{B}$ [**8**]. And then we evaluate the analytic Feynman integral and conditional analytic Feynman integral of functions in $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$.

Let A_1 and A_2 be two nonnegative bounded self-adjoint operators on $\mathcal{H}$. Let $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ be the space of all s-equivalence classes of functions F which for σ in $\mathcal{M}(\mathcal{H})$ have the form

$$F(x, y) = \int_{\mathcal{H}} \exp\{i[(A_1^{1/2}h, x(u))^\sim + (A_2^{1/2}h, y(u))^\sim]\}\, d\sigma(h) \tag{4.1}$$

for (x, y) in $C_0(\mathbf{B}) \times C_0(\mathbf{B})$. As is customary we will identify a function with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ as a class of functions on $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ rather than as a class of equivalence classes.

Now we treat the analog of Theorem 3.2. For $\sigma \in \mathcal{M}(\mathcal{H})$, and for F defined by (4.1), we denote the s-equivalence class of F by $[F]$. Let the operations of addition, multiplication and complex scalar multiplication on the s-equivalence class of $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ be generated by the element-wise operations as in Theorem 3.2. From (4.1) and the fact that $\mathcal{M}(\mathcal{H})$ is an algebra under convolution, it follows that $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ is an algebra.

Theorem 4.1. *Let A_1 and A_2 be nonnegative bounded self-adjoint operators on $\mathcal{H}$. Then the map defined by*

$$\sigma \longmapsto [F] \tag{4.2}$$

sets up an algebra homomorphism between $\mathcal{M}(\mathcal{H})$ and $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$. This is an isomorphism if and only if $R(A_1+A_2)$ is dense in $\mathcal{H}$, where $R(A_1+A_2)$ is the range of A_1+A_2. In this case, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ becomes a Banach algebra under the norm $\|F\|=\|\sigma\|$, and (4.2) is an isomorphism of Banach algebras.

Proof. Let $\mathcal{F}_{A_1,A_2}$ be the space of all s-equivalence classes $[F_1]$ of functions F_1 defined on $\mathbf{B}\times\mathbf{B}$ which for $\sigma\in\mathcal{M}(\mathcal{H})$ have the form

$$F_1(x_1,x_2)=\int_{\mathcal{H}}\exp\{i[(A_1^{1/2}h,x_1)^\sim+(A_2^{1/2}h,x_2)^\sim]\}\,d\sigma(h) \tag{4.3}$$

for $(x_1,x_2)\in\mathbf{B}\times\mathbf{B}$. In view of [**8**, Proposition 3.2], the map defined by

$$\sigma \longmapsto [F_1] \tag{4.4}$$

is a Banach algebra isomorphism between $\mathcal{M}(\mathcal{H})$ and $\mathcal{F}_{A_1,A_2}$ if and only if $R(A_1+A_2)$ is dense in $\mathcal{H}$.

Suppose that $R(A_1+A_2)$ is dense in $\mathcal{H}$. Let F be given by (4.1), and let $F(x,y)=F_1(x(u),y(u))=0$ for s almost everywhere (x,y) in $C_0(\mathbf{B})\times C_0(\mathbf{B})$, where F_1 is given by (4.3). Let $B_1=\{(x_1,x_2)\in\mathbf{B}\times\mathbf{B}: F_1(x_1,x_2)\neq 0\}$, and let $B=\{(x,y)\in C_0(\mathbf{B})\times C_0(\mathbf{B}):(x(u),y(u))\in B_1\}$. Then we have

$$\begin{aligned}
0&=(m_{\mathbf{B}}\times m_{\mathbf{B}})(\sqrt{u}B)\\
&=\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})}\chi_{\sqrt{u}B}(x,y)\,d(m_{\mathbf{B}}\times m_{\mathbf{B}})(x,y)\\
&=\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})}\chi_{\sqrt{u}B_1}(x(u),y(u))\,d(m_{\mathbf{B}}\times m_{\mathbf{B}})(x,y)\\
&=\int_{\mathbf{B}\times\mathbf{B}}\chi_{B_1}\left(\frac{1}{\sqrt{u}}\sqrt{u}x_1,\frac{1}{\sqrt{u}}\sqrt{u}x_2\right)d(m\times m)(x_1,x_2)\\
&=(m\times m)(B_1),
\end{aligned}$$

where the fourth equality follows from Theorem 2.2. By Proposition 3.2 in [**8**], $\sigma = 0$. Thus the map given by (4.2) is one to one and hence it is an isomorphism.

Conversely, suppose that the map given by (4.2) is one to one. To complete the proof, we must show that the map given by (4.4) is one to one. Let $F_1(x_1, x_2) = 0$ for s almost everywhere (x_1, x_2) in $\mathbf{B}\times\mathbf{B}$, where F_1 is given by (4.3). Let $B_1 = \{(x_1, x_2) \in \mathbf{B}\times\mathbf{B} : F_1(x_1, x_2) \neq 0\}$ and $B = \{(x, y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B}) : F_1(x(u), y(u)) \neq 0\}$. For $\lambda_1, \lambda_2 > 0$, let $B_{\lambda_1,\lambda_2} = \{(\lambda_1 x, \lambda_2 y) : (x, y) \in B\}$. Then we have

$$
\begin{aligned}
&(m_{\mathbf{B}} \times m_{\mathbf{B}})(B_{\lambda_1,\lambda_2}) \\
&\qquad = \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} \chi_{B_{\lambda_1,\lambda_2}}(x, y)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&\qquad = \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} \chi_{B_1}\left(\frac{1}{\lambda_1}x(u), \frac{1}{\lambda_2}y(u)\right) d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&\qquad = \int_{\mathbf{B}\times\mathbf{B}} \chi_{B_1}\left(\frac{1}{\lambda_1}\sqrt{u}x_1, \frac{1}{\lambda_2}\sqrt{u}x_2\right) d(m \times m)(x_1, x_2) \\
&\qquad = (m \times m)(\{((\lambda_1/\sqrt{u})x_1, (\lambda_2/\sqrt{u})x_2) : (x_1, x_2) \in B_1\}) \\
&\qquad = 0,
\end{aligned}
$$

where the third equality follows from Theorem 2.2. Since the map given by (4.2) is one to one, we have $\sigma = 0$ and hence the result follows. $\square$

Corollary 4.2. *Let A be a nonnegative bounded self-adjoint operator on $\mathcal{H}$, and let $\mathcal{F}_A(C_0(\mathbf{B}); u)$ be the space of all s-equivalence classes of functions F which for $\sigma \in \mathcal{M}(\mathcal{H})$ have the form given by* (3.7). *Then the map*

$$\sigma \longmapsto [F] \tag{4.5}$$

sets up an algebra homomorphism between $\mathcal{M}(\mathcal{H})$ and $\mathcal{F}_A(C_0(\mathbf{B}); u)$. This is an isomorphism if and only if $R(A)$ is dense in $\mathcal{H}$, where $R(A)$ is the range of A. In this case, $\mathcal{F}_A(C_0(\mathbf{B}); u)$ becomes a Banach algebra under the norm $\|F\| = \|\sigma\|$, and (4.5) *is an isomorphism of Banach algebras.*

Proof. Let $A_1 = A$ and $A_2 = 0$ in Theorem 4.1. For any σ in $\mathcal{M}(\mathcal{H})$, let

$$F_{A,0}(x,y) = \int_{\mathcal{H}} \exp\{i(A^{1/2}h, x(u))^{\sim}\}\, d\sigma(h) \tag{4.6}$$

for (x,y) in $C_0(\mathbf{B}) \times C_0(\mathbf{B})$, and let

$$F_A(x) = \int_{\mathcal{H}} \exp\{i(A^{1/2}h, x(u))^{\sim}\}\, d\sigma(h), \tag{4.7}$$

for x in $C_0(\mathbf{B})$. Define a map from $\mathcal{F}_{A,0}(C_0(\mathbf{B})^2; u)$ to $\mathcal{F}_A(C_0(\mathbf{B}); u)$ by

$$[F_{A,0}] \longmapsto [F_A] \tag{4.8}$$

where $[F_{A,0}], [F_A]$ are s-equivalence classes of $F_{A,0}, F_A$, respectively. Clearly, the map given by (4.8) is well defined and an algebra onto homomorphism. It remains only to show that the map is one to one. Let $F_A(x) = 0$ for s almost everywhere $x \in C_0(\mathbf{B})$, and let $B_A = \{x \in C_0(\mathbf{B}) : F_A(x) \neq 0\}$. Let $B_{A,0} = \{(x,y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B}) : F_{A,0}(x,y) \neq 0\}$. Then we have, for $\lambda_1, \lambda_2 > 0$,

$$\begin{aligned}
(m_{\mathbf{B}} \times m_{\mathbf{B}})(\{(\lambda_1 x, \lambda_2 y) : (x,y) \in B_{A,0}\})& \\
&= (m_{\mathbf{B}} \times m_{\mathbf{B}})(\lambda_1 B_A \times \lambda_2 C_0(\mathbf{B})) \\
&= m_{\mathbf{B}}(\lambda_1 B_A) m_{\mathbf{B}}(\lambda_2 C_0(\mathbf{B})) = 0 \cdot 1 = 0.
\end{aligned}$$

Thus, $F_{A,0}(x,y) = 0$ for s almost everywhere $(x,y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B})$. Thus, we have the result as desired. □

Corollary 4.3. *Let A, A_1 and A_2 be nonnegative bounded self-adjoint operators on $\mathcal{H}$. Suppose both ranges of A and $A_1 + A_2$ are dense in $\mathcal{H}$. Let $\mathcal{S}$ be the Banach algebra given as in* [**2**]*, let $\mathcal{F}(C_0(\mathbf{B}); u)$ be given as in Definition* 3.1*, let $\mathcal{F}$, $\mathcal{F}(\mathcal{H})$ be given as in Corollary* 3.3*, let $\mathcal{F}_{A_1,A_2}$, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ be given as in Theorem 4.1, and let $\mathcal{F}_A(C_0(\mathbf{B}); u)$ be given as in Corollary* 4.2*. Then $\mathcal{M}(\mathcal{H})$, $\mathcal{F}$, $\mathcal{F}(\mathcal{H})$, $\mathcal{F}(C_0(\mathbf{B}); u)$, $\mathcal{F}_{A_1,A_2}$, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ and $\mathcal{F}_A(C_0(\mathbf{B}); u)$ all are isomorphic as Banach algebras. In particular, if $\mathcal{H} = L_2[0,T]$, then they all are isomorphic to $\mathcal{S}$.*

Proof. The results follow from Theorem 4.1, Corollaries 3.3 and 4.2, and Proposition 3.2 in [**8**]. □

Theorem 4.4. *Let $F \in \mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ be given by (4.1). Then $E^{anw_{\lambda_1,\lambda_2}}[F]$ exists for $(\lambda_1,\lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$ and $E^{anf_{q_1,q_2}}[F]$ exists for any $(q_1,q_2) \in \mathbf{R}^2$, ($q_j \neq 0$ for $j = 1,2$). Moreover, we have*

$$E^{anw_{\lambda_1,\lambda_2}}[F] = \int_{\mathcal{H}} \exp\left\{-\frac{u}{2\lambda_1}(A_1h,h) - \frac{u}{2\lambda_2}(A_2h,h)\right\} d\sigma(h)$$

and

$$E^{anf_{q_1,q_2}}[F] = \int_{\mathcal{H}} \exp\left\{-\frac{ui}{2q_1}(A_1h,h) - \frac{ui}{2q_2}(A_2h,h)\right\} d\sigma(h).$$

Proof. For $\lambda_1 > 0$, $\lambda_2 > 0$, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} F^{\lambda_1,\lambda_2}(x,y)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y) \\
&= \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} \int_{\mathcal{H}} \exp\{i[(A_1^{1/2}h, \lambda_1^{-1/2}x(u))^\sim + (A_2^{1/2}h, \lambda_2^{-1/2}y(u))^\sim]\} \\
&\qquad\qquad d\sigma(h)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y) \\
&= \int_{\mathcal{H}} \int_{C_0(\mathbf{B})} \int_{C_0(\mathbf{B})} \exp\{i[(A_1^{1/2}h, \lambda_1^{-1/2}x(u))^\sim + (A_2^{1/2}h, \lambda_2^{-1/2}y(u))^\sim]\} \\
&\qquad\qquad dm_{\mathbf{B}}(x)\, dm_{\mathbf{B}}(y)\, d\sigma(h)
\end{aligned}$$

by Fubini's theorem. By Theorem 2.2 and Lemma 2.5, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} F^{\lambda_1,\lambda_2}(x,y)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y) \\
&= \int_{\mathcal{H}} \int_{\mathbf{B}} \int_{\mathbf{B}} \exp\{i[\lambda_1^{-1/2}(A_1^{1/2}h, \sqrt{u}x_1)^\sim + \lambda_2^{-1/2}(A_2^{1/2}h, \sqrt{u}x_2)^\sim]\} \\
&\qquad\qquad dm(x_1)\, dm(x_2)\, d\sigma(h) \\
&= \int_{\mathcal{H}} \exp\left\{-\frac{u}{2\lambda_1}(A_1h,h) - \frac{u}{2\lambda_2}(A_2h,h)\right\} d\sigma(h).
\end{aligned}$$

The results now follow by Morera's theorem and the dominated convergence theorem. □

Theorem 4.5. *Let $F \in \mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2;u)$ be given by (4.1), and let $X_{\vec{\tau}}$ be given by (2.3). Let*

$$\Gamma_1 = \frac{(t_{p_1}-u)(u-t_{p_1-1})}{t_{p_1}-t_{p_1-1}} \quad \text{and} \quad \Gamma_2 = \frac{(s_{p_2}-u)(u-s_{p_2-1})}{s_{p_2}-s_{p_2-1}} \tag{4.9}$$

if $t_{p_1-1} < u < t_{p_1}$ and $s_{p_2-1} < u < s_{p_2}$, and let $G(h,\vec{\xi}_1,\vec{\xi}_2) = \exp\{i[(A_1^{1/2}h, [\vec{\xi}_1](u))^\sim + (A_2^{1/2}h,[\vec{\xi}_2](u))^\sim]\}$ for $h \in \mathcal{H}$ and for $(\vec{\xi}_1,\vec{\xi}_2) \in (\mathbf{B}^k,\mathbf{B}^l)$. Then $E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}]$ exists for $(\lambda_1,\lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$ and $E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}]$ exists for any (q_1,q_2) in $\mathbf{R}^2$ ($q_j \neq 0$ for $j=1,2$). Moreover, when $t_{p_1-1} < u < t_{p_1}$ for some $p_1 \in \{1,\dots,k\}$ and $s_{p_2-1} < u < s_{p_2}$ for some $p_2 \in \{1,\dots,l\}$, for s almost everywhere $(\vec{\xi}_1,\vec{\xi}_2) \in (\mathbf{B}^k,\mathbf{B}^l)$, we have

$$\begin{aligned} &E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) \\ &\quad = \int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\left\{-\frac{\Gamma_1}{2\lambda_1}(A_1h,h) - \frac{\Gamma_2}{2\lambda_2}(A_2h,h)\right\} d\sigma(h) \end{aligned}$$

and

$$\begin{aligned} &E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) \\ &\quad = \int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\left\{-\frac{i\Gamma_1}{2q_1}(A_1h,h) - \frac{i\Gamma_2}{2q_2}(A_2h,h)\right\} d\sigma(h). \end{aligned}$$

When $t_{p_1-1} < u < t_{p_1}$ for some $p_1 \in \{1,\dots,k\}$ and $u = s_{p_2}$ for some $p_2 \in \{1,\dots,l\}$, for s almost everywhere $(\vec{\xi}_1,\vec{\xi}_2) \in (\mathbf{B}^k,\mathbf{B}^l)$, we have

$$E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) = \int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\left\{-\frac{\Gamma_1}{2\lambda_1}(A_1h,h)\right\} d\sigma(h)$$

and

$$E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) = \int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\left\{-\frac{i\Gamma_1}{2q_1}(A_1h,h)\right\} d\sigma(h).$$

When $u = t_{p_1}$ for some $p_1 \in \{1, \dots, k\}$ and $s_{p_2-1} < u < s_{p_2}$ for some $p_2 \in \{1, \dots, l\}$, for s almost everywhere $(\vec{\xi}_1, \vec{\xi}_2) \in (\mathbf{B}^k, \mathbf{B}^l)$, we have

$$E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2) = \int_{\mathcal{H}} G(h, \vec{\xi}_1, \vec{\xi}_2) \exp\left\{ -\frac{\Gamma_2}{2\lambda_2}(A_2 h, h) \right\} d\sigma(h)$$

and

$$E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2) = \int_{\mathcal{H}} G(h, \vec{\xi}_1, \vec{\xi}_2) \exp\left\{ -\frac{i\Gamma_2}{2q_2}(A_2 h, h) \right\} d\sigma(h).$$

When $u = t_{p_1}$ for some $p_1 \in \{1, \dots, k\}$ and $u = s_{p_2}$ for some $p_2 \in \{1, \dots, l\}$, for any $(\vec{\xi}_1, \vec{\xi}_2) \in (\mathbf{B}^k, \mathbf{B}^l)$, we have

$$E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2) = F([\vec{\xi}_1], [\vec{\xi}_2]) = E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2).$$

Proof. Let $t_{p_1-1} < u < t_{p_1}$ for some $p_1 \in \{1, \dots, k\}$ and $s_{p_2-1} < u < s_{p_2}$ for some $p_2 \in \{1, \dots, l\}$. For $\lambda_1, \lambda_2 > 0$ and for s almost everywhere $(\vec{\xi}_1, \vec{\xi}_2) \in \mathbf{B}^k \times \mathbf{B}^l$, we have

$$\begin{aligned}
&E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(\lambda_1^{-1/2}(x - [x]) + [\vec{\xi}_1], \lambda_2^{-1/2}(y - [y]) + [\vec{\xi}_2])] \\
&= \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} \int_{\mathcal{H}} \exp\{i[(A_1^{1/2}h, \lambda_1^{-1/2}(x(u) - [x](u)) + [\vec{\xi}_1](u))^{\sim} \\
&\quad + (A_2^{1/2}h, \lambda_2^{-1/2}(y(u) - [y](u)) + [\vec{\xi}_2](u))^{\sim}]\}\, d\sigma(h)\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&= \int_{\mathcal{H}} G(h, \vec{\xi}_1, \vec{\xi}_2) \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} \exp\bigg\{ i \bigg[\lambda_1^{-1/2} \bigg(A_1^{1/2}h, x(u) - x(t_{p_1-1}) \\
&\quad - \frac{u - t_{p_1-1}}{t_{p_1} - t_{p_1-1}}(x(t_{p_1}) - x(t_{p_1-1})) \bigg)^{\sim} + \lambda_2^{-1/2} \bigg(A_2^{1/2}h, y(u) - y(s_{p_2-1}) \\
&\qquad - \frac{u - s_{p_2-1}}{s_{p_2} - s_{p_2-1}}(y(s_{p_2}) - y(s_{p_2-1})) \bigg)^{\sim} \bigg] \bigg\}\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y)\, d\sigma(h)
\end{aligned}$$

by Fubini's theorem. Let

$$\alpha_1 = \frac{(t_{p_1} - u)(u - t_{p_1-1})^{1/2}}{t_{p_1} - t_{p_1-1}}, \qquad \beta_1 = -\frac{(u - t_{p_1-1})(t_{p_1} - u)^{1/2}}{t_{p_1} - t_{p_1-1}},$$

$$\alpha_2 = \frac{(s_{p_2} - u)(u - s_{p_2-1})^{1/2}}{s_{p_2} - s_{p_2-1}}, \qquad \beta_2 = -\frac{(u - s_{p_2-1})(s_{p_2} - u)^{1/2}}{s_{p_2} - s_{p_2-1}}.$$

By Theorem 2.2 and Lemma 2.5, we have

$$
\begin{aligned}
&E_{C_0(\mathbf{B})\times C_0(\mathbf{B})}[F(\lambda_1^{-1/2}(x-[x])+[\vec{\xi}_1],\lambda_2^{-1/2}(y-[y])+[\vec{\xi}_2])]\\
&\quad=\int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\\
&\quad\times\int_{\mathbf{B}^2\times\mathbf{B}^2}\exp\{i[\lambda_1^{-1/2}(\alpha_1(A_1^{1/2}h,x_1)^\sim+\beta_1(A_1^{1/2}h,x_2)^\sim)\\
&\qquad+\lambda_2^{-1/2}(\alpha_2(A_2^{1/2}h,y_1)^\sim+\beta_2(A_2^{1/2}h,y_2)^\sim)]\}\\
&\qquad\qquad\qquad d(m^2\times m^2)((x_1,x_2),(y_1,y_2))\,d\sigma(h)\\
&\quad=\int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\bigg\{-\frac{1}{2\lambda_1}(\alpha_1^2+\beta_1^2)|A_1^{1/2}h|^2\\
&\qquad\qquad-\frac{1}{2\lambda_2}(\alpha_2^2+\beta_2^2)|A_2^{1/2}h|^2\bigg\}\,d\sigma(h)\\
&\quad=\int_{\mathcal{H}} G(h,\vec{\xi}_1,\vec{\xi}_2)\exp\bigg\{-\frac{\Gamma_1}{2\lambda_1}(A_1h,h)-\frac{\Gamma_2}{2\lambda_2}(A_2h,h)\bigg\}\,d\sigma(h)
\end{aligned}
$$

where Γ_1 and Γ_2 are given by (4.9). By Morera's theorem and the dominated convergence theorem, the results follow. The other cases are similar. □

Remark 4.6. (1) Let A be a bounded self-adjoint operator on $\mathcal{H}$. Then we can write $A = A^+ - A^-$, where A^+, A^- are both bounded and nonnegative self-adjoint. Take $A_1 = A^+$ and $A_2 = A^-$ in (4.1). By Theorem 4.4, we have

$$
E^{anf_{1,-1}}[F]=\int_{\mathcal{H}}\exp\bigg\{-\frac{ui}{2}(Ah,h)\bigg\}\,d\sigma(h) \tag{4.10}
$$

and, by Theorem 4.5, when $\Gamma_1 = \Gamma_2 \equiv \Gamma$, we have

$$
\begin{aligned}
&E^{anf_{1,-1}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)\\
&\quad=\int_{\mathcal{H}}\exp\{i[((A^+)^{1/2}h,[\vec{\xi}_1](u))^\sim+((A^-)^{1/2}h,[\vec{\xi}_2](u))^\sim]\}\\
&\qquad\qquad\times\exp\bigg\{-\frac{i\Gamma}{2}(Ah,h)\bigg\}\,d\sigma(h).
\end{aligned} \tag{4.11}
$$

(2) If $A^- = 0$, then the righthand sides of (4.10) and (4.11) coincide with (3.8) and (3.9), respectively, with $q = 1$. In particular, if

$A^- = 0$ and A^+ is the identity operator I, then $\mathcal{F}_{I,0}(C_0(\mathbf{B})^2; u)$ is essentially $\mathcal{F}(C_0(\mathbf{B}); u)$ in Definition 3.1 and $E^{anf_{q_1,q_2}}[F_{I,0}] = E^{anf_{q_1}}[F_I]$, $E^{anf_{q_1,q_2}}[F_{I,0}|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2) = E^{anf_{q_1}}[F_I|X_{\tau_1}](\vec{\xi}_1)$, where $F_{I,0}, F_I$ are given by (4.6) and (4.7), respectively, with A replaced with I (also see Corollary 4.2). In this sense, $\mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ contains $\mathcal{F}(C_0(\mathbf{B}); u)$ as a special case.

Our next theorem follows from Theorem 4.5, Fubini's theorem and Lemma 2.5.

Theorem 4.7. *Let $F \in \mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ be given by* (4.1)*, and let $X_{\vec{\tau}}$ be given by* (2.3)*. Let $t_{p_1-1} < u \le t_{p_1}$ for some $p_1 \in \{1, \dots, k\}$ and $s_{p_2-1} < u \le s_{p_2}$ for some $p_2 \in \{1, \dots, l\}$, and let Γ_1, Γ_2 be given by* (4.9) (*possibly, $u = t_{p_1}$ or $u = s_{p_2}$*)*. Let*

$$\alpha_1 = \frac{u - t_{p_1-1}}{t_{p_1} - t_{p_1-1}}, \qquad \beta_1 = \frac{t_{p_1} - u}{t_{p_1} - t_{p_1-1}},$$
$$\alpha_2 = \frac{u - s_{p_2-1}}{s_{p_2} - s_{p_2-1}}, \qquad \beta_2 = \frac{s_{p_2} - u}{s_{p_2} - s_{p_2-1}} \tag{4.12}$$

and for E in $\mathcal{B}(\mathcal{H})$ let

$$\sigma_u(E) = \int_E \exp\left\{-\frac{1}{2}[(\alpha_1^2 + \beta_1^2)(A_1 h, h) + (\alpha_2^2 + \beta_2^2)(A_2 h, h)]\right\} d\sigma(h).$$

For $\lambda_1, \lambda_2 \in \mathbf{C}_+^{\sim} \equiv \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \ge 0\} - \{0\}$ and for $h \in \mathcal{H}$, let $Q(\lambda_1, \lambda_2, h) = \exp\{-(\Gamma_1/2\lambda_1)(A_1 h, h) - (\Gamma_2/2\lambda_2)(A_2 h, h)\}$. Then we have

$$\int_{\mathbf{B}^k \times \mathbf{B}^l} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2)\, d(m^k \times m^l)(\vec{\xi}_1, \vec{\xi}_2) = \int_{\mathcal{H}} Q(\lambda_1, \lambda_2, h)\, d\sigma_u(h)$$

for $(\lambda_1, \lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$ and

$$\int_{\mathbf{B}^k \times \mathbf{B}^l} E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1, \vec{\xi}_2)\, d(m^k \times m^l)(\vec{\xi}_1, \vec{\xi}_2) = \int_{\mathcal{H}} Q(-iq_1, -iq_2, h)\, d\sigma_u(h)$$

for $(q_1, q_2) \in \mathbf{R}^2$ ($q_j \ne 0$ for $j = 1, 2$).

In the following theorems, we regard all complex square roots as the complex number whose real parts are nonnegative.

Theorem 4.8. *Let* $F \in \mathcal{F}_{A_1,A_2}(C_0(\mathbf{B})^2; u)$ *be given by* (4.1)*, and let* $X_{\vec{\tau}}$ *be given by* (2.3)*. For* $(q_1, q_2) \in \mathbf{R}^2$ ($q_j \neq 0$ *with* $j = 1, 2$)*, let* $X_{\vec{\tau}}^{q_1,q_2}(x, y) = X_{\vec{\tau}}((-iq_1)^{-1/2}x, (-iq_2)^{-1/2}y)$ *for* $(x, y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B})$*. Then we have*

$$\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x, y))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) = E^{anw_{\lambda_1,\lambda_2}}[F]$$

for $(\lambda_1, \lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$ *and*

$$\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](X_{\vec{\tau}}^{q_1,q_2}(x, y))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) = E^{anf_{q_1,q_2}}[F].$$

Proof. For $(\vec{x}, \vec{y}) \in \mathbf{B}^k \times \mathbf{B}^l$, let $\vec{x} = (x_1, \ldots , x_k)$ and $\vec{y} = (y_1, \ldots , y_l)$. For $(\lambda_1, \lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x, y)) d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&= \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\lambda_1^{-1/2}(x(t_1), \cdots, x(t_k)), \lambda_2^{-1/2}(y(s_1), \\
&\qquad \cdots, y(s_l)))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&= \int_{\mathbf{B}^k\times\mathbf{B}^l} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}]\left(\lambda_1^{-1/2}\left(\sqrt{t_1}x_1, \cdots , \sum_{j=1}^{k} \sqrt{t_j - t_{j-1}}x_j\right),\right. \\
&\qquad \left.\lambda_2^{-1/2}\left(\sqrt{s_1}y_1, \cdots , \sum_{j=1}^{l} \sqrt{s_j - s_{j-1}}y_j\right)\right) d(m^k \times m^l)(\vec{x}, \vec{y})
\end{aligned}$$

by Theorem 2.2. Let $t_{p_1-1} < u \leq t_{p_1}$ for some $p_1 \in \{1, \ldots , k\}$ and $s_{p_2-1} < u \leq s_{p_2}$ for some $p_2 \in \{1, \ldots , l\}$, and let $Q(\lambda_1, \lambda_2, h)$ be given as in Theorem 4.7. Let α_j be given in (4.12) for $j = 1, 2$. By Lemma 2.5, Theorem 4.5 and Fubini's theorem, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x,y))d(m_{\mathbf{B}}\times m_{\mathbf{B}})(x,y)\\
&= \int_{\mathcal{H}} Q(\lambda_1,\lambda_2,h)\int_{\mathbf{B}^{p_1}\times\mathbf{B}^{p_2}} \exp\biggl\{i\biggl[\lambda_1^{-1/2}\biggl(A_1^{1/2}h, \alpha_1\sqrt{t_{p_1}-t_{p_1-1}}x_{t_{p_1}}\\
&\quad + \sum_{j=1}^{p_1-1}\sqrt{t_j-t_{j-1}}x_j\biggr)^{\sim} + \lambda_2^{-1/2}\biggl(A_2^{1/2}h, \alpha_2\sqrt{s_{p_2}-s_{p_2-1}}y_{p_2}\\
&\quad + \sum_{j=1}^{p_2-1}\sqrt{s_j-s_{j-1}}y_j\biggr)^{\sim}\biggr]\biggr\}\\
&\qquad\qquad\qquad d(m^{p_1}\times m^{p_2})((x_1,\cdots,x_{p_1}),(y_1,\cdots,y_{p_2}))\\
&= \int_{\mathcal{H}} Q(\lambda_1,\lambda_2,h)\exp\biggl\{-\frac{1}{2\lambda_1}\biggl(\sum_{j=1}^{p_1-1}(t_j-t_{j-1}) + \alpha_1^2(t_{p_1}-t_{p_1-1})\biggr)\\
&\times (A_1h,h) - \frac{1}{2\lambda_2}\biggl(\sum_{j=1}^{p_2-1}(s_j-s_{j-1}) + \alpha_2^2(s_{p_2}-s_{p_2-1})\biggr)(A_2h,h)\biggr\}d\sigma(h)\\
&= \int_{\mathcal{H}} \exp\biggl\{-\frac{u}{2\lambda_1}(A_1h,h) - \frac{u}{2\lambda_2}(A_2h,h)\biggr\}d\sigma(h) = E^{anw_{\lambda_1,\lambda_2}}[F]
\end{aligned}$$

by Theorem 4.4. Similarly, we have the other result. □

Let B_1 and B_2 be nonnegative bounded self-adjoint operators on $\mathcal{H}$, and let $\phi \in \mathcal{F}_{B_1,B_2}$ be given by the righthand side of equation (4.3) with A_1, A_2 and σ replaced with B_1, B_2 and $\nu \in \mathcal{M}(\mathcal{H})$, respectively. Let (η_1,η_2) be in $\mathbf{B}\times\mathbf{B}$, and let

$$G(x,y) = F(x,y)\phi((x(T),y(T)) + (\eta_1,\eta_2)) \tag{4.13}$$

for s almost everywhere (x,y) in $C_0(\mathbf{B})\times C_0(\mathbf{B})$ where F is given by (4.1).

Our next theorem follows from Theorem 2.2, Fubini's theorem and Lemma 2.5.

Theorem 4.9. *Let G be given by (4.13), and let $R(h_2) = \exp\{i[(B_1^{1/2}h_2,\eta_1)^{\sim} + (B_2^{1/2}h_2,\eta_2)^{\sim}]\}$ where $h_2 \in \mathcal{H}$ and $\eta_1,\eta_2 \in \mathbf{B}$.*

Then, for s almost every (η_1,η_2) in $\mathbf{B}\times\mathbf{B}$, $E^{anw_{\lambda_1,\lambda_2}}[G]$ exists for $(\lambda_1,\lambda_2)\in\mathbf{C}_+\times\mathbf{C}_+$ and $E^{anf_{q_1,q_2}}[G]$ exists for any (q_1,q_2) in $\mathbf{R}^2$ $(q_j\neq 0$ for $j=1,2)$. Moreover, we have

$$\begin{aligned}E^{anw_{\lambda_1,\lambda_2}}[G] = \int_{\mathcal{H}\times\mathcal{H}} & R(h_2) \\ & \times\exp\bigg\{-\frac{1}{2\lambda_1}[u(A_1h_1,h_1)+2u(A_1^{1/2}h_1,B_1^{1/2}h_2) \\ & + T(B_1h_2,h_2)] - \frac{1}{2\lambda_2}[u(A_2h_1,h_1) \\ & + 2u(A_2^{1/2}h_1,B_2^{1/2}h_2)+T(B_2h_2,h_2)]\bigg\} \\ & d(\sigma\times\nu)(h_1,h_2)\end{aligned}$$

and

$$\begin{aligned}E^{anf_{q_1,q_2}}[G] = \int_{\mathcal{H}\times\mathcal{H}} & R(h_2) \\ & \times\exp\bigg\{-\frac{i}{2q_1}[u(A_1h_1,h_1)+2u(A_1^{1/2}h_1,B_1^{1/2}h_2) \\ & + T(B_1h_2,h_2)] - \frac{i}{2q_2}[u(A_2h_1,h_1) \\ & + 2u(A_2^{1/2}h_1,B_2^{1/2}h_2)+T(B_2h_2,h_2)]\bigg\} \\ & d(\sigma\times\nu)(h_1,h_2).\end{aligned}$$

Theorem 4.10. *Let G be given by (4.13), and let $X_{\vec{\tau}}$ be given by (2.3). Then, for s almost everywhere $(\eta_1,\eta_2)\in\mathbf{B}\times\mathbf{B}$, $E^{anw_{\lambda_1,\lambda_2}}[G|X_{\vec{\tau}}]$ exists for $(\lambda_1,\lambda_2)\in\mathbf{C}_+\times\mathbf{C}_+$ and $E^{anf_{q_1,q_2}}[G|X_{\vec{\tau}}]$ exists for any (q_1,q_2) in $\mathbf{R}^2$ $(q_j\neq 0$ for $j=1,2)$. Moreover, for s almost everywhere $(\vec{\xi}_1,\vec{\xi}_2)\in(\mathbf{B}^k,\mathbf{B}^l)$ we have*

$$E^{anw_{\lambda_1,\lambda_2}}[G|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) = E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)\phi((\xi_{1k},\xi_{2l})+(\eta_1,\eta_2))$$

and

$$E^{anf_{q_1,q_2}}[G|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2) = E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}](\vec{\xi}_1,\vec{\xi}_2)\phi((\xi_{1k},\xi_{2l})+(\eta_1,\eta_2)),$$

where $E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}]$ *and* $E^{anf_{q_1,q_2}}[F|X_{\vec{\tau}}]$ *are given as in Theorem* 4.5 *and* $\vec{\xi}_1 = (\xi_{11}, \dots, \xi_{1k})$, $\vec{\xi}_2 = (\xi_{21}, \dots, \xi_{2l})$.

Proof. Let $\vec{\xi}_1 = (\xi_{11}, \dots, \xi_{1k})$ and $\vec{\xi}_2 = (\xi_{21}, \dots, \xi_{2l})$. For (x, y) in $C_0(\mathbf{B}) \times C_0(\mathbf{B})$ and for $\lambda_1, \lambda_2 > 0$, we have

$$\begin{aligned}
& G(\lambda_1^{-1/2}(x - [x]) + [\vec{\xi}_1], \lambda_2^{-1/2}(y - [y]) + [\vec{\xi}_2]) \\
&= F(\lambda_1^{-1/2}(x - [x]) + [\vec{\xi}_1], \lambda_2^{-1/2}(y - [y]) + [\vec{\xi}_2]) \\
&\times \phi((\lambda_1^{-1/2}(x(T) - [x](T)) \\
&\qquad + [\vec{\xi}_1](T), \lambda_2^{-1/2}(y(T) - [y](T)) + [\vec{\xi}_2](T)) + (\eta_1, \eta_2)) \\
&= F(\lambda_1^{-1/2}(x - [x]) + [\vec{\xi}_1], \lambda_2^{-1/2}(y - [y]) + [\vec{\xi}_2])\phi((\xi_{1k}, \xi_{2l}) + (\eta_1, \eta_2)).
\end{aligned}$$

Thus, the results follow by Theorem 4.5. □

Theorem 4.11. *Let* G *be given by* (4.13), *and let* $X_{\vec{\tau}}$ *be given by* (2.3). *For* $(q_1, q_2) \in \mathbf{R}^2$ ($q_j \neq 0$ *with* $j = 1, 2$), *let* $X_{\vec{\tau}}^{q_1,q_2}(x, y) = X_{\vec{\tau}}((-iq_1)^{-1/2}x, (-iq_2)^{-1/2}y)$ *for* $(x, y) \in C_0(\mathbf{B}) \times C_0(\mathbf{B})$. *Then, for* s *almost everywhere* $(\eta_1, \eta_2) \in \mathbf{B} \times \mathbf{B}$, *we have*

$$\begin{aligned}
\int_{C_0(\mathbf{B}) \times C_0(\mathbf{B})} & E^{anw_{\lambda_1,\lambda_2}}[G|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x, y))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&= E^{anw_{\lambda_1,\lambda_2}}[G]
\end{aligned}$$

for $(\lambda_1, \lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$ *and*

$$\begin{aligned}
\int_{C_0(\mathbf{B}) \times C_0(\mathbf{B})} & E^{anf_{q_1,q_2}}[G|X_{\vec{\tau}}](X_{\vec{\tau}}^{q_1,q_2}(x, y)) d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x, y) \\
&= E^{anf_{q_1,q_2}}[G].
\end{aligned}$$

Proof. For $(\vec{x}, \vec{y}) \in \mathbf{B}^k \times \mathbf{B}^l$, let $\vec{x} = (x_1, \dots, x_k)$ and $\vec{y} = (y_1, \dots, y_l)$.

By Theorems 2.2 and 4.10, for $(\lambda_1, \lambda_2) \in \mathbf{C}_+ \times \mathbf{C}_+$, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[G|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x,y))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y) \\
&= \int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}]((\lambda_1^{-1/2}(x(t_1),\cdots,x(t_k)),\lambda_2^{-1/2}(y(s_1), \\
&\quad \cdots, y(s_l)))\phi((\lambda_1^{-1/2}x(T),\lambda_2^{-1/2}y(T)) + (\eta_1,\eta_2))d(m_{\mathbf{B}}\times m_{\mathbf{B}}))(x,y) \\
&= \int_{\mathbf{B}^k\times\mathbf{B}^l} E^{anw_{\lambda_1,\lambda_2}}[F|X_{\vec{\tau}}]\bigg(\lambda_1^{-1/2}\bigg(\sqrt{t_1}x_1,\cdots,\sum_{j=1}^k\sqrt{t_j-t_{j-1}}x_j\bigg), \\
&\quad \lambda_2^{-1/2}\bigg(\sqrt{s_1}y_1,\cdots,\sum_{j=1}^l\sqrt{s_j-s_{j-1}}y_j\bigg)\bigg)\phi\bigg(\bigg(\lambda_1^{-1/2}\sum_{j=1}^k\sqrt{t_j-t_{j-1}}x_j, \\
&\qquad\qquad \lambda_2^{-1/2}\sum_{j=1}^l\sqrt{s_j-s_{j-1}}y_j\bigg) + (\eta_1,\eta_2)\bigg)\, d(m^k\times m^l)(\vec{x},\vec{y}).
\end{aligned}$$

Let $t_{p_1-1} < u \le t_{p_1}$ for some $p_1 \in \{1,\dots,k\}$ and $s_{p_2-1} < u \le s_{p_2}$ for some $p_2 \in \{1,\dots,l\}$, and let Q, R be given as in Theorems 4.7 and 4.9, respectively. Let α_j be given as in (4.12) for $j = 1, 2$. By Lemma 2.5, Theorem 4.5 and Fubini's theorem, we have

$$\begin{aligned}
&\int_{C_0(\mathbf{B})\times C_0(\mathbf{B})} E^{anw_{\lambda_1,\lambda_2}}[G|X_{\vec{\tau}}](X_{\vec{\tau}}^{\lambda_1,\lambda_2}(x,y))\, d(m_{\mathbf{B}} \times m_{\mathbf{B}})(x,y) \\
&= \int_{\mathbf{B}^k\times\mathbf{B}^l}\bigg[\int_{\mathcal{H}} Q(\lambda_1,\lambda_2,h)\exp\bigg\{i\bigg[\lambda_1^{-1/2}\bigg(A_1^{1/2}h, \alpha_1\sqrt{t_{p_1}-t_{p_1-1}}x_{t_{p_1}} \\
&\quad + \sum_{j=1}^{p_1-1}\sqrt{t_j-t_{j-1}}x_j\bigg)^{\sim} + \lambda_2^{-1/2}\bigg(A_2^{1/2}h, \alpha_2\sqrt{s_{p_2}-s_{p_2-1}}y_{s_{p_2}} \\
&\quad + \sum_{j=1}^{p_2-1}\sqrt{s_j-s_{j-1}}y_j\bigg)^{\sim}\bigg]\bigg\}d\sigma(h)\bigg] \\
&\times\bigg[\int_{\mathcal{H}}\exp\bigg\{i\bigg[\bigg(B_1^{1/2}h, \lambda_1^{-1/2}\sum_{j=1}^k\sqrt{t_j-t_{j-1}}x_j\bigg)^{\sim} \\
&\quad + \bigg(B_2^{1/2}h, \lambda_2^{-1/2}\sum_{j=1}^l\sqrt{s_j-s_{j-1}}y_j\bigg)^{\sim}\bigg]\bigg\}R(h)\, d\nu(h)\bigg] d(m^k\times m^l)(\vec{x},\vec{y})
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathcal{H}\times\mathcal{H}} Q(\lambda_1,\lambda_2,h_1)R(h_2) \\
&\times \int_{\mathbf{B}^k\times\mathbf{B}^l} \exp\biggl\{ i\biggl[\lambda_1^{-1/2}\biggl[\sum_{j=1}^{p_1-1}\sqrt{t_j-t_{j-1}}(A_1^{1/2}h_1+B_1^{1/2}h_2,x_j)^\sim \\
&\quad + \sqrt{t_{p_1}-t_{p_1-1}}(\alpha_1 A_1^{1/2}h_1+B_1^{1/2}h_2,x_{p_1})^\sim \\
&\quad + \sum_{j=p_1+1}^{k}\sqrt{t_j-t_{j-1}}(B_1^{1/2}h_2,x_j)^\sim\biggr] \\
&\quad + \lambda_2^{-1/2}\biggl[\sum_{j=1}^{p_2-1}\sqrt{s_j-s_{j-1}}(A_2^{1/2}h_1+B_2^{1/2}h_2,y_j)^\sim \\
&\quad + \sqrt{s_{p_2}-s_{p_2-1}}(\alpha_2 A_2^{1/2}h_1+B_2^{1/2}h_2,y_{p_2})^\sim \\
&\quad + \sum_{j=p_2+1}^{l}\sqrt{s_j-s_{j-1}}(B_2^{1/2}h_2,y_j)^\sim\biggr]\biggr]\biggr\} d(m^k\times m^l)(\vec{x},\vec{y})d(\sigma\times\nu)(h_1,h_2) \\
&= \int_{\mathcal{H}\times\mathcal{H}} Q(\lambda_1,\lambda_2,h_1)R(h_2) \\
&\times \exp\biggl\{-\frac{1}{2\lambda_1}(t_{p_1-1}|A_1^{1/2}h_1+B_1^{1/2}h_2|^2+(t_{p_1}-t_{p_1-1}) \\
&\times |\alpha_1 A_1^{1/2}h_1+B_1^{1/2}h_2|^2+(T-t_{p_1})|B_1^{1/2}h_2|^2) \\
&\quad -\frac{1}{2\lambda_2}(s_{p_2-1}|A_2^{1/2}h_1+B_2^{1/2}h_2|^2+(s_{p_2}-s_{p_2-1})|\alpha_2 A_2^{1/2}h_1+B_2^{1/2}h_2|^2 \\
&\qquad\qquad + (T-s_{p_2})|B_2^{1/2}h_2|^2)\biggr\} d(\sigma\times\nu)(h_1,h_2) \\
&= \int_{\mathcal{H}\times\mathcal{H}} R(h_2)\exp\biggl\{-\frac{1}{2\lambda_1}[u(A_1h_1,h_1)+2u(A_1^{1/2}h_1,B_1^{1/2}h_2) \\
&\quad + T(B_1h_2,h_2)] - \frac{1}{2\lambda_2}[u(A_2h_1,h_1)+2u(A_2^{1/2}h_1,B_2^{1/2}h_2) \\
&\qquad\qquad + T(B_2h_2,h_2)]\biggr\} d(\sigma\times\nu)(h_1,h_2) = E^{anw_{\lambda_1,\lambda_2}}[G]
\end{aligned}
$$

where the last equality follows from Theorem 4.9.

The proof of the other case is similar. □

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