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THE CHARACTERIZATION OF MOORE-PENROSE INVERSE MODULE MAPS AND THEIR CONTINUITY

LUNCHUAN ZHANG

ABSTRACT. In this paper we will introduce the concept of *Moore-Penrose inverse module map* and an equivalent characterization, provided that E and F are both (right) Hilbert C^*-modules over a C^*-algebra A, with $L(E,F)$ the set of all adjointable A-module maps from E to F. Then we use the concept as a main tool in obtaining a Douglas type factorization theorem about certain important bounded module maps. Thus, we come to the discussion of the continuity of Moore-Penrose inverse module maps depending upon a parameter: Let X be a topological space and $x \mapsto T_x : X \mapsto L(E)$ a continuous map, with $R(T_x)$ a closed submodule in E for each $x \in X$. Then the Moore-Penrose inverse module map T_x^+ of T_x is continuous if and only if $||T_x^+||$ is locally bounded. Furthermore, this is equivalent to the following statement:

For any x_0 in X, there exists a neighborhood U_0 of x_0 and a positive number λ such that $(0, \lambda^2) \subseteq \mathbf{C} \setminus \sigma(T_x^* T_x)$ for all $x \in U_0$, where $\sigma(T)$ denotes the spectrum of the operator T.

1. Introduction. Hilbert C^*-modules constitute a frequently used tool in operator theory and operator algebras. Research fields benefiting from it include K-theory, index theory for operator-valued conditional expectations, group representation theory, operator-valued free probability and investigations into compact quantum groups, generalized Atiyah-Singer index theorems and topological invariants. Besides these, the theory of Hilbert C^*-modules is very interesting in its own right.

It is well known that Moore-Penrose inverse matrices and Moore-Penrose inverse operators play an important role in matrix theory and in operator theory, respectively. Meanwhile, in the study of factorization of Hilbert C^*-module maps, there is no suitable tool to use. Motivated by the above observation, in this paper we will introduce the concept of *Moore-Penrose inverse module maps* and an

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equivalent characterization, see Theorem 1. Then we use it as the main tool in obtaining a Douglas type factorization theorem of certain important bounded module maps, see Theorem 2. In the end, we describe explicitly the continuity of Moore-Penrose inverse module maps depending upon a parameter, see Theorem 3. We also obtain some other interesting results.

For the basic theory of Hilbert C^*-modules, see [**3**].

2. The main results and proofs. Let E be a Hilbert C^*-module over a C^*-algebra A. The *orthogonal complement* of a closed submodule F of E is $F^\perp = \{y \in E : \langle x, y\rangle = 0, \text{ for all } x \in F\}$, and F is said to be *complemented* if $E = F \oplus F^\perp$.

Definition 1. Let E and F be Hilbert C^*-modules over a C^*-algebra A, and let $T \in L(E, F)$. If there exists a $T^+ \in L(F, E)$ such that

(1) $TT^+T = T$,

(2) $T^+TT^+ = T^+$,

(3) $(TT^+)^* = TT^+$,

(4) $(T^+T)^* = T^+T$,

then T^+ is called the *Moore-Penrose inverse module* map of T.

Remark 1. If A degenerates to complex field $\mathbf{C}$, then T^+ will reduce to the usual Hilbert space Moore-Penrose inverse. On the other hand, if $E = F$, since $L(E)$ is a C^*-algebra, see [**3**], then T^+ will agree with the Moore-Penrose inverse in a C^*-algebra, see [**2**].

Remark 2. From the conditions of the above Definition 1 we have that T^+ is unique. Indeed, suppose that there is another module operator $T' \in L(F, E)$, which is also a Moore-Penrose inverse of T. With Definition 1 and routine calculation we have $R(TT^+) = R(T) = R(TT')$ and $R(T^+T) = R(T^*) = R(T'T)$. Note that $TT^+, T^+T, TT', T'T$ are all projections, hence $TT^+ = TT'$ and $T^+T = T'T$. Therefore, $T^+ = T^+TT^+ = T^+TT' = T'TT' = T'$.

The following theorem gives an equivalent characterization of the Moore-Penrose inverse module map:

Theorem 1. *Let E, F be Hilbert C^*-modules over a C^*-algebra A, and let $T \in L(E,F)$. Then the following statements are equivalent:*

(1) *T has a Moore-Penrose inverse module map in $L(F,E)$,*

(2) *$R(T)$ is a closed submodule in F.*

Proof. 1) $\Rightarrow$ (2) is clear, see the Remark 2.

(2) $\Rightarrow$ (1). Suppose that $R(T)$ is closed. By [**3**, Theorem 3.2], $N(T)$ and $R(T)$ are both complemented submodules in E and in F, respectively, where $N(T)$ is the kernel of T; so $E = N(T) \oplus R(T^*)$ and $F = N(T^*) \oplus R(T)$. Define a linear map $T^+ : F \mapsto E$ by

$$T^+x = \begin{cases} (T|N(T)^\perp)^{-1}x & x \in R(T) \\ 0 & x \in N(T^*), \end{cases}$$

and a linear map $(T^+)^* : E \mapsto F$ by

$$(T^+)^*x = \begin{cases} (T^*|N(T^*)^\perp)^{-1}x & x \in R(T^*) \\ 0 & x \in N(T). \end{cases}$$

To prove that T^+ is the Moore-Penrose inverse of T, the key step is to prove $T^+ \in L(F,E)$, equivalently, $\langle T^+x, y\rangle = \langle x, (T^+)^*y\rangle$, $x \in F$, $y \in E$. The verification of this identity is straightforward using the orthogonal direct sum decompositions, and so is the verification of conditions (1)–(4) of Definition 1. □

Following Theorem 1 above, the main result of [**2**] becomes a simple consequence, that is,

Corollary 1 (see [**2**]). *Let A be a unital C^*-algebra, and let $a \neq 0$ in A. Then a has a Moore-Penrose inverse a^+ in A if and only if aA is a closed right ideal in A.*

Proof. Apply Theorem 1 to the operator of left multiplication by a; then Corollary 1 is completed. □

In the following, we will apply the Moore-Penrose inverse module map to obtain a factorization result which is the analogue for C^*-modules of a well-known result of Douglas, see [**1**]:

Theorem 2. *Given a Hilbert C^*-module E over a C^*-algebra A, and $T, S \in L(E)$ with $R(S)$ closed, the following statements are equivalent*:

(1) $R(T) \subset R(S)$,

(2) $TT^* \leq \lambda^2 SS^*$, *for some* $\lambda \geq 0$,

(3) *there exists a* $Q \in L(E)$ *such that* $T = SQ$.

Proof. (3) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (3). Suppose that (1)holds. For each $x \in E$, we have $T(x) \in R(S)$. Since S has closed range, $N(S)$ is a complemented submodule, see [**3**, Theorem 3.2], so there is a unique $y \in N(S)^{\perp}$ such that $T(x) = S(y)$. Define a linear map Q by $Q(x) = y$. It is obvious that $T = SQ$, and the only thing left to show is $Q \in L(E)$, which is accomplished by the computation $Q = S^+SQ = S^+T$.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Since $R(S)$ is closed, so is $R(S^*)$, see [**3**, Theorem 3.2]. Define a linear map Q_1 on E by $Q_1S^*(x) = T^*(x)$, for $x \in E$ and $Q_1 = 0$ on $(S^*E)^{\perp}$. Q_1 is well defined since

$$||T^*x||^2 = ||\langle TT^*x, x\rangle|| \leq \lambda^2||\langle SS^*x, x\rangle|| = \lambda^2||S^*x||^2.$$

By construction,

$$T^* = Q_1S^* = Q_1S^*(S^+)^*S^* = T^*(S^+)^*S^*,$$

so $T = SQ$, where $Q = S^+T \in L(E)$. □

Remark 3. We point out that much of the proof of Theorem 2 above is modeled after Douglas's.

From Theorem 2 we can obtain directly the following factorization result in C^*-algebras:

Corollary 2. *Let A be a C^*-algebra, $a, b \in A$, and suppose bA is a closed right ideal of A. Then the following statements are equivalent*:

(1) *there is a c in A such that $a = bc$*,

(2) $aa^* \leq \lambda^2 bb^*$, *for some constant* $\lambda \geq 0$.

Remark 4. First, we point out that the above corollary has been known for some time, for example it follows from Theorem 1.5.2 in Pedersen's book (where the context is closed left ideal), see [**4**, Theorem 1.5.2]. Second, in Corollary 2 the implication "(1) $\Rightarrow$ (2)" does not require the assumption that bA is closed. But the converse implication is not correct in general if the condition that bA being closed is removed. The following is an easy counterexample:

Counterexample 1. In $C[0,1]$, define f and g as follows: $f(x) = |x - 1/2|$; $g(x) = f(x)$ for $0 \le x \le 1/2$, and $g(x) = 2f(x)$ for $1/2 < x \le 1$. Then $ff^* \le gg^*$. But there is no h in $C[0,1]$ such that $f = gh$.

To finish, we will describe the continuity of T_x^+, where $x \mapsto T_x : X \mapsto L(E)$ is a continuous map, and X is a topological space. We can obtain the following result:

Theorem 3. *Suppose that $x \mapsto T_x : X \mapsto L(E)$ is a continuous map, and $R(T_x)$ is a closed submodule in E, for each x in X, where X is a topological space. Then the following statements are equivalent*:

(1) *$x \mapsto T_x^+ : X \mapsto L(E)$ is continuous.*

(2) *$||T_x^+||$ is locally bounded, that is, for any $x_0 \in X$, there exists a constant $M > 0$ and a neighborhood U_0 of x_0 such that $||T_x^+|| \le M$ for all $x \in U_0$.*

(3) *For any $x_0 \in X$, there exists a neighborhood U_0 of x_0 and a positive number λ such that $(0, \lambda^2) \subseteq \mathbf{C} \setminus \sigma(T_x^* T_x)$ for all $x \in U_0$, where $\sigma(T_x^* T_x)$ is the spectrum of the operator $T_x^* T_x$.*

In order to prove Theorem 3 we need the following lemma:

Lemma 1. *Suppose that $R(T)$ is a closed submodule in a Hilbert C^*-module E, where T is a positive operator in $L(E)$. Then $0 < \lambda \le ||T^+||^{-1}$ if and only if $(0, \lambda) \subseteq \mathbf{C} \setminus \sigma(T)$.*

Proof. Suppose that $0 < \lambda \le ||T^+||^{-1}$. For any $k \in (0, \lambda)$, we have $||kT^+|| = k||T^+|| < 1$, so $T^+(T - k) = T^+T - kT^+$ is invertible on

$R(T^+T)$. Note that T^+ is invertible on $R(T^+T)$, and $T-k$ is self-adjoint; hence, $T-k$ is invertible on $R(T^+T)$. On the other hand, $(T-k)|_{N(T)} = -k$ is invertible on $N(T)$, where $N(T)$ is the kernel of T. Since $E = N(T) \oplus R(T^*) = N(T) \oplus R(T^+T)$, it follows that $(T-k)$ is invertible on E. Therefore $k \in \mathbf{C} \setminus \sigma(T)$, that is, $(0, \lambda) \subseteq \mathbf{C} \setminus \sigma(T)$.

Conversely, suppose that $(0, \lambda) \subseteq \mathbf{C} \setminus \sigma(T)$. Since T is positive, we have $\inf\{\sigma(T) \setminus \{0\}\} \geq \lambda$. For any $k \in (0, \lambda)$, note that the restriction T_0 of T to $R(T^+T)$ is invertible, so we have $\inf\{\sigma(T_0)\} \geq \lambda > k$ on $R(T^+T)$, which implies $T^+T_0 - kT^+ \geq 0$ on $R(T^+T)$. Note again that T is positive, so $T^+|_{N(T)} = T^+|_{N(T^*)} = 0$. Since $R(T^+T) \oplus N(T) = E$, it follows that $||kT^+|| \leq ||T^+T_0|| = ||T^+T|| = 1$ on E; therefore, $\lambda \leq ||T^+||^{-1}$. □

Proof of Theorem 3. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Suppose that, for any $x_0 \in X$, there exists a constant $M > 0$ and a neighborhood U_0 of x_0 such that $||T_x^+|| \leq M$ for all $x \in U_0$. Since $x \mapsto T_x$ is continuous, for each $\varepsilon > 0$ there is an open neighborhood U of x_0 such that $||T_x - T_{x_0}|| < \varepsilon$ for all $x \in U$. The properties of Definition 1 lead to $T_x^+(T_x^*)^+T_x^* = T_x^+$ and $T_x^*(T_x^*)^+T_x^+ = T_x^+$; using these relations and routine calculation we can get the following equation:

$$\begin{aligned} T_x^+ - T_{x_0}^+ = &-T_x^+(T_x - T_{x_0})T_{x_0}^+ + T_x^+(T_x^*)^+(T_x^* - T_{x_0}^*)(1 - T_{x_0}T_{x_0}^+) \\ &+ (1 - T_x^+T_x)(T_x^* - T_{x_0}^*)T_{x_0}^{*+}T_{x_0}^+. \end{aligned}$$

Set $\overline{U} = U_0 \cap U$; combining the above proof with this equation we can find a constant $M > 0$ such that $||T_x^+ - T_{x_0}^+|| < M\varepsilon$ for all $x \in \overline{U}$, showing that $x \mapsto T_x^+$ is continuous at x_0. Here we omit the details.

(2) $\Rightarrow$ (3). Suppose that, for any $x_0 \in X$, there exists a constant $M > 0$ and a neighborhood U_0 of x_0 such that $||T_x^+|| \leq M$ for all $x \in U_0$; hence, $||T_x^+||^{-1} \geq M^{-1}$. Since $||(T_x^*T_x)^+|| = ||T_x^+T_x^{*+}|| = ||T_x^+T_x^{+*}|| = ||T_x^+||^2$ (here we use the basic fact that $(T^*)^+ = (T^+)^*$, and $T^+(T^*)^+$ is the Moore-Penrose inverse module map of T^*T in $L(E)$ by routine calculation), we have $M \leq ||T_x^+||^{-1} \Leftrightarrow M^2 \leq ||(T_x^*T_x)^+||^{-1}$. Then it follows from Lemma 1 that $(0, M^{-2}) \subseteq C \setminus \sigma(T_x^*T_x)$ for all $x \in U$.

(3) $\Rightarrow$ (2). Suppose that for any $x_0 \in X$, there exists a neighborhood U_0 of x_0 and a positive number λ such that $(0, \lambda^2) \subseteq \mathbf{C} \setminus \sigma(T_x^*T_x)$ for

all $x \in U_0$. It follows from Lemma 1 that $||(T_x^*T_x)^+||^{-1} \geq \lambda^2$, noticing again that $0 < \lambda \leq ||T_x^+||^{-1} \Leftrightarrow 0 < \lambda^2 \leq ||(T_x^*T_x)^+||^{-1}$; hence, $||T_x^+|| \leq \lambda^{-1}$ for all $x \in U_0$, showing that $||T_x^+||$ is locally bounded. $\square$

Counterexample 2. In $M_2(\mathbf{C})$, the invertible matrices $\begin{pmatrix} n/(n+1) & 0 \\ 0 & 1/n \end{pmatrix}$ converge to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, which has a Moore-Penrose inverse; however, the inverses $\begin{pmatrix} (n+1)/n & 0 \\ 0 & n \end{pmatrix}$ are unbounded, hence do not converge. Thus, the boundedness condition of Theorem 3 is necessary.

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School of Information Science, Renmin University of China, Beijing 100090, P.R. China
Email address: zhanglc@ruc.edu.cn