ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 1, 2008

ON RECURRENCE RELATIONS FOR THE EXTENSIONS OF EULER SUMS

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ABSTRACT. We consider the extended Euler sums

$$E_{p,q}=\sum_{k=1}^{\infty}\frac{1}{k^q}\bigg(\sum_{j=1}^{2k}\frac{1}{j^p}\bigg) \quad \text{and} \quad T_{p,q}=\sum_{k=1}^{\infty}\frac{1}{k^q}\bigg(\sum_{j=1}^{[k/2]}\frac{1}{j^p}\bigg)$$

and obtain the explicit values of $E_{p,q}$ and $T_{p,q}$ when the weight $p+q$ is odd via integral transformations of Bernoulli identities involving Bernoulli polynomials. Two families of Bernoulli identities are transformed into explicit formulæ of Euler sums and extended Euler sums.

1. Preliminaries. The sequence of Bernoulli numbers ($n = 0,1,2,\dots$) is defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}, \quad |t|<2\pi. \tag{1.1}$$

It is a sequence of rational numbers and it can be evaluated through the following recursive formula:

$$\begin{cases} B_0=1, \\ \binom{n}{1}B_{n-1}+\cdots+\binom{n}{n}B_0=0, n\ge 2. \end{cases} \tag{1.2}$$

In particular, we get $B_1=-1/2$ from the relation

$$2B_1+B_0=0.$$

2000 AMS *Mathematics subject classification.* Primary 11M06, Secondary 11B68, 11M35.
Keywords and phrases. Euler sums, Bernoulli polynomials, Hurwitz zeta function.
The first author was supported by the Department of Accounting and Information Systems, Chang Jung Christian University, and the National Science Council of Taiwan, Republic of China. The second author was supported by the Department of Mathematics, National Chung Cheng University, and the National Science Council of Taiwan, Republic of China. The first author is the corresponding author.
Received by the editors on September 12, 2005, and in revised form on February 13, 2006.
DOI:10.1216/RMJ-2008-38-1-225

Also $B_{2k+1} = 0$ for $k \geq 1$ since the function

$$F(t) = \frac{t}{e^t - 1} + \frac{t}{2} \tag{1.3}$$

is an even function of t by a direct verification. Bernoulli numbers are used to express the special values of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1, \tag{1.4}$$

namely,

$$\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{m-1}B_{2m}}{2(2m)!}, \quad m \geq 1. \tag{1.5}$$

Given the functional equation of $\zeta(s)$, this is equivalent to

$$\zeta(1-n) = -\frac{B_n}{n}, \quad n \geq 1. \tag{1.6}$$

On the other hand, Bernoulli polynomials are polynomials with Bernoulli numbers as coefficients defined by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \quad n = 0, 1, 2, \dots , \tag{1.7}$$

or equivalently,

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}, \quad |t| < 2\pi. \tag{1.8}$$

Bernoulli polynomials can be evaluated one by one through the differential equation

$$\frac{d}{dx} B_n(x) = nB_{n-1}(x), \quad B_n(0) = B_n, \quad n \geq 1, \tag{1.9}$$

and $B_0(x) = 1$. The first few Bernoulli polynomials are

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}. \tag{1.10}$$

Bernoulli polynomials satisfy the relation

$$B_n(1-x) = (-1)^n B_n(x). \tag{1.11}$$

They have the following Fourier expansions [**10**]:

$$\sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n} B_{2n}(x)}{2(2n)!}, n \geq 1, 0 \leq x < 1, \tag{1.12}$$

$$\sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n+1}} = \frac{(-1)^{n+1}(2\pi)^{2n+1} B_{2n+1}(x)}{2(2n+1)!}, n \geq 1, 0 \leq x < 1. \tag{1.13}$$

Bernoulli polynomials are used to express the special values at negative integers of Hurwitz zeta function defined as below [**12**]:

$$\zeta(s;x) = \sum_{n=0}^{\infty} (n+x)^{-s}, \quad \operatorname{Re} s > 1, x > 0. \tag{1.14}$$

Such a zeta function has its analytic continuation in the complex plane as a function of s and, for each positive integer m,

$$\zeta(1-m;x) = -\frac{B_m(x)}{m}. \tag{1.15}$$

Zeta functions related to linear forms were considered by the second author [**5, 6, 7**] as generalizations of Riemann zeta and Hurwitz zeta functions. Their special values at negative integers are given by polynomial functions in Bernoulli numbers and Bernoulli polynomials.

Theorem A [**5, 7**]. *Let $a_1, a_2, \ldots, a_r$ be positive numbers. Define the zeta function $Z_1(s)$ by*

(1.16)

$$Z_1(s) = \sum_{n=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} [a_1 n_1 + a_2 n_2 + \cdots + a_r n_r]^{-s}, \quad \operatorname{Re} s > r.$$

Then $Z_1(s)$ has its analytic continuation in the whole complex plane as a function of s. Furthermore, for each positive integer $m \geq r$,

(1.17)

$$Z_1(r-m) = \sum_{|p|=m} \frac{(m-r)!}{p_1! p_2! \cdots p_r!} a_1^{p_1-1} a_2^{p_2-1} \cdots a_r^{p_r-1} B_{p_1} B_{p_2} \cdots B_{p_r},$$

where $p = (p_1, \ldots, p_r)$ ranges over all r-tuples of nonnegative integers and $|p| = p_1 + \cdots + p_r$.

Theorem B [5, 7]. *Let $a_1, a_2, \ldots, a_r$ be positive integers, and let $x = (x_1, x_2, \ldots, x_r)$ be r-tuples of nonnegative numbers such that $x_1 + x_2 + \cdots + x_r > 0$. Define the zeta function $Z_2(s; x)$ by*

$$Z_2(s;x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} [a_1(n_1 + x_1) + a_2(n_2 + x_2) + \cdots + a_r(n_r + x_r)]^{-s}, \quad \mathrm{Re}\, s > r. \tag{1.18}$$

Then $Z_2(s; x)$ has its analytic continuation in the whole complex plane as a function of s. Furthermore, for each positive integer $m \geq r$,

$$Z_2(r-m;x) = (-1)^r \sum_{|p|=m} \frac{(m-r)!}{p_1! p_2! \cdots p_r!} a_1^{p_1-1} a_2^{p_2-1} \cdots a_r^{p_r-1} B_{p_1}(x_1) B_{p_2}(x_2) \cdots B_{p_r}(x_r). \tag{1.19}$$

There is a famous identity on Bernoulli numbers due to Euler [**1**]:

$$\sum_{k=1}^{n-1} \frac{(2n)!}{(2k)!(2n-2k)!} B_{2k} B_{2n-2k} = -(2n+1) B_{2n}, \quad n \geq 2, \tag{1.20}$$

which is equivalent to the identity

$$\sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) = \frac{2n+1}{2} \zeta(2n). \tag{1.21}$$

One way to prove the above identity is to consider the zeta function

$$Z_3(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (n_1 + n_2)^{-s}, \quad \mathrm{Re}\, s > 2. \tag{1.22}$$

Note that $Z_3(s)$ is a specialization of $Z_1(s)$.

With $n = n_1 + n_2$ as a new dummy variable in place of n_1, n_2, we get the identity

$$Z_3(s) = \sum_{n=2}^{\infty} (n-1)n^{-s} = \zeta(s-1) - \zeta(s). \tag{1.23}$$

Now there are at least two ways to evaluate $Z_3(s)$ at negative integers: one way is by means of Theorem A, and the other way is by means of the special values of Riemann zeta function. Setting $s = 2 - 2n$ with $n \geq 2$ in the identity (1.23), we get

$$\sum_{k=0}^{n} \frac{(2n-2)!}{(2k)!(2n-2k)!} B_{2k} B_{2n-2k} = -\frac{B_{2n}}{2n}. \tag{1.24}$$

A rearrangement then yields the identity (1.20).

Similar considerations lead to identities of a similar kind such as the following which appeared in [**6**].

$$\begin{aligned} \sum_{p+q+r=n} & \frac{(2n)!}{(2p)!(2q)!(2r)!} B_{2p} B_{2q} B_{2r} \\ &= \frac{(2n+1)(2n+2)}{2} B_{2n} + \frac{n(n-1)}{2} B_{2n-2}, \quad n \geq 3, \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} \sum_{p+q+r+l=n} & \frac{(2n)!}{(2p)!(2q)!(2r)!(2l)!} B_{2p} B_{2q} B_{2r} B_{2l} \\ &= \frac{(2n+1)(2n+2)(2n+3)}{6} B_{2n} + \frac{4n^2(2n-1)}{3} B_{2n-2}, \end{aligned} \tag{1.26}$$

where $p, q, r, l \geq 1$ in the above summations.

To obtain identities involving Bernoulli polynomials, we simply consider zeta functions of the form

$$Z_4(s; x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} [(n_1 + x) + (n_2 + x) + n_3 + \cdots + n_r]^{-s},$$

which is equal to

$$\frac{1}{(r-1)!}\sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+r-1)(n+2x)^{-s}.$$

The above zeta function is a linear combination of shifted Hurwitz zeta functions, so that we have two ways to evaluate $Z_4(s;x)$ at negative integers and hence identities involving Bernoulli polynomials follow.

For example, consider the zeta function

$$Z_5(s;x)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(n_1+x)+(n_2+x)]^{-s},$$

which is equal to

$$\sum_{n=0}^{\infty}(n+1)(n+2x)^{-s}.$$

A slight observation then leads to the identity

$$Z_5(s;x)=\mathbf{Z}(s-1,2x)+(1-2x)\mathbf{Z}(s;2x).$$

By setting $s=1-2n$ with $n\geq 1$, one gets that

(1.27)
$$\sum_{p+q=2n+1}\frac{(2n-1)!}{p!q!}B_p(x)B_q(x)=-\frac{B_{2n+1}(2x)}{2n+1}+2B_1(x)\frac{B_{2n}(2x)}{2n}.$$

Here are just a few.

(1.28)
$$\sum_{p+q+r=2n+1}\frac{(2n-2)!}{p!q!r!}B_p(x)B_q(x)B_r$$
$$=\frac{1}{2}\frac{B_{2n+1}(2x)}{2n+1}+\frac{(3-4x)}{2}\frac{B_{2n}(2x)}{2n}+(2x^2-3x+1)\frac{B_{2n-1}(2x)}{2n-1},$$

and

(1.29)
$$\sum_{p+q+r+l=2n+1}\frac{(2n-3)!}{p!q!r!l!}B_p(x)B_q(x)B_rB_l$$
$$=-\frac{1}{6}\frac{B_{2n+1}(2x)}{2n+1}+(x-1)\frac{B_{2n}(2x)}{2n}$$
$$-\frac{(12x^2-24x+11)}{6}\frac{B_{2n-1}(2x)}{2n-1}$$
$$+\frac{(8x^3-24x^2+22x-6)}{6}\frac{B_{2n-2}(2x)}{2n-2}.$$

Identities such as those above can be transformed into relations among Euler sums and extended Euler sums. We shall introduce a new relation in the next section by multiplying both sides by $\cot \pi x$ and then integrating with respect to x from 0 to 1/2.

2. Extended Euler sums and main theorems. For a pair of positive integers p and q with $q > 1$, the classical Euler sum is defined as [**9**]:

$$S_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \left(\sum_{j=1}^{k} \frac{1}{j^p} \right). \tag{2.1}$$

$S_{p,q}$ is also denoted by $S_{p,q}^{++}$. Other alternating Euler sums are defined as follows [**9**].

$$S_{p,q}^{+-} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^q} \left(\sum_{j=1}^{k} \frac{1}{j^p} \right), \tag{2.2}$$

$$S_{p,q}^{-+} = \sum_{k=1}^{\infty} \frac{1}{k^q} \left(\sum_{j=1}^{k} \frac{(-1)^{j-1}}{j^p} \right), \tag{2.3}$$

and

$$S_{p,q}^{--} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^q} \left(\sum_{j=1}^{k} \frac{(-1)^{j-1}}{j^p} \right). \tag{2.4}$$

The evaluation of $S_{p,q}$ in terms of values of Riemann zeta functions are known when $p = 1$, $(p, q) = (2, 4)$ or $(4, 2)$, $p = q$, or $p + q$ is odd. In particular, we have the following theorems.

Theorem C [**1, 9**]. *For each positive integer n with $n \geq 2$, we have*

$$S_{1,n} = \frac{n+2}{2} \zeta(n+1) - \frac{1}{2} \sum_{k=2}^{n-1} \zeta(k) \zeta(n+1-k). \tag{2.5}$$

Theorem D [**9**]. *For positive integers m and n, we have*

$$S_{2m,2n+1} = \frac{1}{2}\zeta(2m+2n+1) + \sum_{k=0}^{m} \binom{2\bar{k}}{2n} \zeta(2k)\zeta(2\bar{k}+1) + \sum_{k=0}^{n} \binom{2\bar{k}}{2m-1} \zeta(2k)\zeta(2\bar{k}+1), \tag{2.6}$$

where $\bar{k} = m+n-k$. *Here* $\zeta(s)$ *is the Riemann zeta function with* $\zeta(0) = -1/2$.

It is interesting to note that $S_{2m,2n+1}$ comes from the integral transformation of a product of two Bernoulli polynomials. Indeed, we have in Proposition 6 that

$$\int_0^1 B_{2m}(x)B_{2n+1}(x)\cot \pi x\, dx = \frac{(-1)^{m+n}4(2m)!(2n+1)!}{(2\pi)^{2m+2n+1}} \left\{ S_{2m,2n+1} - \frac{1}{2}\zeta(2m+2n+1) \right\}. \tag{2.7}$$

Consequently, another way of verifying Theorem D follows from the Bernoulli identity

$$B_{2m}(x)B_{2n+1}(x) = (2n+1)\sum_{k=0}^{m} \binom{2m}{2k} B_{2k} \frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1} + (2m)\sum_{k=0}^{n} \binom{2n+1}{2k} B_{2k} \frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1} \tag{2.8}$$

with $\bar{k} = m+n-k$, and the well-known formula [**4**]

$$\int_0^1 B_{2n+1}(x)\cot \pi x\, dx = \frac{2(-1)^{n+1}(2n+1)!}{(2\pi)^{2n+1}}\zeta(2n+1). \tag{2.9}$$

Remark 1. The Bernoulli identity (2.8) comes from the identity of the zeta function

$$\begin{aligned}(2.10)\quad \zeta(ps,x)\zeta(qs,x)+\zeta(ps+qs,x)\\ &=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(n_1+n_2+x)^p(n_2+x)^q]^{-s}\\ &\quad+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(n_1+n_2+x)^q(n_2+x)^p]^{-s},\end{aligned}$$

which follows from

$$\begin{aligned}(2.11)\quad &\zeta(ps,x)\zeta(qs,x)\\ &=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(n_1+x)^p(n_2+x)^q]^{-s}\\ &=\sum_{n_1\geq n_2}[(n_1+x)^p(n_2+x)^q]^{-s}+\sum_{n_1\leq n_2}[(n_1+x)^p(n_2+x)^q]^{-s}\\ &\quad-\sum_{n_1=n_2}[(n_1+x)^p(n_2+x)^q]^{-s}.\end{aligned}$$

In this paper, we shall transform the Bernoulli identities involving Bernoulli polynomials such as the identities (1.27), (1.28) and (1.29) into the explicit evaluations of the extensions of Euler sums defined by

$$E_{p,q}=\sum_{k=1}^{\infty}\frac{1}{k^q}\left(\sum_{j=1}^{2k}\frac{1}{j^p}\right) \tag{2.12}$$

and

$$T_{p,q}=\sum_{k=1}^{\infty}\frac{1}{k^q}\left(\sum_{j=1}^{[k/2]}\frac{1}{j^p}\right). \tag{2.13}$$

Here and throughout, we name $E_{p,q}$ and $T_{p,q}$ the extended Euler sums.

The following relationships

$$S_{p,q}-S_{p,q}^{+-}=2^{1-q}E_{p,q}, \tag{2.14}$$

$$S_{p,q}-S_{p,q}^{-+}=2^{1-p}T_{p,q}, \tag{2.15}$$

and for $p, q \geq 2$,

$$E_{p,q} + T_{q,p} = 2^{-p}\zeta(p+q) + \zeta(p)\zeta(q), \tag{2.16}$$

are all elementary by the definitions of $E_{p,q}$ and $T_{p,q}$. It is known that the values of $S_{p,q}$ and $S_{p,q}^{-+}$ were already given in [**9**] when $p = 1$ or $p + q$ is odd, so that $T_{p,q}$ can be obtained from (2.15) when $p = 1$ or $p + q$ is odd. Moreover, we obtain the value of $E_{p,q}$ from $T_{q,p}$ when $p + q$ is odd and $p, q \geq 2$ with the relationship (2.16).

For other explicit values of certain extended Euler sums such as $E_{1,2n}$, $E_{2,2n-1}$ and $T_{2n-2,3}$, we state our main theorems as follows. The proofs will be given in Sections 4 to 6. First, we obtain the evaluation of $E_{1,2n}$ from the identity (1.27).

Theorem 1. *For each positive integer n, we have*

$$E_{1,2n} = \frac{2n+1}{4}\zeta(2n+1) - \sum_{k=0}^{n-1} 2^{2n-2k}\zeta(2k)\zeta(2n+1-2k). \tag{2.17}$$

Next, we obtain the value of $E_{2,2n-1}$ from the identity (1.28).

Theorem 2. *For positive integers n with $n \geq 2$, we have*

(2.18)
$$\begin{aligned} E_{2,2n-1} &= \frac{1}{8}\zeta(2n+1) \\ &\quad + \sum_{k=0}^{n-1}(2n-2k)2^{2n-1-2k}\zeta(2k)\zeta(2n+1-2k) \\ &\quad + 2^{2n-2}\sum_{k=0}^{1}\binom{2n-2k}{2n-2} \\ &\quad \times [\zeta(2k)\zeta(2n+1-2k) + \eta(2k)\eta(2n+1-2k)], \end{aligned}$$

where

$$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{-s}} = (1-2^{1-s})\zeta(s)$$

with $\eta(1) = \log 2$.

Finally, we obtain the explicit evaluation of $T_{2n-2,3}$ from the identity (1.29).

Theorem 3. *For positive integers n with $n \geq 2$, we have*

(2.19)
$$\begin{aligned} T_{2n-2,3} &= \frac{1}{16}\zeta(2n+1) \\ &+ \sum_{k=0}^{n-1} \binom{2n-2k}{2} 2^{2n-2-2k}\zeta(2k)\zeta(2n+1-2k) \\ &+ 2^{2n-3} \sum_{k=0}^{1} \binom{2n-2k}{2n-3} \\ &\times [\zeta(2k)\zeta(2n+1-2k) + \eta(2k)\eta(2n+1-2k)] \end{aligned}$$

with the same $\eta(s)$ defined as in Theorem 2.

A direct verification reveals that the evaluation of $E_{1,2n}$ in Theorem 1 comes from the relation (2.14) with $S_{1,2n}$ and $S_{1,2n}^{+,-}$ given in [**9**]. However, what we provide in this work is a more systematic way of evaluating $E_{1,2n}$ as well as deriving the explicit values of $E_{2,2n-1}$ and $T_{2n-2,3}$, which so far had not yet been considered. Furthermore, the explicit formulæ of the general extended Euler sums of odd weight which will be shown in Section 7 can also be carried out.

Our method in this paper is based on the integral transformation of certain kinds of Bernoulli identities into $E_{p,q}$ and $T_{p,q}$ as one shall see in Section 3. It is worth noting that in contrast to the similar technique anticipated by Crandall and Buhler [**3**], we develop a more systematic way of deriving the desired identities among Bernoulli polynomials by showing how they arise from the special values at the negative integers of the certain zeta functions [**7**].

3. The integral transformations. Like the classical Euler sum $S_{p,q}$, both $E_{p,q}$ and $T_{p,q}$ come from the integral transformations of products of Bernoulli polynomials when the weight $p+q$ is odd.

Proposition 1. *For positive integers m and n, we have*

$$\int_0^{1/2} B_{2m}(x)B_{2n+1}(2x)\cot\pi x\,dx \tag{3.1}$$
$$= \frac{(-1)^{m+n}2(2m)!(2n+1)!}{(2\pi)^{2m+2n+1}}\{E_{2m,2n+1} - 2^{-(2m+1)}\zeta(2m+2n+1)\}.$$

Proof. For $0 \le x < 1/2$, we have from (1.13) that

$$B_{2n+1}(2x) = \frac{(-1)^{n+1}2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin 4k\pi x}{k^{2n+1}}.$$

Note that

$$\sin 4k\pi x\cot\pi x = 1 + 2\sum_{j=1}^{2k}\cos 2j\pi x - \cos 4k\pi x.$$

In order to evaluate the integral, it suffices to evaluate

$$\int_0^{1/2} B_{2m}(x)\,dx \quad \text{and} \quad \int_0^{1/2} B_{2m}(x)\cos 2j\pi x\,dx.$$

The anti-derivative of $B_{2m}(x)$ is $B_{2m+1}(x)/2m+1$, so it follows that

$$\int_0^{1/2} B_{2m}(x)\,dx = \frac{1}{2m+1}\left\{B_{2m+1}\left(\frac{1}{2}\right) - B_{2m+1}(0)\right\} = 0.$$

For each $1 \le j \le 2k$, we have

$$\begin{aligned}\int_0^{1/2} & B_{2m}(x)(2\cos 2j\pi x)\,dx \\ &= \frac{(-1)^{m-1}2(2m)!}{(2\pi)^{2m}}\sum_{l=1}^{\infty}\frac{1}{l^{2m}}\int_0^{1/2} 2\cos 2l\pi x\cos 2j\pi x\,dx \\ &= \frac{(-1)^{m-1}2(2m)!}{(2\pi)^{2m}}\frac{1}{j^{2m}}\cdot\frac{1}{2}\end{aligned}$$

since

$$\int_0^{1/2} 2\cos 2l\pi x \cos 2j\pi x \, dx = \begin{cases} 0 & \text{if } l \neq j, \\ 1/2 & \text{if } l = j. \end{cases}$$

Thus our assertion follows. □

In exactly the same way, we obtain the following.

Proposition 2. *For positive integers m and n, we have*

$$\begin{aligned} (3.2) \quad \int_0^{1/2} & B_{2m}(2x) B_{2n+1}(x) \cot \pi x \, dx \\ &= \frac{(-1)^{m+n} 2(2m)!(2n+1)!}{(2\pi)^{2m+2n+1}} \\ &\quad \times \{T_{2m,2n+1} - 2^{-(2n+2)} \zeta(2m+2n+1)\}. \end{aligned}$$

Here is an exceptional case excluded from the above considerations.

Proposition 3. *For each positive integer n, we have*

$$\begin{aligned} (3.3) \quad \int_0^{1/2} & B_1(x) \{B_{2n}(2x) - B_{2n}\} \cot \pi x \, dx \\ &= \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n+1}} \left\{ E_{1,2n} - \frac{1}{4} \zeta(2n+1) \right\}. \end{aligned}$$

Proof. For $0 \leq x < 1/2$, we have

$$B_{2n}(2x) - B_{2n} = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 4k\pi x - 1}{k^{2n}}$$

in light of (1.12). Note that

$$(\cos 4k\pi x - 1) \cot \pi x = -2 \sum_{j=1}^{2k} \sin 2j\pi x + \sin 4k\pi x.$$

Also we have
$$\int_0^{1/2} B_1(x)(-2\sin 2j\pi x)\,dx = \frac{1}{2\pi j}$$
by an integration by parts. It follows that
$$\int_0^{1/2} B_1(x)\{B_{2n}(2x) - B_{2n}\}\cot \pi x\,dx$$
$$= \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \left\{ \sum_{j=1}^{2k} \frac{1}{2\pi j} - \frac{1}{2} \cdot \frac{1}{4\pi k} \right\}$$
$$= \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n+1}} \left\{ E_{1,2n} - \frac{1}{4}\zeta(2n+1) \right\}. \qquad \square$$

In the same manner, we get the following propositions.

Proposition 4. *For each positive integer n, we have*
$$\int_0^{1/2} B_1(2x)\{B_{2n}(x) - B_{2n}\}\cot \pi x\,dx = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n+1}} \{T_{1,2n} - 2^{-(2n+1)}\zeta(2n+1)\}. \tag{3.4}$$

Proposition 5. *For each positive integer n, we have*
$$\int_0^{1/2} B_1(x)\{B_{2n}(x) - B_{2n}\}\cot \pi x\,dx = \frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n+1}} \left\{ S_{1,2n} - \frac{1}{2}\zeta(2n+1) \right\}. \tag{3.5}$$

The following proposition has a similar proof to Proposition 1.

Proposition 6. *For positive integers m and n, we have*
$$\int_0^{1/2} B_{2m}(x)B_{2n+1}(x)\cot \pi x\,dx = \frac{(-1)^{m+n}2(2m)!(2n+1)!}{(2\pi)^{2m+2n+1}} \left\{ S_{2m,2n+1} - \frac{1}{2}\zeta(2m+2n+1) \right\}. \tag{3.6}$$

4. The evaluation for $E_{1,2n}$. To obtain the evaluation of $E_{1,2n}$, we need Bernoulli identities involving the particular term $B_1(x)\{B_{2n}(2x)-B_{2n}\}$. So we begin with identity (1.27). By removing the terms corresponding to $p=0$, $p=1$, $q=0$, and $q=1$ from the lefthand side of the identity to the righthand side, we get the identity

$$\begin{aligned}2\sum_{k=1}^{n-1}&\frac{(2n-1)!}{(2k+1)!(2n-2k)!}B_{2k+1}(x)B_{2n-2k}(x)\\&=-2\frac{(2n-1)!}{(2n+1)!}B_{2n+1}(x)-\frac{B_{2n+1}(2x)}{2n+1}\\&\quad+2B_1(x)\frac{B_{2n}(2x)}{2n}-2B_1(x)\frac{B_{2n}(x)}{2n},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sum_{k=1}^{n-1}&\frac{(2n-1)!}{(2k+1)!(2n-2k)!}B_{2k+1}(x)B_{2n-2k}(x)\\&=-\frac{(2n-1)!}{(2n+1)!}B_{2n+1}(x)-\frac{1}{2}\frac{B_{2n+1}(2x)}{2n+1}\\&\quad+B_1(x)\frac{1}{2n}\{B_{2n}(2x)-B_{2n}\}\\&\quad-B_1(x)\frac{1}{2n}\{B_{2n}(x)-B_{2n}\}.\end{aligned}\tag{4.1}$$

Each term in the above identity is ready to be applied with an integral transformation. Multiplying both sides by $\cot\pi x$ and then integrating with respect to x from 0 to $1/2$, we get the identity

$$\begin{aligned}&\left\{E_{1,2n}-\frac{1}{4}\zeta(2n+1)\right\}-\left\{S_{1,2n}-\frac{1}{2}\zeta(2n+1)\right\}\\&\quad=\frac{n+1}{2}\zeta(2n+1)-\sum_{k=1}^{n-1}\left\{S_{2n-2k,2k+1}-\frac{1}{2}\zeta(2n+1)\right\}\end{aligned}\tag{4.2}$$

after a cancelation of a constant. With the help of the formulæ (2.5) and (2.6) mentioned in Section 2, and the well-known formula

$$\sum_{k=0}^{m}\binom{m}{k}=2^m$$

for any positive integer m, we get our explicit value of $E_{1,2n}$. Thus, the proof of Theorem 1 is completed.

5. The evaluation for $E_{2,2n-1}$. The evaluation of $E_{2,2n-1}$ comes from the integral transformation of $B_2(x)B_{2n-1}(2x)$, so we choose the identity (1.28) as a possible candidate. The identity is as follows.

$$\sum_{p+q+r=2n+1} \frac{(2n-2)!}{p!q!r!} B_p(x)B_q(x)B_r = \frac{1}{2}\frac{B_{2n+1}(2x)}{2n+1} + \frac{(3-4x)}{2}\frac{B_{2n}(2x)}{2n} + (2x^2-3x+1)\frac{B_{2n-1}(2x)}{2n-1}. \tag{5.1}$$

Again we want to remove the terms corresponding to $r=1$, $p=1$ and $q=1$ from the lefthand side of the identity to the righthand side.

The term corresponding to $r=1$ is equal to

$$\left(-\frac{1}{2}\right)\sum_{p+q=2n} \frac{(2n-2)!}{p!q!} B_p(x)B_q(x). \tag{5.2}$$

With the identity (1.27), but replacing $2n+1$ with $2n$ instead, the terms (5.2) are equal to

$$\left(-\frac{1}{2}\right)\left\{-\frac{B_{2n}(2x)}{2n} + 2B_1(x)\frac{B_{2n-1}(2x)}{2n-1}\right\}. \tag{5.3}$$

The terms corresponding to $p=1$, $q=1$ and r even are given by

$$2B_1(x)\left\{\sum_{q+r=2n} \frac{(2n-2)!}{q!r!} B_q(x)B_r + \frac{1}{2}\frac{B_{2n-1}(x)}{2n-1}\right\}, \tag{5.4}$$

which is equal to

$$2B_1(x)\left\{-\frac{B_{2n}(x)}{2n} + (x-1)\frac{B_{2n-1}(x)}{2n-1} + \frac{1}{2}\frac{B_{2n-1}(x)}{2n-1}\right\}. \tag{5.5}$$

So, after the above rearrangement, we get the following Bernoulli identity.

(5.6)
$$2\sum_{k=0}^{n-2}\sum_{l=1}^{n-k-1}\frac{(2n-2)!}{(2k)!(2l+1)!(2n-2k-2l)!}B_{2k}B_{2l+1}(x)B_{2n-2k-2l}(x)$$
$$+2\sum_{k=0}^{n-1}\frac{(2n-2)!}{(2k)!(2n+1-2k)!}B_{2k}B_{2n+1-2k}(x)$$
$$=\frac{1}{2}\frac{B_{2n+1}(2x)}{2n+1}-2B_1(x)\frac{1}{2n}\{B_{2n}(2x)-B_{2n}\}$$
$$+2B_1(x)\frac{1}{2n}\{B_{2n}(x)-B_{2n}\}+2B_2(x)\frac{B_{2n-1}(2x)}{2n-1}$$
$$-2B_2(x)\frac{B_{2n-1}(x)}{2n-1}+\frac{1}{6}\frac{B_{2n-1}(2x)}{2n-1}-\frac{1}{6}\frac{B_{2n-1}(x)}{2n-1}.$$

Multiplying both sides by $\cot\pi x$ and integrating with respect to x from 0 to 1/2, we get the following identity.

(5.7)
$$E_{2,2n-1}-\frac{1}{8}\zeta(2n+1)$$
$$=\frac{2n(2n-1)}{16}\zeta(2n+1)$$
$$-\frac{2n-1}{2}\Big\{\Big[E_{1,2n}-\frac{1}{4}\zeta(2n+1)\Big]$$
$$-\Big[S_{1,2n}-\frac{1}{2}\zeta(2n+1)\Big]\Big\}$$
$$+\Big\{S_{2,2n-1}-\frac{1}{2}\zeta(2n+1)\Big\}$$
$$+\frac{1}{2}\sum_{k=0}^{n-1}\zeta(2k)\zeta(2n+1-2k)$$
$$-\sum_{k=0}^{n-2}\sum_{l=1}^{n-k-1}\zeta(2k)\Big\{S_{2n-2k-2l,2l+1}-\frac{1}{2}\zeta(2n+1-2k)\Big\},$$

for each positive integer $n\geq 2$.

Consequently, the formula (5.7) can be verified to be equivalent to (2.18) in light of (4.2) and the repeated use of (2.6). This gives the proof of Theorem 2.

6. The evaluation for $T_{2n-2,3}$. The evaluation of $T_{2n-2,3}$ depends on the integral transformation of $B_{2n-2}(2x)B_3(x)$, so we choose the identity (1.29) as a possible candidate. The identity is given by

$$\sum_{p+q+r+l=2n+1} \frac{(2n-3)!}{p!q!r!l!} B_p(x)B_q(x)B_rB_l = -\frac{1}{6}\frac{B_{2n+1}(2x)}{2n+1} + (x-1)\frac{B_{2n}(2x)}{2n} - \frac{(12x^2-24x+11)}{6}\frac{B_{2n-1}(2x)}{2n-1} + \frac{(8x^3-24x^2+22x-6)}{6}\frac{B_{2n-2}(2x)}{2n-2}. \tag{6.1}$$

Let

$$a(k)B_{2k} = \sum_{r+l=k} \frac{(2k)!}{(2r)!(2l)!} B_{2r}B_{2l}.$$

By applying with the identity (1.24), we have that

$$a(k) = \begin{cases} 1 & \text{if } k=0, \\ 2 & \text{if } k=1, \\ -(2k-1) & \text{if } k>1. \end{cases}$$

Therefore, the following identity follows after we remove the terms corresponding to $p=1$, $q=1$, $r=1$, and $l=1$ from the lefthand side of the identity (6.1) to the righthand side.

$$\begin{aligned} 2\sum_{k=0}^{n-2}\sum_{l=1}^{n-k-1} &\frac{(2n-3)!}{(2k)!(2l+1)!(2n-2k-2l)!} a(k)B_{2k}B_{2l+1}(x)B_{2n-2k-2l}(x) \\ &+ 2\sum_{k=0}^{n-1}\frac{(2n-3)!}{(2k)!2n+1-2k!} a(k)B_{2k}B_{2n+1-2k}(x) \\ &= -\frac{1}{6}\frac{B_{2n+1}(2x)}{2n+1} + B_1(x)\frac{1}{2n}\{B_{2n}(2x)-B_{2n}\} \\ &\quad - B_1(x)\frac{1}{2n}\{B_{2n}(x)-B_{2n}\} \end{aligned} \tag{6.2}$$

$$
\begin{aligned}
&+\frac{1}{2}B_1(x)\frac{1}{2n-2}\{B_{2n-2}(2x)-B_{2n-2}\}\\
&-\frac{1}{2}B_1(x)\frac{1}{2n-2}\{B_{2n-2}(x)-B_{2n-2}\}+2B_2(x)\frac{B_{2n-1}(2x)}{2n-1}\\
&-2B_2(x)\frac{B_{2n-1}(x)}{2n-1}+\frac{4}{3}B_3(x)\frac{B_{2n-2}(x)}{2n-2}-B_3(x)\frac{B_{2n-2}(2x)}{2n-2}\\
&+\frac{1}{4}\frac{B_{2n-1}(2x)}{2n-1}-\frac{1}{6}\frac{B_{2n-1}(x)}{2n-1}.
\end{aligned}
$$

Applying the same integral transformation to (6.2), we get the following identity.

$$
\begin{aligned}
&(6.3)\quad T_{2n-2,3}-\frac{1}{16}\zeta(2n+1)\\
&=\frac{4}{3}\Big\{S_{2n-2,3}-\frac{1}{2}\zeta(2n+1)\Big\}+\frac{2(2n-2)}{3}\\
&\quad\times\Big\{\Big(E_{2,2n-1}-\frac{1}{8}\zeta(2n+1)\Big)-\Big(S_{2,2n-1}-\frac{1}{2}\zeta(2n+1)\Big)\Big\}\\
&\quad-\frac{(2n)(2n-1)}{6}\Big\{\Big(E_{1,2n}-\frac{1}{4}\zeta(2n+1)\Big)-\Big(S_{1,2n}-\frac{1}{2}\zeta(2n+1)\Big)\Big\}\\
&\quad+2\zeta(2)\Big\{\Big(E_{1,2n-2}-\frac{1}{4}\zeta(2n-1)\Big)-\Big(S_{1,2n-2}-\frac{1}{2}\zeta(2n-1)\Big)\Big\}\\
&\quad+\frac{(2n)(2n-1)(2n-2)}{72}\zeta(2n+1)-\frac{1}{6}(2n-2)\zeta(2)\zeta(2n-1)\\
&\quad+\frac{2}{3}\sum_{k=0}^{n-2}\sum_{l=1}^{n-k-1}a(k)\zeta(2k)\Big\{S_{2n-2k-2l,2l+1}-\frac{1}{2}\zeta(2n+1-2k)\Big\}\\
&\quad-\frac{1}{3}\sum_{k=0}^{n-1}a(k)\zeta(2k)\zeta(2n+1-2k)
\end{aligned}
$$

for each positive integer $n \geq 2$.

Consequently, Theorem 3 can be verified with the same argument as in the proof of Theorem 2.

7. Explicit formulæ of $E_{2m,2n+1}$ and $T_{2m,2n+1}$. In the final section, we give the explicit generalized formulæ for the extended Euler sums of the forms $E_{2m,2n+1}$ and $T_{2m,2n+1}$ in Theorem 4.

For positive integers p and q, we consider the product of two Hurwitz zeta functions defined as

$$\zeta(ps,2x)\zeta(qs,x)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(n_1+2x)^p(n_2+x)^q]^{-s},\quad \operatorname{Re}s>1. \tag{7.1}$$

Now, we decompose the above double series into three subseries according to the conditions

(1) $n_1\geq 2n_2$,

(2) n_1 is even and $n_1<2n_2$,

(3) n_1 is odd and $n_1<2n_2$.

When $n_1\geq 2n_2$, we let $n_1=n_1'+2n_2$. When n_1 is even and $n_1<2n_2$, we let $n_1=2n_1'$, $n_2=n_1'+n_2'+1$. When n_1 is odd and $n_1<2n_2$, we let $n_1=2n_1'+1$, $n_2=n_1'+n_2'+1$. This leads to the following identity of zeta functions.

$$\begin{aligned}(7.2)\quad &\zeta(ps,2x)\zeta(qs,x)\\ &=2^{-ps}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(\frac{1}{2}n_1+(n_2+x)\right)^p(n_2+x)^q\right]^{-s}\\ &\quad+2^{-ps}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[(n_1+x)^p((n_1+x)+(n_2+1))^q\right]^{-s}\\ &\quad+2^{-ps}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\left(n_1+x+\frac{1}{2}\right)^p\left(\left(n_1+x+\frac{1}{2}\right)+\left(n_2+\frac{1}{2}\right)\right)^q\right]^{-s}.\end{aligned}$$

A new kind of zeta functions then appear on the righthand side of the above identity. Here we give the evaluation at negative integers of this new zeta function.

Proposition 7. *For positive integers p and q and numbers a,x,y with $a>0$, $x\geq 0$, $y>0$, define the zeta function as*

$$Z_{p,q}(s;a,x,y)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[(a(n_1+x)+(n_2+y))^p(n_2+y)^q]^{-s},\quad \operatorname{Re}s>1.$$

Then $Z_{p,q}$ has its analytic continuation in the whole complex plane as a function of s. Furthermore, for each positive integer m,

$$Z_{p,q}(-m;a,x,y) = \frac{1}{pm+1}\sum_{k=0}^{pm+1}\binom{pm+1}{k}a^{k-1}B_k(x)\frac{B_{pm+qm+2-k}(y)}{pm+qm+2-k} + \frac{p(-1)^{qm}a^{pm+qm+1}}{p+q}(pm)!(qm)!\frac{B_{pm+qm+2}(x)}{(pm+qm+2)!}.$$

Proof. Here we only give an outline. For $\operatorname{Re} s > 1$, we have

$$Z_{p,q}(s;a,x,y)\Gamma(ps)\Gamma(qs) = \int_0^\infty\int_0^\infty t_1^{ps-1}t_2^{qs-1}\left(\frac{e^{(1-y)(t_1+t_2)}}{e^{t_1+t_2}-1}\right)\left(\frac{e^{(1-x)at_1}}{e^{at_1}-1}\right)dt_1\,dt_2,$$

where

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt, \quad \operatorname{Re} s > 1$$

is the classical Gamma function.

Under the transformation $t_1 = tu$, $t_2 = tv$ with $t > 0$, $u, v \geq 0$ and $u + v = 1$, the integral is transformed into

$$\int_0^\infty t^{ps+qs-1}\,dt\int_0^1 u^{ps-1}v^{qs-1}\frac{e^{(1-y)t}e^{(1-x)atu}}{(e^t-1)(e^{atu}-1)}\,du.$$

The special value at the negative integer $s = -m$ is just the coefficient of t^{pm+qm} of the integration with respect to u. The result is

$$Z_{p,q}(-m;a,x,y) = \frac{(-1)^m m!}{p+q}\sum_{\alpha+\beta=pm+qm+2}\frac{a^{\beta-1}B_\alpha(1-y)B_\beta(1-x)}{\alpha!\beta!}G_\beta(-m)$$

where for $\operatorname{Re} s > 0$,

$$G_\beta(s) = \frac{\Gamma(s)}{\Gamma(ps)\Gamma(qs)}\int_0^1 u^{ps-1}v^{qs-1}u^{\beta-1}\,du = \frac{\Gamma(s)\Gamma(ps+\beta-1)}{\Gamma(ps)\Gamma(ps+qs+\beta-1)}.$$

Our assertion then follows from

$$G_\beta(-m) = \begin{cases} [(pm)!/m!][(pm+qm+1-\beta)!/ \\ \quad (pm+1-\beta)!](p+q)(-1)^{m+pm+qm} \\ \qquad\qquad\qquad \text{if } 0 \le \beta \le pm+1, \\ [(pm)!/m!][(qm)!/1]p(-1)^{m+pm} \\ \qquad\qquad\qquad \text{if } \beta = pm+qm+2, \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \quad \square$$

Setting $s = -m$ in the identity (7.2), we get that

$$\begin{aligned} (7.3)\quad & B_{pm+1}(2x)B_{qm+1}(x) \\ &= (qm+1)\sum_{k=0}^{pm+1}\binom{pm+1}{k}2^{pm+1-k}B_k\frac{B_{pm+qm+2-k}(x)}{pm+qm+2-k} \\ &\quad + \left(\frac{1}{2}\right)^{qm+1}\frac{p(-1)^{qm}}{p+q}(pm+1)!(qm+1)!\frac{B_{pm+qm+2}}{(pm+qm+2)!} \\ &\quad + 2^{pm}(pm+1)\sum_{k=0}^{qm+1}\binom{qm+1}{k}B_k(1)\frac{B_{pm+qm+2-k}(x)}{pm+qm+2-k} \\ &\quad + 2^{pm}\frac{q(-1)^{pm}}{p+q}(pm+1)!(qm+1)!\frac{B_{pm+qm+2}(1)}{(pm+qm+2)!} \\ &\quad + 2^{pm}(pm+1)\sum_{k=0}^{qm+1}\binom{qm+1}{k}B_k\left(\frac{1}{2}\right)\frac{B_{pm+qm+2-k}(x+(1/2))}{pm+qm+2-k} \\ &\quad + 2^{pm}\frac{q(-1)^{pm}}{p+q}(pm+1)!(qm+1)!\frac{B_{pm+qm+2}(1/2)}{(pm+qm+2)!}. \end{aligned}$$

In particular, we have for positive integers m and n that

$$\begin{aligned} (7.4)\quad & \\ & B_{2m}(x)B_{2n+1}(2x) \\ &\qquad = 2m\sum_{k=0}^{n}\binom{2n+1}{2k}2^{2n+1-2k}B_{2k}\frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1} \end{aligned}$$

$$+ 2^{2n}(2n+1)\sum_{k=0}^{m}\binom{2m}{2k}B_{2k}\frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1}$$
$$+ 2^{2n}(2n+1)\sum_{k=0}^{m}\binom{2m}{2k}B_{2k}\left(\frac{1}{2}\right)\frac{B_{2\bar{k}+1}(x+(1/2))}{2\bar{k}+1}$$

and

(7.5)
$$\begin{aligned}B_{2m}(2x)&B_{2n+1}(x)\\ &= (2n+1)\sum_{k=0}^{m}\binom{2m}{2k}2^{2m-2k}B_{2k}\frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1}\\ &\quad+ 2^{2m-1}(2m)\sum_{k=0}^{n}\binom{2n+1}{2k}B_{2k}\frac{B_{2\bar{k}+1}(x)}{2\bar{k}+1}\\ &\quad+ 2^{2m-1}(2m)\sum_{k=0}^{n}\binom{2n+1}{2k}B_{2k}(\frac{1}{2})\frac{B_{2\bar{k}+1}(x+(1/2))}{2\bar{k}+1},\end{aligned}$$

where $\bar{k} = m+n-k$. Under the integral transformation by multiplying $\cot \pi x$ and integrating with respect to x from 0 to $1/2$, we get our last main theorem.

Theorem 4. *For positive integers m and n, we have*

$$\begin{aligned}E_{2m,2n+1} &= 2^{-(2m+1)}\zeta(2m+2n+1)\\ &\quad+\sum_{k=0}^{n}\binom{2m+2n-2k}{2m-1}2^{2n+1-2k}\zeta(2k)\zeta(2m+2n+1-2k)\\ &\quad+2^{2n}\sum_{k=0}^{m}\binom{2m+2n-2k}{2n}\zeta(2k)\zeta(2m+2n+1-2k)\\ &\quad+2^{2n}\sum_{k=0}^{m}\binom{2m+2n-2k}{2n}\eta(2k)\eta(2m+2n+1-2k)\end{aligned}$$

and

$$\begin{aligned}
T_{2m,2n+1}& \\
&= 2^{-(2n+2)}\zeta(2m+2n+1)\\
&\quad+\sum_{k=0}^{m}\binom{2m+2n-2k}{2n}2^{2m-2k}\zeta(2k)\zeta(2m+2n+1-2k)\\
&\quad+2^{2m-1}\sum_{k=0}^{n}\binom{2m+2n-2k}{2m-1}\zeta(2k)\zeta(2m+2n+1-2k)\\
&\quad+2^{2m-1}\sum_{k=0}^{n}\binom{2m+2n-2k}{2m-1}\eta(2k)\eta(2m+2n+1-2k),
\end{aligned}$$

where

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{-s}}=(1-2^{1-s})\zeta(s)$$

with $\eta(1)=\log 2$.

Corollary 1. *For positive integers m and n we have*

$$\begin{aligned}
S^{+\,-}_{2m,2n+1} &= \frac{1}{2}\eta(2m+2n+1)\\
&\quad+\sum_{k=0}^{m}\binom{2m+2n-2k}{2n}\eta(2k)\zeta(2m+2n+1-2k)\\
&\quad-\sum_{k=0}^{n}\binom{2m+2n-2k}{2m-1}\eta(2k)\eta(2m+2n+1-2k)
\end{aligned}$$

and

$$\begin{aligned}
S^{-\,+}_{2m,2n+1} &= \frac{1}{2}\eta(2m+2n+1)\\
&\quad+\sum_{k=0}^{n}\binom{2m+2n-2k}{2m-1}\eta(2k)\zeta(2m+2n+1-2k)\\
&\quad-\sum_{k=0}^{m}\binom{2m+2n-2k}{2n}\eta(2k)\eta(2m+2n+1-2k).
\end{aligned}$$

Proof. It follows from

$$S_{2m,2n+1} - S^{+\,-}_{2m,2n+1} = 2^{-2n} E_{2m,2n+1}$$

and

$$S_{2m,2n+1} - S^{-\,+}_{2m,2n+1} = 2^{1-2m} T_{2m,2n+1}. \qquad \square$$

Remark 2. As a generalization of the extended Euler sums $E_{p,q}$, we may change the range of the inner summation into $j = 1$ to kl or $[kr]$ with k being a positive integer and r being a rational number. Namely,

$$E^{(k)}_{p,q} = \sum_{l=1}^{\infty} \frac{1}{l^q} \left(\sum_{j=1}^{kl} \frac{1}{j^p} \right)$$

and

$$E^{(r)}_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \left(\sum_{j=1}^{[kr]} \frac{1}{j^p} \right).$$

Then we obtain countable families of these new sums. The following results can be found in [**8**].

Theorem I. *For positive integers k, we have*

$$\begin{aligned} E^{(k)}_{1,2n} = \frac{2n+1}{2k}\zeta(2n+1) &- \sum_{l=0}^{n-1} k^{2n-2l}\zeta(2l)\zeta(2n+1-2l) \\ &- \sum_{j=1}^{k-1} k^{2n-1} S\left(1, \frac{j}{k}\right) S\left(2n, \frac{j}{k}\right), \end{aligned}$$

where $S(s,x) = \sum_{n=1}^{\infty} (\sin 2\pi x / n^s)$.

Theorem II. *Suppose that $r = a/b$ with a, b being relatively prime positive integers. Then for each positive integer n,*

$$E_{1,2n}^{(r)} = \left(\frac{1}{2ab^{2n}} + \frac{nb}{a}\right)\zeta(2n+1) - \sum_{l=0}^{n-1} a^{2n-2l}\zeta(2l)\zeta(2n+1-2l)$$

$$- \sum_{u=1}^{a-1} a^{2n-1} S\left(1, \frac{bu}{a}\right) S\left(2n, \frac{u}{a}\right)$$

$$- \sum_{v=1}^{b-1} \left[C\left(2n, \frac{av}{b}\right) - \zeta(2n)\right] C\left(1, \frac{v}{b}\right)$$

$$- \sum_{v=1}^{b-1}\sum_{l=0}^{n-1} a^{2n-2l} C\left(2l, \frac{av}{b}\right) C\left(2n+1-2l, \frac{v}{b}\right)$$

$$- \sum_{v=1}^{b-1}\sum_{l=0}^{n-1} a^{2n-1-2l} S\left(2l+1, \frac{av}{b}\right) S\left(2n-2l, \frac{v}{b}\right).$$

Here $\bar{l} = m + n - l$,

$$C(s,x) = \sum_{n=1}^{\infty} \frac{\cos 2\pi x}{n^s},$$

and

$$S(s,x) = \sum_{n=1}^{\infty} \frac{\sin 2\pi x}{n^s}$$

in the above summations with $C(0,x) = \zeta(0) = -1/2$ and $C(1,x) = -\log(2\sin(\pi x))$ for $0 < x < 1$.

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