

ON ORDERINGS, VALUATIONS AND 0-PRIMES OF A COMMUTATIVE RING

K.G. VALENTE

1. Introduction. In this paper we examine the interplay between the orderings and valuations of a commutative ring. In the field case, this well-known area of study, which has influenced work in both quadratic forms and real geometry, is developed quite nicely in several sources. Among these we make special note of the recent monographs of Lam [5] and Prestel [7], while, as an overview of semi-real rings and orderings, [4] serves well. We will also consider the 0-primes of a commutative ring, which were introduced in [8], and their relationship to orderings.

As we will see, our investigation is somewhat encumbered by the fact that there are two notions of a valuation of a commutative ring; the first can be found in Bourbaki [1] and the second is a refinement due to Manis [6]. It is not clear to the author which choice (if any) should be made; both have their advantages as well as shortcomings (although the latter is most preferred in the literature).

Throughout this paper, R will denote a commutative ring with unity (assuming all of the standard conventions). We will employ the following notation: $\sigma(R)$ denotes the sums of squares of R ; $X(R)$, the space of all orderings of R and $\text{RSpec}(R) := \{P \cap -P \mid P \in X(R)\}$. Also, \mathbf{Q}^+ (respectively \mathbf{R}^+) will refer to the set of all non-negative rational (respectively real) numbers, and, for A and B arbitrary sets, $A \setminus B := \{a \in A \mid a \notin B\}$. Lastly, given a totally ordered abelian group G (which we will always consider additively), we set $G^* = G \cup \{\infty\}$ while adopting the conventions that ∞ is larger than any element of G and $g + \infty = \infty + g = \infty$ for all $g \in G$.

2. Valuations.

DEFINITION. By a *valuation* of a commutative ring we will mean a map $v : R \rightarrow G^*$, where G is a totally ordered abelian group (possibly trivial), satisfying

- (1) $v(ab) = v(a) + v(b)$,
- (2) $v(a + b) \geq \min[v(a), v(b)]$,
- (3) $v(1) = 0$ and $v(0) = \infty$.

If $\text{im}(v) \setminus \{\infty\}$ is a subgroup of G , then we say that v is an *M-valuation* (or *Manis valuation*). In either case we may and shall assume that G is generated by $\text{im}(v) \setminus \{\infty\}$.

For $v : R \rightarrow G^*$ and $w : R \rightarrow H^*$ valuations of R , we say that v is equivalent to w if there is an order-isomorphism, $\theta : G \rightarrow H$, with $w = \theta^* \circ v$ (where θ^* is the natural extension of θ to G^*). Let $\text{Val}(R)$ (respectively $\text{MVal}(R)$) denote the set of (equivalence classes of) valuations (respectively M-valuations) of R .

For $v \in \text{Val}(R)$ we identify three subsets of R :

$$A_v = \{a \in R \mid v(a) \geq 0\},$$

$$p_v = \{b \in R \mid v(b) > 0\}$$

and

$$I_v = \{c \in R \mid v(c) = \infty\}.$$

One can show that A_v is a subring of R , p_v is a prime ideal of A_v and I_v is a prime ideal of R . We note that if v is an M-valuation, then it is completely determined (up to equivalence) by the pair (A_v, p_v) . Further, a pair (B, q) determines an M-valuation of R if and only if the following condition is met:

$$r \in R \setminus B \Rightarrow \exists x \in q \quad \text{with } rx \in B \setminus q$$

(see [6]). In general a valuation cannot be characterized by the associated pair or triple (see [9]).

PROPOSITION 2.1. *Given $v : R \rightarrow G^*$ in $\text{Val}(R)$, there is a valuation $\mathcal{V} : F_v \rightarrow G^*$, where F_v is the quotient field of R/I_v , with*

$$\mathcal{V}(\bar{a}/\bar{b}) = v(a) - v(b)$$

for all $a \in R$ and $b \notin I_v$.

PROOF. One checks or sees [1]. \square

It is easily verified that

$$d_v := A_v/p_v \hookrightarrow k_v := A_v/p_v$$

in a natural way. If k_v is the quotient field of d_v , we then call v *special*.

PROPOSITION 2.2. *Every M-valuation is special.*

PROOF. See [8]. \square

It should be noted that the converse to our last proposition is false (see Example 6.1) and that there exist non-special valuations.

One might ask under what circumstances all valuations are in fact M-valuations. Clearly this is the case if we assume that R is a field. More generally, it has been shown that if R has a large Jacobson radical (i.e., the only prime ideals which contain $J(R)$ are maximal) then every valuation is, with slight modification, surjective. One may refer to [3] and [9] for details.

3. Valuations, orderings and 0-primes.

DEFINITION. A 0-prime of R is an M-valuation pair (A_v, p_v) together with a subset $T \subseteq A_v$ such that

- (1) $T \in X(A_v)$ and
- (2) $\text{Supp}(T) := T \cap -T = p_v$.

We denote the set of 0-primes of R by $P_0(R)$.

By the remarks above, T induces an ordering, \bar{T} , of d_v with $\text{Supp}(\bar{T}) = \{0\}$ and hence an ordering, \bar{T}' , of k_v . Letting

$$T' = \{\alpha \in A_v \mid \bar{\alpha} \in \bar{T}'\}$$

we see that $(A_{\mathcal{V}}, p_{\mathcal{V}})$ and T' form a 0-prime of F_v which we will refer to as the *associated field with 0-prime*. Finally, we call a 0-prime *archimedean* when the following criterion is met:

$$a \in T \text{ and } b \in T \setminus (-T) \Rightarrow \exists n \in \mathbf{N} \text{ with } nb - a \in T.$$

If $R = F$ is a field, then a 0-prime is essentially what is known as a real-valuation of F (Theorem 1.4 of [8]). In this case, it is well-documented that F is formally real if and only if it admits a real-valuation. Given a commutative ring, the connection between orderings and 0-primes is illuminated in our next result.

THEOREM 3.1. *For R a commutative ring, the following are equivalent:*

- (1) R is semi-real (i.e., $-1 \notin \sigma(R)$),
- (2) $X(R) \neq \emptyset$ and
- (3) $P_0(R) \neq \emptyset$.

PROOF. The equivalence of (1) and (2) has been established in [4]. We therefore need only concern ourselves with conditions (2) and (3).

It is easily shown that the second statement implies the third since, given any $P \in X(R)$, we have $(R, \text{Supp}(P))$ is an M-valuation pair.

Now let (A_v, T) be in $P_0(R)$ with $A_v = (A_v, p_v)$. We know that $(A_{\mathcal{V}}, T')$ is a 0-prime of F_v . Thus, by the remarks above, F_v is formally real and, as such, admits an ordering, say \bar{P} . Setting

$$P = \{r \in R \mid \bar{r} \in \bar{P}\}$$

one easily checks that P is a preordering of R . Now, since every preordering is contained in some ordering (see [4]), our proof is complete. \square

Assume that R has a large Jacobson radical and $P \in X(R)$. Since P induces an ordering, P' , on $F_P := qf(R/\text{Supp}(P))$, there is a valuation, \mathcal{V} , of F_P with $k_{\mathcal{V}} \subseteq \mathbf{R}$. Letting v denote the “pull-back”

(M-) valuation of R , we see that (A_v, p_v) admits an archimedean \mathcal{O} -prime. With this construction, one can prove the following fact.

THEOREM 3.2. *If R possesses a large Jacobson radical, then the following are equivalent:*

- (1) R is semi-real,
- (2) $X(R) \neq \emptyset$,
- (3) R admits an archimedean \mathcal{O} -prime,
- (4) There exists an M -valuation pair (A_v, p_v) and an injective ring homomorphism $d_v \hookrightarrow \mathbf{R}$ and
- (5) $P_0(R) \neq \emptyset$.

4. Valuations compatible with orderings. We now wish to turn our attention to the question of “compatibility” between valuations and orderings.

DEFINITION. For $P \in X(R)$ and $v \in \text{Val}(R)$, we say that v is *strongly compatible*, or simply *s-compatible*, with P if

- (1) $I_v = \text{Supp}(P)$ and
- (2) $0 < a \leq b$ implies $v(a) \geq v(b)$.

If the weaker condition

- (2') $1 + p_v \subseteq P$

is satisfied, we then say that v is *compatible* with P .

We note that if R is a field, then v is *s-compatible* with P if and only if v is compatible with P (see [5, §2]).

In general, the following series of results insure that, to each ordering of R , there is an *s-compatible* valuation.

LEMMA 4.1. *Fix $P \in X(R)$ and $v \in \text{Val}(R)$ with $I_v = \text{Supp}(P)$. Write \mathcal{V} and P' for the naturally associated valuation and ordering of*

F_v . Then v is s -compatible with P if and only if \mathcal{V} is compatible with P' .

PROPOSITION 4.2. For P as above and v compatible with P ,

$$\bar{P} := (P \cap A_v)/p_v$$

is a preordering of d_v . If v is s -compatible with P , then \bar{P} is an ordering of d_v with $\text{Supp}(\bar{P}) = \{0\}$.

THEOREM 4.3. Given $P \in X(R)$, there exists a $v \in \text{Val}(R)$ which satisfies all of the following properties:

- (1) v is s -compatible with P ,
- (2) given $w \in \text{Val}(R)$ s -compatible with $P, A_v \subseteq A_w$,
- (3) $A_v = \{a \in R \mid \exists n \in \mathbf{N}, 0 \neq m \in \mathbf{N} \text{ with } n \pm ma \in P\}$,
- (4) (d_v, \bar{P}) can be order embedded in $(\mathbf{R}, \mathbf{R}^+)$ and
- (5) $(k_{\mathcal{V}}, \bar{P}')$ can be order embedded in $(\mathbf{R}, \mathbf{R}^+)$.

Moreover, v is unique with respect to the final property.

A SKETCH OF THE PROOF. The valuation we seek is obtained by considering the ordering $P' \subseteq F_P$ which is induced by P and pulling back the smallest valuation of F_P which is compatible with P' [5, Theorem 2.6]. The uniqueness statement is a direct consequence of a theorem due to Brown and Dubois (see [5, Theorem 2.11]). \square

We refer to the valuation described in this theorem as the canonical valuation associated to P , writing $v(P)$ for v and $A(P)$ for A_v . It is important to note that, in general, $v(P)$ is not an M-valuation. Also, there are situations in which $v(P)$ is not unique with respect to the property (4) above (see Example 6.1), although this is the case when R has a large Jacobson radical (in particular, when R is a field).

From Proposition 4.2 it is clear that, for an M-valuation pair (A_v, p_v) which is s -compatible with some $P \in X(R)$, we have $(A_v, (P \cap A_v) + p_v)$ is a 0-prime of R . That is, for (A_v, p_v) to admit a 0-prime, it suffices

that v be s -compatible with some ordering. This is also a necessary condition.

PROPOSITION 4.4. *For $(A_v, p_v) \in MVal(R)$, the pair admits a 0-prime if and only if v is s -compatible with some $P \in X(R)$.*

5. Two subrings and a topological space. For R a semi-real ring, set

$$M^\#(R) = \{(v, i) | v \in \text{Val}(R) \text{ and } i : k_v \hookrightarrow \mathbf{R}\}$$

and

$$SV^\#(R) = \{v \in \text{Val}(R) | k_v \text{ is formally real}\}.$$

THEOREM 5.1. *With the notation as above,*

$$\bigcap_{v \in M^\#(R)} A_v = \bigcap_{w \in SV^\#(R)} A_w = \bigcap_{P \in X(R)} A(P) \subseteq \bigcap_{(A,T) \in P_0(R)} A.$$

We denote the first subring by $H(R)$ and the latter by $H_M(R)$.

For $R = F$, a field, $H_M(F) = H(F)$ and the subring is called the *holomorphy ring* of F . For a commutative ring, the inclusion above may be strict as is the case for $R = \mathbf{Q}[X]$. Here one can show that $H_M(R) = R$ while $H(R) = \mathbf{Q}$ (see Example 4.2 of [8]).

There is another characterization for $H(R)$ of which we should make note. For $I \in \mathbf{R} \text{Spec}(R)$, we have a natural homomorphism

$$n_I : R \rightarrow F_I := qf(R/I).$$

With this it can be shown that

$$H(R) = \bigcap_{I \in \mathbf{R} \text{Spec}(R)} n_I^{-1}(H(F_I)).$$

For $S \subseteq R$ with $S + S \subseteq S, SS \subseteq S$ and $1 \in S$, the set

$$A(S) := \{r \in R | \exists n \in \mathbf{N}, 0 \neq m \in \mathbf{N} \text{ with } n \pm mr \in S\}$$

is a subring of R . For a field it is well-documented that $H(F)$ coincides with $A(\sigma(F))$. Our next result generalizes this fact.

THEOREM 5.2. *For R any of the following:*

- (a) $k[X_1, \dots, X_n]$ for some (formally real) field k ,
- (b) a semi-local dedekind domain with 2 a unit, or
- (c) a ring having $|\mathbf{R} \operatorname{Spec}(R)| < \infty$,

we have $H(R) = A(\cap_{P \in X(R)} P)$. Further, in (a), $H(R) = H(k)$.

PROOF. In (a) we will make use of the following construction. For a semi-real domain, S , and $P \in X(S)$,

$$T_1(P) := \{0\} \cup \{f = s_0 + \dots + s_n X^n \mid n = \deg(f) \text{ and } s_n \in P\}$$

is an ordering of $S[X]$ and $T_1(P) \cap S = P$. With this, given $P \in X(k)$, let $T_n(P)$ be the ordering of $R = k[X_1, \dots, X_n]$ induced by P and successive applications of the above.

Let $f \in A(\sigma(k))$. Then there exist $p, q \in \mathbf{N}$ with $p \pm qf \in \sigma(k)$. So $f \in k$. Just suppose that $p \pm qf \notin T$ for some $T \in X(R)$. Then, setting $T' = T \cap k, p \pm qf \notin T'$ and we have a contradiction to the choice of f . Hence $A(\sigma(k)) \subseteq A(\cap_{X(R)} P)$.

Now choose $g \in H(R)$ and $P \in X(k)$. By definition, $g \in A(T_n(P))$ and $p' \pm q'g \in T_n(P)$ for some $p', q' \in \mathbf{N}$. Again $g \in k$ and as such $p' \pm q'g \in T_n(P) \cap k = P$, this for all $P \in X(k)$. Thus $g \in \cap_{X(k)} A(P) = H(k)$, and we have settled (a).

The case (b) quickly follows from the fact that

$$\sigma(R) = \sigma(F) \cap R,$$

where F is the field of quotients of R (see [2]).

The remaining case is handled somewhat differently. Let $a \in H(R)$ and $I \in \mathbf{R} \operatorname{Spec}(R)$. By the above, $\bar{a} \in H(F_I) = A(\sigma(F_I))$ and there exists $0 < r_I \in \mathbf{Q}$ with $r_I \pm \bar{a} \in P'$ for all $P' \in X(F_I)$. Consider $r = \max\{r_I \mid I \in \mathbf{R} \operatorname{Spec}(R)\}$. Writing $r = n/m$, we have $n \pm ma \in P$ for all $P \in X(R)$. This completes the proof. \square

We have a well-defined mapping $\Lambda : X(R) \rightarrow M^\#(R)$ given by

$$P \mapsto (v(P), i(P)),$$

where $v(P)$ is the canonical valuation introduced in Theorem 4.3 and $i(P)$ is the unique embedding of k_ν into \mathbf{R} induced by \overline{P}' . For $(w, j) \in M^\#(R)$, there is an ordering, T' , of F_w with $\overline{T}' = j^{-1}(\mathbf{R}^+)$. Letting T denote the pull-back ordering on R , we have $\Lambda(P) = (w, j)$. Thus the mapping is surjective, and, with the Harrison topology on $X(R)$, we may consider the quotient topology, \mathcal{T} , on $M^\#(R)$ which is induced by Λ .

Given $a \in R$, we may also define a map $\hat{a} : M^\#(R) \rightarrow \mathbf{R} \cup \{\infty\}$ via

$$(v, i) \mapsto \lambda_\nu(a),$$

where λ_ν represents the real place determined by (v, i) on F_v . Note that, by our definition, if $a \in H(R)$, then $\text{im}(\hat{a}) \subseteq \mathbf{R}$. With this we may define two more topologies on $M^\#(R)$. Namely,

\mathcal{S} : the smallest topology making \hat{a} continuous for all $a \in R$ and

\mathcal{S}_H : The smallest topology making \hat{a} continuous for all $a \in H(R)$.

Clearly \mathcal{S} is a finer topology than \mathcal{S}_H .

PROPOSITION 5.3. *The map $\hat{a} : (M^\#(R), \mathcal{T}) \rightarrow \mathbf{R}$ is continuous for all $a \in H(R)$.*

A proof of this proposition can be obtained from a slight modification of the proof in the field case (see Theorem 9.7 of [5]). This shows that \mathcal{T} is also finer than \mathcal{S}_H .

When dealing with fields, it is well-known that these topologies agree on $M^\#(F)$, making it a compact and Hausdorff space. In general, this is not the case.

Let $I \in \text{R Spec}(R)$ and set

$$M_I^\#(R) = \{(v, i) \in M^\#(R) \mid I_v = I\}$$

and

$$X_I(R) = \{P \in X(R) \mid \text{Supp}(P) = I\}.$$

With this notation, we have the following commuting diagram:

$$\begin{array}{ccc}
 X_I(R) & \xrightarrow{\Lambda_I} & M_I^\#(R) \\
 (\cdot)' \downarrow & & \downarrow \text{ nat} \\
 X(F_I) & \xrightarrow{\lambda} & M^\#(F_I)
 \end{array}$$

where Λ_I denotes the restriction of Λ to $X_I(R)$; P' , the natural ordering of F_I induced by $P \in X(R)$ and $\text{nat}(v, i) = (\mathcal{V}, i)$. Letting \mathcal{T}_I be the quotient topology induced by Λ_I on $M_I^\#(R)$ and, using the restricted Harrison topology on $X_I(R)$, one can show that the vertical maps are both homeomorphisms.

Lastly, since $M^\#(R) = \cup(M_I^\#(R), \mathcal{T}_I)$, we can introduce a disjoint union topology, \mathcal{U} , on $M^\#(R)$. It is easily verified that $\mathcal{T} \subseteq \mathcal{U}$. Also, for $a \in R$,

$$\hat{a} : (M_I^\#(R), \mathcal{T}_I) \rightarrow \mathbf{R} \cup \{\infty\}$$

is continuous. Thus,

$$\hat{a} : (M^\#(R), \mathcal{U}) \rightarrow \mathbf{R} \cup \{\infty\}$$

is as well, and $\mathcal{S} \subseteq \mathcal{U}$. We may arrange these topologies in the following diagram:

$$\begin{array}{ccc}
 \mathcal{S} & \subseteq & \mathcal{U} \\
 \cup | & & \cup | \\
 \mathcal{S}_H & \subseteq & \mathcal{T}
 \end{array}$$

Now $M^\#(R)$ with \mathcal{T} is a compact (but not necessarily Hausdorff) space, while, with the topology \mathcal{U} , it becomes locally compact and Hausdorff.

PROPOSITION 5.4. *Let $|\mathbf{R} \text{Spec}(R)| < \infty$. Then the above topologies coincide if and only if $(M^\#(R), \mathcal{S}_H)$ is Hausdorff.*

In particular, for R as in the proposition, the topologies all agree if $H(R)$ separates points of $M^\#(R)$. As we will see in the next section, this is not always the case.

6. Examples.

EXAMPLE 6.1. Let $R = \mathbf{Q}[X]$, $w : R \rightarrow \mathbf{Z}^*$ given by $w(f) = -\deg(f)$ and

$$P = \{0\} \cup \{a_m X^m + \cdots + a_n X^n \mid m \leq n \text{ and } a_m > 0\}.$$

One can check that w is a valuation of R which is compatible with P , but not s -compatible (consider $a = X^2$ and $b = X$). Hence, $w \neq v(P)$, although we have $(d_w, \overline{P}) \hookrightarrow (\mathbf{Q}, \mathbf{Q}^+)$ in a natural way.

In this example, $v(P)$ is the valuation obtained from the X -adic valuation of $\mathbf{Q}(X)$.

EXAMPLE 6.2. Let $R = \mathbf{Q}[X]_{(X^2+1)}$. We note in this example that, with $F = \mathbf{Q}(X)$, $H(R) = R \cap H(F) \neq H(F)$ (since $1/(X^2 + 1) \in H(F) \setminus H(R)$). Further, it can be shown that $H(R)$ separates points of $M^\#(R)$. Thus, $(M^\#(R), \mathcal{S}_H)$ is a compact, Hausdorff space.

EXAMPLE 6.3. Consider $R = \mathbf{Q}[X]_{(X^2-5)}$. In contrast to the above, we now have $H(R) = H(F)$. Let v be the trivial valuation with $I_v = (X^2 - 5)$ and w be the $(X^2 - 5)$ -adic valuation of $\mathbf{Q}(X)$ restricted to R . Letting $i : \mathbf{Q}[\sqrt{5}] \rightarrow \mathbf{R}$ denote the identity map, we see that (v, i) and (w, i) are distinct points of $M^\#(R)$, but, for any $a \in H(R)$ (or even R), $\hat{a}(v, i) = \hat{a}(w, i)$. Thus $(M^\#(R), \mathcal{S}_H)$ is not Hausdorff.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720

Currently: DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY, HAMILTON, NY 13346